FINSLER SPACES FROM CONFORMAL $\beta$-CHANGE

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Abstract: We have considered the conformal $\beta$-change of the Finsler metric given by $L(x, y) \rightarrow \tilde{L}(x, y) = e^{\sigma(x)}f(L(x, y), \beta(x, y))$, where $\sigma(x)$ is a function of $x$, $\beta(x, y) = h_i(x)y^i$ is a $1$-form on the underlying manifold $M^n$, and $f(L(x, y), \beta(x, y))$ is a homogeneous function of degree one in $L$ and $\beta$. Let $F^n$ and $\tilde{F}^n$ denote Finsler spaces with metric functions $L$ and $\tilde{L}$ respectively. It has been investigated how $S_3$-likeness and $S_4$-likeness of $F^n$ are linked with corresponding properties of $\tilde{F}^n$. Further, necessary and sufficient conditions for a Killing vector field of $F^n$ to be a vector field of the same kind in $\tilde{F}^n$ have been obtained.

Keywords: Finsler metric, conformal $\beta$-change, $S_3$-like Finsler space, $S_4$-like Finsler space, $\nu$-curvature tensor, Killing vector field.

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1. Introduction

Let $F^n = (M^n, L)$ be an $n$-dimensional Finsler space on the differentiable manifold $M^n$ equipped with the fundamental function $L(x, y)$. Prasad and Kumari [11] and Shibata [12] have studied the general case of $\beta$-change, that is, $L'(x, y) = f(L, \beta)$, where $f$ is a positively homogeneous function of degree one in $L$ and $\beta$, and $\beta$ given by $\beta(x, y) = h_i(x)y^i$ is a one-form on $M^n$. The $\beta$-change of special Finsler spaces has been studied by Shukla, Pandey and Mandal [14].

The conformal theory of Finsler space was initiated by Knebelman [8] in 1929 and has been investigated in detail by many authors (Hashiguchi [3], Izumi [5, 6] and Kitayama [7]). The conformal change is defined as $L'(x, y) = e^{\sigma(x)}L(x, y)$, where $\sigma(x)$ is a function of position only and is known as conformal factor. In 2008, Abed [1, 2] introduced the change $L''(x, y) = e^{\sigma(x)}L(x, y) + \beta(x, y)$, which he called a $\beta$-conformal change, thus he generalized the conformal and Randers changes. Moreover, he studied some special Finsler spaces under this change such as C-reducible and $S_3$-like Finsler
spaces. In 2009 and 2010, Youssef, Abed and Elgendi [18,19] introduced the transformation $L''(x,y) = f(e^\beta L, \beta)$, which is general $\beta$-change of conformally changed Finsler metric $L$. They have not only established the relationships between some important tensors of $(M^n, L)$ and the corresponding tensors of $(M^n, L'')$, but have also studied several properties of this change.

Shukla and Mishra [13] have changed the order of combination of the above two changes, i.e., they have applied $\beta$-change first and conformal change afterwards as follows:

$$\tilde{L}(x, y) = e^{\sigma(x)}f(L(x, y), \beta(x, y)), \quad (1)$$

where $\sigma(x)$ is a function of $x$ and $\beta(x, y) = b_i(x)y^i$ is a 1-form. They have called this change as conformal $\beta$-change of Finsler metric. In this paper they have investigated the condition under which a conformal $\beta$-change of Finsler metric leads a Douglas space into a Douglas space. They have also found the necessary and sufficient conditions for this change to be a projective change.

They have studied quasi-C-reducibility, C-reducibility and semi-C-reducibility of the Finsler space with this metric in their paper [15], wherein they have also calculated the T-tensor [4] of $\tilde{F}^n$. When $\sigma = 0$, it reduces to a $\beta$-change. When $\sigma = constant$, it becomes a homothetic $\beta$-change. When $f(L, \beta)$ has special forms as $L + \beta, \frac{v^2}{L-\beta}, \frac{v^2}{\beta^m}, \frac{v^{m+1}}{\beta^m}$ ($m \neq 0, -1$), one obtains conformal Randers change, conformal Matsumoto change, conformal Kropina change, conformal generalized Kropina change of Finsler metric respectively.

In the present paper, we investigate some other properties of conformal $\beta$-change. The Finsler space equipped with the metric $\tilde{L}$ given by (1) will be denoted by $\tilde{F}^n$. Throughout the paper the quantities corresponding to $\tilde{L}$ will be denoted by putting bar on the top of them.

