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A STUDY ON $\mathcal{W}_6$ AND $\mathcal{W}_8$ CURVATURE TENSORS IN (ε)-LP-SASAKIAN MANIFOLDS WITH QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract: The aim of the present paper is to study the $\mathcal{W}_6$-curvature tensors on (ε)-LP-Sasakian manifolds with quarter-symmetric metric connection. In this paper we studied the flatness of $\mathcal{W}_6$-Curvature tensor in (ε)-LP-Sasakian manifolds with quarter-symmetric metric connection. Moreover we construct a relationship between $\mathcal{W}_6$-curvature tensor in (ε)-LP-Sasakian manifolds with quarter-symmetric metric connection and $\mathcal{W}_8$-curvature tensor in (ε)-LP-Sasakian manifolds with Levi-Civita connection. At last, we have shown that an (ε)-LP Sasakian manifold with quarter-symmetric metric connection satisfying $\widetilde{\mathcal{W}}_8.\widetilde{\mathfrak{R}} = 0$ and $\widetilde{\mathcal{W}}_6.\widetilde{\mathfrak{R}} = 0$, is a special type of $\mathfrak{R}$-Einstein manifold.

Key Words and phrases: (ε)-LP-Sasakian manifold, Quarter-symmetric metric connection, $\mathcal{W}_6$-Curvature tensor, $\mathcal{W}_8$-Curvature tensor, flat manifold, $\mathcal{W}_6$-flat manifold.

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1. Introduction

In 1969, Takahashi [14] introduced an almost contact manifold equipped with an associated pseudo-Riemannian metric. In particular, he studied Sasakian manifolds equipped with an associated pseudo-Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are known as (ε)-almost contact metric manifolds and (ε)-Sasakian manifolds, respectively [2], [14] [10]. In 1989, Motsumoto [9] replaced the structure vector field ξ by $-\xi$ in an almost para-contact manifold and associated a Lorentzian metric with the resulting structure and gave a notion of Lorentzian para-Sasakian manifold. Mihai and Roska [8] and others studied Lorentzian para-Sasakian manifolds. Recently, Prasad and Shrivastava [13] introduced the notion of Lorentzian para-Sasakian manifolds with indefinite metric which also include usual LP-Sasakian manifolds. Such manifolds are known to be indefinite Lorentzian para-Sasakian manifolds or (ε)-Lorentzian para-Sasakian manifolds.

A quarter-symmetric metric connection has been studied by many authors in many ways to different extent as Mandal and De [6], Ahmad et al. [1], Prasad and Haseeb [12] and others.

This paper has been organized as follows: Section 1 is introductory. Section 2 is devoted to some results used in the sequel. In sections 3 and 4, we introduced an idea of (ε)-LP-Sasakian manifold with quarter-symmetric metric connection and $\mathcal{W}_6$ and $\mathcal{W}_8$ curvature tensors are defined on this manifold. In section 5, we have established that a $\mathcal{W}_6$-flat (ε)-LP-Sasakian manifold with quarter-symmetric metric connection need not be flat manifold. Further, we established a relationship between $\mathcal{W}_6$-curvature tensor and $\mathcal{W}_8$-curvature tensor in (ε)-LP-Sasakian manifold with respect to the quarter-symmetric metric connection and Levi Civita connection respectively.

2. Preliminaries

An n-dimensional differentiable manifold M, admitting a $(1,1)$-tensor field F, a covariant vector field ξ, a 1-form $\mathfrak{N}$, satisfying a Lorentzian metric g such that

$$F^2X = X + \mathfrak{N}(X)\xi,$$
$$\mathfrak{N} \circ F = 0,$$
$$F\xi = 0,$$
$$\mathfrak{N}(\xi) = -1, \tag{1}$$
$$\mathrm{g}(X,\xi) = \varepsilon\ \mathfrak{N}(X),$$
$$\mathrm{g}(X,FY) = \mathrm{g}(FX,Y),$$
$$\mathrm{g}(FX,FY) = \mathrm{g}(X,Y) + \varepsilon\ \mathfrak{N}(X)\mathfrak{N}(Y)$$

for all vector fields $X, Y \in \chi(M)$, is called (ε)-Lorentzian para-Sasakian manifold or simply as (ε)-LP-Sasakian manifold. Here $\mathfrak{D}$ denotes the Levi-civita connection with respect to g and ε is 1 or -1 according to the vector field ξ being timelike or spacelike.