Homogeneity of $f$ gives

$$Lf_1 + \beta f_2 = f, \quad (2)$$

where subscripts “1” and “2” denote the partial derivatives with respect to $L$ and $\beta$ respectively. Differentiating above equation with respect to $L$ and $\beta$ respectively, we get

$$Lf_{11} + \beta f_{21} = 0 \text{ and } Lf_{12} + \beta f_{22} = 0. \quad (3)$$

Hence we have

$$\frac{f_{11}}{\beta^2} = \frac{-f_{12}}{L}\beta = \frac{f_{22}}{L^2}, \quad (4)$$

which gives

$$f_{11} = \beta^2 \omega, f_{12} = -L\beta \omega, f_{22} = L^2 \omega, \quad (5)$$

where Weierstrass function $\omega$ is positively homogeneous of degree-3 in $L$ and $\beta$. Therefore

$$L\omega_1 + \beta \omega_2 + 3\omega = 0, \quad (6)$$
where $\omega_1$ and $\omega_2$ are positively homogeneous of degree $-4$ in $L$ and $\beta$. Throughout the paper we frequently use the above equations without quoting them. Also we have assumed that $f$ is not a linear function of $L$ and $\beta$ so that $\omega \neq 0$.

Killing equations play an important role in the study of a Finsler space whose points undergo an infinitesimal transformation. In fact, they give a characterization for the transformation to preserve distances. In 1979, Singh et al. [17] have discussed Killing correspondence between Randers space $(M^n, L = \alpha + \beta)$ and the space $(M^n, L_1)$, where $L_1^2 = L^2 + \beta^2$. In 2014, Shukla and Gupta [16] have discussed Killing correspondence between Randers space $(M^n, L)$ and the space $(M^n, L^*)$, where $L^* = f(L, \beta)$. Kumbar et al. [9] have studied Killing correspondence between $(M^n, L)$ and $(M^n, L'')$, where $L'' = f(e^\theta L, \beta)$.

The aim of this paper is to study some special Finsler spaces arising from conformal $\beta$-change of Finsler metric, viz., $S_3$-like and $S_4$-like Finsler spaces. Further, we study Killing correspondence between the Finsler spaces $F^n$ and $\tilde{F}^n$.

2. Fundamental quantities of $\tilde{F}^n$

Differentiating equation (1) with respect to $y^i$ we have

$$\tilde{l}_i = e^\sigma (f_1 l_i + f_2 b_i).$$

Differentiating (7) with respect to $y^j$, we have

$$\tilde{h}_{ij} = e^{2\sigma} \left( \frac{f_{ij}}{L} h_{ij} + f L^2 \omega m_i m_j \right),$$

where $m_i = b_i - \frac{\beta}{L} L_i$.

From (7) and (8) we get the following relation between metric tensors of $F^n$ and $\tilde{F}^n$:

$$\tilde{g}_{ij} = e^{4\sigma} \left[ \frac{f_{ij}}{L} g_{ij} - \frac{\beta}{L} l_i l_j + p(l_i b_j + b_i l_j) + (f L^2 \omega + f z^2) b_i b_j \right],$$

where $p = f_1 f_2 - f L \beta \omega$.

The contravariant components $\tilde{g}^{ij}$ of the metric tensor of $\tilde{F}^n$, obtainable from $\tilde{g}^{ij} \tilde{g}_{jk} = \delta^i_k$, are as follows:

$$\tilde{g}^{ij} = e^{-2\sigma} \left[ \frac{L}{f_{ij}} g^{ij} + \frac{\beta}{L} l^i l^j - \omega L^2 \frac{f_{ij}}{f_{kl}} b^i b^j - \frac{\beta}{L} f_{ij} (l^i b^j + b^i l^j) \right],$$

where $l^i = g^{ij} l_j$, $b^i = g^{ij} b_j$, $b^2 = b^i b_i$, $g^{ij}$ is the reciprocal tensor of $g_{ij}$ of $F^n$, and $\Delta = b^2 - \frac{\beta^2}{L^2}$, $t = f_1 + L^2 \omega \Delta$.