Also we have

$$(\mathfrak{D}_X F)(Y) = \mathrm{g}(X,Y)\xi + \varepsilon\mathfrak{N}(Y)X + 2\varepsilon\mathfrak{N}(X)\mathfrak{N}(Y)\xi. \tag{2}$$
$$\mathfrak{D}_X\xi = \varepsilon FX \tag{3}$$

Now since the 1-form $\mathfrak{N}$ is closed in an (ε)-LP-Sasakian manifold, so we have

$$(\mathfrak{D}_X\mathfrak{N})(Y) = \mathrm{g}(FX,Y), \quad \mathrm{g}(FX,\xi) = 0 \tag{4}$$

for all vector fields $X, Y \in \chi(M)$.

Moreover, we have the curvature tensor R, the Ricci operator Q and the Ricci tensor $\mathcal{S}$ in (ε)-LP-Sasakian manifolds with the Levi-Civita connection $\mathfrak{D}$ as

$$\mathfrak{R}(X,Y)\xi = \mathfrak{N}(Y)X - \mathfrak{N}(X)Y, \tag{5}$$
$$\mathfrak{R}(\xi,X)Y = \varepsilon\mathrm{g}(X,Y)\xi - \mathfrak{N}(Y)X, \tag{6}$$

$$\mathfrak{R}(\xi,X)\xi = X + \mathfrak{N}(X)\xi = -\mathfrak{R}(X,\xi)\xi, \tag{7}$$

$$\mathfrak{N}(\mathfrak{R}(X,Y)Z) = \varepsilon[\mathrm{g}(Y,Z)\mathfrak{N}(X) - \mathrm{g}(X,Z)\mathfrak{N}(Y)], \tag{8}$$

$$\mathcal{S}(X,\xi) = (n-1)\mathfrak{N}(X), \tag{9}$$

$$QX = \varepsilon(n-1)X. \tag{10}$$

where $X,Y,Z \in \chi(M)$ and $\mathrm{g}(QX,Y) = \mathcal{S}(X,Y)$.

Definition 2.1: A linear connection $\widetilde{\mathfrak{D}}$ in a Riemannian manifold M is said to be a quarter-symmetric connection [13] if the torsion tensor T of the connection $\widetilde{\mathfrak{D}}$ is defined as

$$T(X,Y) = \widetilde{\mathfrak{D}}_X Y - \widetilde{\mathfrak{D}}_Y X - [X,Y] = \mathfrak{N}(Y)FX - \mathfrak{N}(X)FY \tag{11}$$

Further the quarter-symmetric connection is called quarter-symmetric metric connection if it satisfies

$$\widetilde{\mathfrak{D}}g = 0 \tag{12}$$

3. $(\boldsymbol{\varepsilon})$-$\boldsymbol{LP}$-Sasakian manifolds with quarter-symmetric metric connection

Let $\widetilde{\mathfrak{D}}$ be linear connection and $\mathfrak{D}$ be Riemannian connection of an n-dimensional (ε)-LP-Sasakian manifold. This linear connection $\widetilde{\mathfrak{D}}$ defined by [6]

$$\widetilde{\mathfrak{D}}_X Y = \mathfrak{D}_X Y + \mathfrak{N}(Y)FX - \varepsilon \mathrm{g}(FX,Y)\xi, \tag{13}$$

where $\mathfrak{N}$ is 1-form and $X,Y \in \chi(M)$, denotes the quarter-symmetric metric connection.

In an n-dimensional (ε)-LP-Sasakian manifold let $\widetilde{\mathfrak{R}}$ be curvature tensor with respect to quarter-symmetric metric connection $\widetilde{\mathfrak{D}}$, then

$$\widetilde{\mathfrak{R}}(X,Y)Z = \widetilde{\mathfrak{D}}_X \widetilde{\mathfrak{D}}_Y Z - \widetilde{\mathfrak{D}}_Y \widetilde{\mathfrak{D}}_X Z - \widetilde{\mathfrak{D}}_{[X,Y]} Z. \tag{14}$$

Now using (1), (5), (13) in (14) we have

$$\widetilde{\mathfrak{R}}(X,Y)Z = \mathfrak{R}(X,Y)Z + (2-\varepsilon)[\mathrm{g}(FX,Z)FY - \mathrm{g}(FY,Z)FX] \\ + [\mathfrak{N}(X)\mathrm{g}(Y,Z) - \mathfrak{N}(Y)\mathrm{g}(X,Z)]\xi + \varepsilon[\mathfrak{N}(Y)X - \mathfrak{N}(X)Y]\mathfrak{N}(Z). \tag{15}$$

for $X,Y,Z \in \chi(M)$.

Let $\{e_1, e_2, e_3, \dots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields on any point of the manifold. The Ricci tensor $\tilde{\mathcal{S}}$ and the scalar curvature $\bar{\tau}$ of the manifold with a quarter-symmetric metric connection $\widetilde{\mathfrak{D}}$ are defined by

$$\tilde{\mathcal{S}}(X,Y) = \sum_{i=1}^{n} \varepsilon_i \mathrm{g}(\widetilde{\mathfrak{R}}(e_i,X)Y, e_i)$$

and

$$\bar{\tau} = \sum_{i=1}^{n} \varepsilon_i \tilde{\mathcal{S}}(e_i, e_i).$$

Also, we have

$$\mathrm{g}(X, Y) = \sum_{i=1}^{n} \varepsilon_i \mathrm{g}(X, e_i)\mathrm{g}(Y, e_i).$$

Contracting (15) with respect to X, we have

$$\tilde{\mathcal{S}}(Y, Z) = \mathcal{S}(Y, Z) + (1-\varepsilon)\mathrm{g}(Y, Z) + (n\varepsilon - 1)\mathfrak{N}(Y)\mathfrak{N}(Z) - (2-\varepsilon)\mathrm{g}(FY, Z)\psi. \quad (16)$$

where $\psi = traceF$ and have value $\psi = \sum_{i=1}^{n} \varepsilon_i \mathrm{g}(Fe_i, e_i)$. Replacing $Z = \xi$ in equation (16) and using (1) and (9)

$$\tilde{\mathcal{S}}(Y, \xi) = (1-\varepsilon)(n-1)\mathfrak{N}(Y). \quad (17)$$

And replacing $X = \xi$ in equation (15) and using (1) and (6), we get

$$\tilde{\mathfrak{R}}(\xi, Y)Z = -(1-\varepsilon)(\mathrm{g}(Y, Z)\xi + \mathfrak{N}(Z)Y). \quad (18)$$

Again replacing $X = \xi$ in equation (15) and using (5) we have

$$\tilde{\mathfrak{R}}(X, Y)\xi = (1-\varepsilon)[\mathfrak{N}(Y)X - \mathfrak{N}(X)Y]. \quad (19)$$

for any vector fields $X, Y \in \chi(M)$.

4. $\mathcal{W}_6$-Curvature tensor and $\mathcal{W}_8$-Curvature tensor

Pokhariyal and Mishra [1] have introduced new tensor fields $\mathcal{W}$ and defined the $\mathcal{W}_6$-Curvature tensor field on Riemannian manifolds as [12]

$$\mathcal{W}_6(X, Y)Z = \mathfrak{R}(X, Y)Z + \frac{1}{n-1}[\mathrm{g}(X, Y)QZ - \mathcal{S}(Y, Z)X] \quad (20)$$

Again, Pokhariyal defined the $\mathcal{W}_8$-Curvature tensor field on Riemannian manifolds as [12]

$$\mathcal{W}_8(X, Y)Z = \mathfrak{R}(X, Y)Z + \frac{1}{n-1}[\mathcal{S}(X, Y)Z - \mathcal{S}(Y, Z)X] \quad (21)$$

Definition 4.1: The $\mathcal{W}_6$-curvature tensor in (ε)-LP-Sasakian manifolds with quarter-symmetric metric connection $\tilde{\mathfrak{D}}$ is

$$\tilde{\mathcal{W}}_6(X, Y)Z = \tilde{\mathfrak{R}}(X, Y)Z + \frac{1}{n-1}[\mathrm{g}(X, Y)\bar{Q}Z - \tilde{\mathcal{S}}(Y, Z)X] \quad (22)$$

Definition 4.2: The $\mathcal{W}_8$-curvature tensor in (ε)-LP-Sasakian manifolds with Levi-Civita connection $\mathfrak{D}$ is

$$\mathcal{W}_8(X, Y)Z = \mathfrak{R}(X, Y)Z - [\mathrm{g}(X, Y)Z - \mathrm{g}(Y, Z)X] \quad (23)$$