Cartan’s covariant C-tensor $C_{ijk}$ of $F^n$ is defined by

$$C_{ijk} = \frac{1}{4} \partial_i \partial_j g^{kl} L^2 = \frac{1}{2} \partial_i g_{ij}$$

and Cartan’s C-vector is defined as follows:
\[ C_i = C_{ijk} g^j. \]  

Under the conformal change (1) we get the following relation between Cartan's C-tensors of \( F^n \) and \( \tilde{F}^n \):

\[ \tilde{C}_{ijk} = e^{2\sigma} \left[ \frac{f_f}{L} C_{ijk} + \frac{p}{2L} (h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + qL^2 - m_im_jm_km_k \right], \]

where \( = 3f_2\omega + f\omega_2. \)

We have

(a) \( m_il^i = 0 \), (b) \( m_ib^i = b^2 - \frac{p^2}{L^2} = \Delta = b_lm_l \), (c) \( g_{ij}m^i = h_{ij}m^i = m_j. \)

From (7), (9), (10) and (13), we get

\[ \tilde{C}^i_{jk} = C^i_{jk} + \frac{p}{2ff_3} (h_{jk}m^i + h_{j}^i m_k + h_{k}^i m_j) - \frac{pL\Delta}{2f^2f_3^2} h_{jk}m^n - \frac{(2pL + qL^4\Delta)}{2f^2f_3^2} m_im_jm_kn^l \]

\[ - \frac{l}{f_3} C_{ijk}n^l + \frac{qL^3}{2ff_3} m_jm_kn^l, \]

where \( n^h = fL^2\omega b^h + pt^h \) and \( h_{ij} = g_{ij}h_{ij}, C_{jk} = C_{ijk}b^i, C_{i} = C_{ijk}b^j b^k \) and so on.

**Proposition 2.1.** The normalized supporting element \( \tilde{l}_i \), angular metric tensor \( \tilde{g}_{ij} \), fundamental metric tensor \( \tilde{g}_{ijk} \) and (h)v-torsion tensor \( \tilde{C}_{ijk} \) of \( \tilde{F}^n \) are given by (7),(8), (9) and (13) respectively.

From (10), (12), (13) and (14) we get the following relations between the C-vectors of \( F^n \) and \( \tilde{F}^n \):

\[ \tilde{C}_i = C_i - L^2\omega C_L + \mu m_i, \]

where \( = \frac{p(n+1)-3pL^3\omega + qL^4\Delta(1-L^3\omega)}{2ff_3}. \)

3. **Expression for v-curvature tensors of \( \tilde{F}^n \)**

The v-curvature tensor of Finsler space with fundamental function L is given by

\[ S_{hijk} = C_{ijr} \tilde{C}_{hkr} - C_{ikr} \tilde{C}_{hjr}. \]

Therefore the v-curvature tensor of conformally \( \beta \)-changed Finsler space \( \tilde{F}^n \) is given by

\[ \tilde{S}_{hijk} = \tilde{C}_{ijr} \tilde{C}_{hkr} - \tilde{C}_{ikr} \tilde{C}_{hjr}. \]

From equations (13) and (15), we have

\[ \tilde{C}_{ijk} \tilde{C}_{hk} = e^{2\sigma} \left[ \frac{f_f}{L} C_{ijk} \tilde{C}_{hk} + \frac{p}{2L} (C_{ijk}m_h + C_{ijk}m_k + C_{ikh}m_j + C_{hjk}m_i) \right] \]

\[ - \frac{pf_f}{2Lc} (C_{ijk}h_{hk} + C_{ijk}h_{ij}) - \frac{f_fL^2\omega}{c} C_{ijk}C_{hk} + \frac{pL^3\Delta}{4ff_3} h_{hk}h_{ij}. \]
Interchanging \( j \) and \( k \) in (18) and subtracting the equation thus obtained from (18) and using (17), we get

\[
\bar{S}_{hijk} = e^{2\sigma} \left[ \frac{ff_{s}}{L} S_{hijk} + \Theta_{jk}(d_{hk}h_{ij} + d_{ij}h_{hk} + E_{hk}C_{ij} + E_{ij}C_{hk}) \right],
\]

where \( d_{ij} = QC_{ij} + RH_{ij} + PM_{i}m_{j} \),

\[
E_{ij} = I m_{i}m_{j} + T C_{ij},
\]

\[
P = \frac{p(p(f_{s} - L^{3}A^{3}) + L^{3}Q)}{4t \ell}, \quad Q = \frac{f_{s}p}{2t \ell}, \quad R = \frac{p^{2}A}{8t \ell},
\]

\[
I = \frac{L^{2}(f_{s}q - 2p \omega)}{2t}, \quad T = \frac{f_{s}L^{2}Q}{2t},
\]

and \( \Theta_{jk} \) denotes interchange of \( j \) and \( k \) and subtraction.