Definition 4.3: The $\mathcal{W}_8$-curvature tensor in (ε)-LP-Sasakian manifolds with quarter-symmetric metric connection $\tilde{\mathfrak{D}}$ is

$$\widetilde{W}_8(X,Y)Z = \widetilde{\mathfrak{R}}(X,Y)Z + \frac{1}{n-1}[\tilde{S}(X,Y)Z - \tilde{S}(Y,Z)X] \tag{24}$$

5. $\mathcal{W}_6$-flat (ε)-LP-Sasakian manifolds with quarter-symmetric metric connection

In this section, we consider $\mathcal{W}_6$-flat in (ε)-LP-Sasakian manifolds with quarter-symmetric metric connection.

Definition 5.1 An (ε)-LP-Sasakian manifold with quarter-symmetric metric connection is $\mathcal{W}_6$-flat if $\mathcal{W}_6$-curvature tensor vanishes. i.e.

$$\widetilde{W}_6(X,Y)Z = 0. \tag{25}$$

for all $X, Y, Z \in \chi(M)$.

Now Expanding equation (22) with respect to V, we have

$$\widetilde{W}_6(X,Y,Z,V) = \widetilde{\mathfrak{R}}(X,Y,Z,V) + \frac{1}{n-1}[\mathfrak{g}(X,Y)\mathfrak{g}(\bar{Q}Z,V) - \tilde{S}(Y,Z)\mathfrak{g}(X,V)] \tag{26}$$

By using equation (25), (26) reduces to

$$0 = \widetilde{\mathfrak{R}}(X,Y,Z,V) + \frac{1}{n-1}[\mathfrak{g}(X,Y)\mathfrak{g}(\bar{Q}Z,V) - \tilde{S}(Y,Z)\mathfrak{g}(X,V)] \tag{27}$$

$$\widetilde{\mathfrak{R}}(X,Y,Z,V) = \frac{1}{n-1}[\tilde{S}(Y,Z)\mathfrak{g}(X,V) - \mathfrak{g}(X,Y)\mathfrak{g}(\bar{Q}Z,V)] \tag{28}$$

Taking $Z = \xi$ in (28), we have

$$\widetilde{\mathfrak{R}}(X,Y,\xi,V) = \frac{1}{n-1}[\tilde{S}(Y,\xi)\mathfrak{g}(X,V) - \mathfrak{g}(X,Y)\mathfrak{g}(\bar{Q}\xi,V)] \tag{29}$$

Using equations (17) in (29),we have

$$\widetilde{\mathfrak{R}}(X,Y,\xi,V) = \frac{1}{n-1}[(1-\varepsilon)(n-1)\mathfrak{N}(Y)\mathfrak{g}(X,V) - \mathfrak{g}(X,Y)(1-\varepsilon)(n-1)\mathfrak{N}(V)] \tag{30}$$

$$\widetilde{\mathfrak{R}}(X,Y,\xi,V) = (1-\varepsilon)[\mathfrak{N}(Y)\mathfrak{g}(X,V) - \mathfrak{N}(V)\mathfrak{g}(X,Y)] \neq 0 \tag{31}$$

Hence, we have:

Theorem 5.1: A $\mathcal{W}_6$-flat (ε)-LP-Sasakian manifolds with quarter-symmetric metric connection is not flat.

6. Relationship between $\mathcal{W}_6$-curvature tensor and $\mathcal{W}_8$-curvature tensors in (ε)-LP-Sasakian manifolds

The $\mathcal{W}_6$-curvature tensor in (ε)-LP-Sasakian manifolds with quarter-symmetric metric connection is given by (22) and expanding it with respect to V as

$$\widetilde{W}_6(X,Y,Z,V) = \widetilde{\mathfrak{R}}(X,Y,Z,V) + \frac{1}{n-1}[\mathfrak{g}(X,Y)\tilde{S}(Z,V) - \tilde{S}(Y,Z)\mathfrak{g}(X,V)]. \tag{32}$$

Putting $Z = \xi$ in (32), we have

$$\widetilde{W}_6(X,Y,\xi,V) = \widetilde{\mathfrak{R}}(X,Y,\xi,V) + \frac{1}{n-1}[\mathfrak{g}(X,Y)\tilde{S}(\xi,V) - \tilde{S}(Y,\xi)\mathfrak{g}(X,V)]. \tag{33}$$