**Proposition 3.1** The relation between \( v \)-curvature tensors of \( F^{n} \) and \( \bar{F}^{n} \) is given by (19).

We get the following expressions for the vertical Ricci tensor \( \bar{S}_{ik} \) and the vertical scalar curvature \( \bar{S} \) associated with the transformed space \( \bar{F}^{n} \):

\[
\bar{S}_{ik} = S_{ik} + K h_{ik} + \left\{ L^{2}Q \left[ \frac{ff_{s}}{ff_{s}} - \frac{(n-3)k}{ff_{s}} \right] d_{ik} + \varphi_{ik} \right\},
\]

where \( K = \frac{L^{3}Q}{ff_{s}} \),

\[
\varphi_{ik} = \frac{L}{ff_{s}} \left[ E_{rk}C_{r}^{i} + E_{ri}C_{r}^{k} - \left( L^{2}A - \frac{f_{s}L^{2}Q}{2t} \right) C_{ik} \right]
\]

\[
- \frac{L^{2}Q}{ff_{s}} \left( d_{k}m_{i} + d_{i}m_{k} + E_{k}C_{L} + E_{i}C_{c_{k}} - E_{c_{k}}C_{i} - E_{i}C_{k} - E_{k}C_{i} \right)
\]

\[
+ \frac{ff_{s}}{L} \bar{S}_{hijk}b^{h}b^{j},
\]

\( d_{-} = d_{h}b^{h}b^{j}, E_{-} = E_{h}b^{h}b^{j} \)

and

\[
\bar{S} = e^{-2\sigma} \left[ \frac{L}{ff_{s}} S - \frac{2L}{ff_{s}} \left( L^{2}Q - (n-2) \right) \right] - \frac{L^{2}Q}{ff_{s}} \varphi_{-} + \frac{L}{ff_{s}} \varphi - \frac{L^{2}Q}{ff_{s}} \bar{S}_{hijk}b^{h}b^{j},
\]

where \( \varphi_{-} = \varphi_{h}b^{h}b^{j}, \varphi = \varphi_{h}g^{h}b^{j} \)

and \( \varphi_{ij} \) is symmetric and indicatory.
4. The $S_3$-likeness and $S_4$-likeness

In this section, following Matsumoto [10], we shall investigate special cases of $\bar{F}^n$.

**Definition 4.1.** A Finsler space $(M^n, L)$ with dimension $n \geq 3$ is called $S_3$-like if the $\nu$-curvature tensor $S_{hijk}$ satisfies

$$S_{hijk} = \frac{S}{(n-1)(n-2)} (h_{ik}h_{kj} - h_{hk}h_{ij}),$$

where scalar $S$ is vertical scalar curvature.

Define the tensor

$$K_{hijk} = S_{hijk} - \frac{S}{(n-1)(n-2)} (h_{ik}h_{kj} - h_{hk}h_{ij}).$$

It is clear that the tensor $K_{hijk}$ vanishes iff $F^n$ is $S_3$-like.

**Proposition 4.1.** Under the conformal $\beta$-change (1), the tensor $K_{hijk}$ associated with the space $\bar{F}^n$ has the form

$$\bar{K}_{hijk} = e^{2\sigma} \frac{f_{ij}}{L} K_{hijk} + U_{hijk},$$

where

$$U_{hijk} = e^{2\sigma} \Theta_{ik} \left[ d_{ik}h_{ij} + d_{ij}h_{ik} + E_{ik}c_{ij} + E_{ij}c_{ik} - \frac{\rho f^2 f^2_{ij}}{L^2 (n-1)(n-2)} h_{ik}h_{kj} - \frac{f_{ij}^2 \omega}{(n-1)(n-2)} \left( S + \rho f^2 f^2_{ij} \right) h_{ik}h_{kj} + h_{ik}m_{ik}m_{kj} \right],$$

$$\rho = \left[ \frac{L}{f_{ij}} \varphi - \frac{f_{ij}^2 \omega}{f_{ij}^2 \tau} \left( \varphi + S_{ik}b^ib^k \right) - \frac{2Lk}{f_{ij}} \left( \frac{f^2 \omega}{\tau} - (n-2) \right) \right].$$

From (24) we have the following theorem:

**Theorem 4.1.** Conformally $\beta$-changed Finsler space $\bar{F}^n$ is $S_3$-like iff $F^n$ is $S_3$-like and the tensor $U_{hijk}$ given by (25) vanishes identically.

**Definition 4.2.** A Finsler space $(M^n, L)$ with dimension $n \geq 4$ is called $S_4$-like if the $\nu$-curvature tensor $S_{hijk}$ satisfies

$$S_{hijk} = \Theta_{ik} (h_{ij}K_{ik} + h_{ik}K_{ij}),$$

where

$$K_{ik} = \frac{1}{(n-3)} \left( S_{ik} - \frac{S}{2(n-2)} h_{ik} \right).$$

Define the tensor

$$H_{hijk} = S_{hijk} - \Theta_{ik} (h_{ij}K_{ik} + h_{ik}K_{ij}).$$

Then $F^n$ is $S_4$-like iff $H_{hijk}$ vanishes.

**Proposition 4.2.** Under the conformal $\beta$-change (1), the tensor $H_{hijk}$
associated with the space $\tilde{F}^n$ has the form
\[ \tilde{H}_{hijk} = e^{2\sigma} \frac{f_{\xi}}{L} H_{hijk} + V_{hijk}, \] (26)
where
\[ V_{hijk} = \mathcal{Z}_{hijk} \{ E_{hk} C_{ij} + f L^2 \omega K_{hk} m_i m_j \] 
\[ + \frac{f_{\xi}}{L(n-3)} \left( \frac{\Delta L^2 \omega}{f_{\xi} t} d_{hk} h_{ij} + K h_{ij} h_{hk} + \varphi_{hk} h_{ij} \right) - \frac{f L^2 \omega S}{2(n-2)(n-3)} m_h m_k h_{ij} \] 
\[ + \frac{\rho f_{\xi} h_{ij}}{2L(n-2)(n-3)} \left( \frac{f_{\xi}}{L} h_{hk} + f L^2 \omega m_{hk} \right) + \frac{f L^2 \omega h_{hk} m_i m_j}{(n-3)} \] 
\[ + \frac{f_{\xi}^2}{f_{\xi}(n-3)} \left( \varphi_{hk} + \left( \frac{\Delta L^2 \omega}{t} - (n-3) \right) d_{hk} - \frac{\rho f_{\xi}^2}{2L^2(n-2)} h_{hk} \right) m_i m_j \] (27)
and $\mathcal{Z}_{hijk} (X_{hk} Y_{ij})$ denotes $X_{hk} Y_{ij} + X_{ij} Y_{hk} - X_{hj} Y_{ik} - X_{ik} Y_{hj}$.

From (26) we have the following theorem:

**Theorem 4.2.** Conformally $\beta$-changed Finsler space $\tilde{F}^n$ is $S_4$-like iff $F^n$ is $S_4$-like and the tensor $V_{hijk}$ given by (27) vanishes identically.

5. **Killing correspondence of $F^n$ and $\tilde{F}^n$**

Let us consider an infinitesimal transformation
\[ x'^i = x^j + \varepsilon v^i(x), \] (28)
where $\varepsilon$ is an infinitesimal constant and $v^i(x)$ is a contravariant vector field. This vector field $v^i(x)$ is said to be a Killing vector field in $F^n$ if the metric tensor of the Finsler space with respect to the infinitesimal transformation (28) is Lie invariant, i.e. if
\[ E_v g_{ij} = 0, \] (29)
where $E_v$ is the operator of Lie differentiation. The condition (29) is equivalent to
\[ v_{ij} + v_{jl} + 2C^p_{ij} v_{h|0} = 0, \] (30)
where $v_i = g_{il} v^l$ and the symbol $|$ denotes h-covariant differentiation with respect to the Cartan’s connection $\mathcal{C}'$.

In this section we investigate a necessary and sufficient condition for Killing vector field in $F^n$ to be a Killing vector field in $\tilde{F}^n$. For the aforesaid investigation some preliminaries are required, which are given below.