Also contracting (15) with respect to V and putting $Z=\xi$, we have

$$\widetilde{\mathfrak{R}}(X,Y,\xi,V)=\mathfrak{R}(X,Y,\xi,V)+(2-\varepsilon)[\mathrm{g}(FX,\xi)\mathrm{g}(FY,V)-\mathrm{g}(FY,\xi)\mathrm{g}(FX,V)]$$
$$+[\mathfrak{N}(X)\mathrm{g}(Y,\xi)\mathrm{g}(\xi,V)-\mathfrak{N}(Y)\mathrm{g}(X,\xi)\mathrm{g}(\xi,V)]$$
$$+\varepsilon\mathfrak{N}(\xi)[\mathfrak{N}(Y)\mathrm{g}(X,V)-\mathfrak{N}(X)\mathrm{g}(Y,V)]. \quad (34)$$

using (1), equation (34) reduces to

$$\widetilde{\mathfrak{R}}(X,Y,\xi,V)=\mathfrak{R}(X,Y,\xi,V)-\varepsilon[\mathfrak{N}(Y)\mathrm{g}(X,V)-\mathfrak{N}(X)\mathrm{g}(Y,V)]. \quad (35)$$

Now using equation (35) and (17), equation (33) becomes

$$\widetilde{\mathcal{W}}_6(X,Y,\xi,V)=\mathfrak{R}(X,Y,\xi,V)+\varepsilon[\mathfrak{N}(X)\mathrm{g}(Y,V)-\mathfrak{N}(Y)\mathrm{g}(X,V)]$$
$$+(\varepsilon-1)[\mathrm{g}(X,V)\mathfrak{N}(Y)-\mathrm{g}(X,Y)\mathfrak{N}(V)]. \quad (36)$$

Interchanging V and ξ, we have

$$-\widetilde{\mathcal{W}}_6(X,Y,V,\xi)=-\mathfrak{R}(X,Y,V,\xi)+\varepsilon[\mathfrak{N}(X)\mathrm{g}(Y,V)-\mathfrak{N}(Y)\mathrm{g}(X,V)]$$
$$+(\varepsilon-1)[\mathrm{g}(X,V)\mathfrak{N}(Y)-\mathrm{g}(X,Y)\mathfrak{N}(V)]. \quad (37)$$

Contracting (37) with respect to ξ, we have

$$\widetilde{\mathcal{W}}_6(X,Y)V=\mathfrak{R}(X,Y)V-\varepsilon[\mathrm{g}(Y,V)X-\mathrm{g}(X,V)Y]$$
$$-(\varepsilon-1)[\mathrm{g}(X,V)Y-\mathrm{g}(X,Y)V]. \quad (38)$$

Replacing $V=Z$ in (38), we have

$$\widetilde{\mathcal{W}}_6(X,Y)Z=\mathfrak{R}(X,Y)Z-\varepsilon[\mathrm{g}(Y,Z)X-\mathrm{g}(X,Z)Y]$$
$$-(\varepsilon-1)[\mathrm{g}(X,Z)Y-\mathrm{g}(X,Y)Z]. \quad (39)$$

Using equation (8) and simplifying (39), we get

$$\widetilde{\mathcal{W}}_6(X,Y)Z=\varepsilon[\mathrm{g}(Y,Z)X-\mathrm{g}(X,Z)Y]-\varepsilon[\mathrm{g}(Y,Z)X-\mathrm{g}(X,Z)Y]$$
$$-(\varepsilon-1)[\mathrm{g}(X,Z)Y-\mathrm{g}(X,Y)Z]. \quad (40)$$

Hence, equation (40) becomes

$$\widetilde{\mathcal{W}}_6(X,Y)Z=(\varepsilon-1)[\mathrm{g}(X,Y)Z-\mathrm{g}(X,Z)Y]. \quad (41)$$