We put
\[ 2 r_{ij} = b_{i|j} + b_{j|i}, \quad 2 s_{ij} = b_{i|j} - b_{j|i}. \]

The transformed Christoffel symbols of the Finsler space $\tilde{F}^n$ are given by
\[
\tilde{y}^i_{jk} = \frac{1}{2} \tilde{g}^{ir} \left( \partial_j \tilde{g}_{kr} + \partial_k \tilde{g}_{jr} - \partial_r \tilde{g}_{jk} \right).
\] (31)

Now we deal with the well-known functions \( G^i(x, y) \) which are \((2p)\)-homogeneous in \( y^i \) and are given by
\[
G^i = \frac{1}{2} y^i_{jk} y^j y^k.
\] (32)

Using (9), (10), (31) and (32), we have
\[
\tilde{G}^i = \frac{1}{2} \tilde{y}^i_{jk} y^j y^k = G^i + D^i,
\] (33)
where the vector \( D^i \) is given by
\[
D^i = \frac{f_{2L}}{f_1} s^i_0 - \frac{k}{ff_1} (f_1 r_{00} - 2Lf_2 s_{r_0 b^r}) \left( py^i - L^2 \omega f b^i \right) + \sigma_0 y^i - \frac{1}{2} f^2 \sigma^i,
\] (34)
in which \( s^i_0 = g^{ir} s_{rj} y^j \).

Let \( C = (F_{jk}, \tilde{N}^i, \tilde{C}_{jk}) \) be the Cartan’s connection on the space \( F^n \).

For coefficient \( \tilde{N}^i = \hat{\partial}^i \tilde{G}^i \) to the non-linear connection, we differentiate (33) with respect to \( y^j \) and get
\[
\tilde{N}^i = N^i_j + D^i_j,
\] (35)
where the tensor \( D^i_j = \hat{\partial}^i D^i_j \) is given by
\[
D^i_j = \frac{le^{2\sigma}}{f_1 f} A^i_j - Q^i_{Arj} b^r + \frac{pl f_2}{f^2 f_1^2} b_{0ij} \left\{ -L f_1 b^i + (f \beta - \Delta L f_2) y^i \right\} + \sigma_j y^i - f \sigma^i (f_1 l_j + f_2 b_j),
\] (36)
in which
\[
A^i_{ij} = \frac{1}{2} r_{00} B_{ij} + e^{2\sigma} f f_2 s_{ij} + s_{0j} Q_i - \left( \frac{e^{2\sigma} f f_1}{k} C_{imj} + V_{ijm} \right) D^n,
\]
\[
A^i_j = g^{ir} A_{rj} , V_{ijm} = g_{sj} V_{im}^s , Q_i = e^{2\sigma} \left( py^i + f L^2 \omega y_i + f_2^2 b_i \right),
\]
\[
B_{jk} = \frac{1}{2} e^{2\sigma} \left( p h_{jk} + q L^2 m_j m_k \right), \hat{\partial}_k Q_j = \frac{1}{2} B_{mk},
\]
\[
V_{ijk} = \frac{e^{2\sigma}}{n + 1} \pi_{ijkl} \left\{ (n + 1) \left( \alpha_1 h_{ij} + \alpha_2 m_i m_j \right) m_k + \omega L^2 m_i m_j C_k \right\} + \omega L^2 \left( f f_1 h_{ij} + L^3 \omega m_i m_j \right) C_k,
\]
\[
\alpha_1 = \frac{p}{2L} - \frac{\mu f f_1}{L(n + 1)} , \quad \alpha_2 = \frac{q L^2}{6} - \frac{\mu \omega L^2}{(n + 1)}
\]
and \( \pi_{ijkl} \) represents cyclic permutation and sum over the indices \( i, j \) and \( k \).
Let $B\bar{\Gamma} = (\bar{G}^j_{jk}, \bar{N}^j_k, 0)$ be the Berwald connection on the space $F^n$. Differentiating (35) with respect to $y^k$, we have connection coefficients $\bar{G}^j_{jk} = \partial_k \bar{N}^j_k$ of $B\bar{\Gamma}$, which are given by

$$\bar{G}^j_{jk} = G^j_{jk} + B^j_{jk}, \quad B^j_{jk} = \partial_k D^j_j,$$

where $G^j_{jk}$ are connection coefficients of $B\Gamma$ on $F^n$. Substituting from (31), (13), (10), (35) and (15) in

$$\bar{F}^i_{jk} = \bar{\gamma}^i_{jk} + \bar{C}^j_{kr} \bar{N}^r_m g^{im} - \bar{C}^j_{kr} \bar{N}^r_j - \bar{C}^j_{rj} \bar{N}^r_k$$

we obtain connection coefficients $\bar{F}^i_{jk}$ of Cartan's connection $C\bar{\Gamma}$ on $F^n$ as

$$\bar{F}^i_{jk} = F^i_{jk} + D^i_{jk},$$

where

$$D^i_{jk} = \left[ \frac{e^{-2\sigma} L}{f f_1} g^{is} - Q^i b^s + y^s \frac{e^{-2\sigma} p L}{f^3 f_1 t} \{-L f b^i + (f \beta - \Delta f^2 f_2) y^i\} \right].$$

(37)

$$\begin{align*}
(B_{sj} b^0_{0j} + B_{sk} b^0_{0j} - B_{kj} b^0_{0s} + s_{sj} Q_k + s_{sk} Q_j + \tau_{kj} Q_s + \frac{e^2 f f_1}{L} C_{jkr} D^r_s) \\
+ V_{jkr} D^r_s - \frac{e^2 f f_1}{L} C_{skm} D^m_j - V_{sjm} D^m_k - \frac{e^2 f f_1}{L} C_{sjm} D^m_k - V_{skm} D^m_j \right) \\
- e^{-2\sigma} \sigma^i \bar{B}^i_{0j}.
\end{align*}$$

(39)

The tensor $D^i_{jk}$, called the difference tensor, has the following properties:

(a) $D^i_{j0} = B^i_{j0} = D^i_j$, (b) $D^i_{00} = 2 D^i$. (40)

**Theorem 5.1.** A Killing vector field $v^i(x)$ in $F^n$ is Killing vector field in $\bar{F}^n$ if and only if

$$W^i_{lj} v_{l0} - v_r D^i_{lj} - \bar{C}^h_{lj} v_r D^i_j = 0,$$

(41)

where $\bar{C}^h_{lj}$ is the associate Cartan tensor of $\bar{F}^n$ and $W^i_{lj} = \bar{C}^h_{lj} - C^h_{lj}$.

**Proof.** Assume that $v^i(x)$ is Killing vector field in $F^n$. Then condition (30) is satisfied. The h-covariant derivatives of $v^i(x)$ with respect to $C\Gamma$ and $C\bar{\Gamma}$ are respectively given as

(a) $v_{ilj} = \partial_j v^i - v_r F^r_{ij}$, (b) $v_{ilj} = \partial_j v^i - v_r F^r_{ij}$

(42)

where $\partial_j \equiv \frac{\partial}{\partial x^j}$.

By virtue of (42)(a) and (38), the equation (42)(b) takes the form

$$v_{ilj} = v_{ilj} - v_r D^i_{lj}.$$

(43)

From (43) and (40)(a) we have
\[ v_{i|j} + v_{j|i} + 2\tilde{C}^h_{ij}v_{h|i} = v_{i|j} + v_{j|i} - 2\nu_r D^r_{ij} + 2\tilde{C}^h_{ij}v_{h|0} - 2\tilde{C}^h_{ij}v_r D^r_h. \] (44)

Using condition (30) in (44) and putting \( W_{ij}^h = \tilde{C}^h_{ij} - C^h_{ij} \), we get
\[ v_{i|j} + v_{j|i} + 2\tilde{C}^h_{ij}v_{h|i} = 2W_{ij}^h v_{h|0} - 2\nu_r D^r_{ij} - 2\tilde{C}^h_{ij}v_r D^r_h. \] (45)

From (45) it follows that \( v^i(x) \) is Killing vector field in \( \tilde{F}^n \) iff (41) holds.

Transvecting (41) by \( y^i \) and \( y^j \) and noting equation 40(b) and the fact that \( W_{ij}^h y^i y^j = 0 = \tilde{C}^h_{ij}y^i y^j, \) we get
\[ v_r D^r = 0. \]

Thus we have the following corollary:

**Corollary 5.1.** If \( v^i(x) \) is Killing vector field in \( F^n \) and \( \tilde{F}^n \) both, then it is orthogonal to the vector \( D^i(x, y) \).

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**References**