The $\mathcal{W}_8$ curvature tensor in (ε)-LP-Sasakian manifolds with Levi-Civita connection is given by (23).Expanding (23) with respect to V, we have

$$\mathcal{W}_8(X,Y,Z,V)=\mathfrak{R}(X,Y,Z,V)-[\mathrm{g}(X,Y)\mathrm{g}(Z,V)-\mathrm{g}(Y,Z)\mathrm{g}(X,V)] \quad (42)$$

putting $Z=\xi$ in above equation

$$\mathcal{W}_8(X,Y,\xi,V)=\mathfrak{R}(X,Y,\xi,V)-[\mathrm{g}(X,Y)\mathrm{g}(\xi,V)-\mathrm{g}(Y,\xi)\mathrm{g}(X,V)] \quad (43)$$

$$\mathcal{W}_8(X,Y,\xi,V)=\mathfrak{R}(X,Y,\xi,V)-[\mathrm{g}(X,Y)\varepsilon\mathfrak{N}(V)-\varepsilon\mathfrak{N}(Y)\mathrm{g}(X,V)]. \quad (44)$$

Now interchanging V and ξ in above equation, we have

$$-\mathcal{W}_8(X,Y,V,\xi) = -\mathfrak{R}(X,Y,V,\xi) - \varepsilon[\mathrm{g}(X,Y)\mathfrak{N}(V) - \mathfrak{N}(Y)\mathrm{g}(X,V)]. \quad (45)$$

Contracting (45) with respect to ξ, we have

$$\mathcal{W}_8(X,Y)V = \mathfrak{R}(X,Y)V + \varepsilon[\mathrm{g}(X,Y)V - \mathrm{g}(X,V)Y]. \quad (46)$$

Replace V by Z

$$\mathcal{W}_8(X,Y)Z = \mathfrak{R}(X,Y)Z + \varepsilon[\mathrm{g}(X,Y)Z - \mathrm{g}(X,Z)Y]. \quad (47)$$

Simplifying (47), we have

$$\mathcal{W}_8(X,Y)Z = \varepsilon[\mathrm{g}(Y,Z)X - \mathrm{g}(X,Z)Y] + \varepsilon[\mathrm{g}(X,Y)Z - \mathrm{g}(X,Z)Y]. \quad (48)$$

Now using equation(41) in equation (48), we get

$$\mathcal{W}_8(X,Y)Z = \frac{\varepsilon}{\varepsilon-1}\widetilde{\mathcal{W}}_6(X,Y)Z + \varepsilon[\mathrm{g}(Y,Z)X - \mathrm{g}(X,Z)Y]. \quad (49)$$

Thus we conclude:

Theorem 6.1: *A $\widetilde{\mathcal{W}}_6$-curvature tensor on a quarter-symmetric metric connection is connected to a $\mathcal{W}_8$ -curvature tensor with the Levi-Civita connection in an (ε)-LP-Sasakian manifolds as* $\mathcal{W}_8(X,Y)Z = \frac{\varepsilon}{\varepsilon-1}\widetilde{\mathcal{W}}_6(X,Y)Z + \varepsilon[\mathrm{g}(Y,Z)X - \mathrm{g}(X,Z)Y]$.

7. (ε)-LP-Sasakian manifolds with quarter-symmetric metric connection satisfying $\widetilde{\mathcal{W}}_8.\widetilde{\mathfrak{R}} = 0$

In this section, we discussed on (ε)-LP-Sasakian manifold with quarter-symmetric metric connection satisfying $\widetilde{\mathcal{W}}_8.\widetilde{\mathfrak{R}} = 0$. Then, we have

$$\widetilde{\mathcal{W}}_8(\xi,U)\widetilde{\mathfrak{R}}(X,Y)Z - \widetilde{\mathfrak{R}}(\widetilde{\mathcal{W}}_8(\xi,U)X,Y)Z - \widetilde{\mathfrak{R}}(X,\widetilde{\mathcal{W}}_8(\xi,U)Y)Z - \widetilde{\mathfrak{R}}(X,Y)\widetilde{\mathcal{W}}_8(\xi,U)Z = 0. \quad (50)$$

Replacing Z by ξ in (50), we have

$$\widetilde{\mathcal{W}}_8(\xi,U)\widetilde{\mathfrak{R}}(X,Y)\xi - \widetilde{\mathfrak{R}}\big(\widetilde{\mathcal{W}}_8(\xi,U)X,Y\big)\xi - \widetilde{\mathfrak{R}}\big(X,\widetilde{\mathcal{W}}_8(\xi,U)Y\big)\xi - \widetilde{\mathfrak{R}}(X,Y)\widetilde{\mathcal{W}}_8(\xi,U)\xi = 0. \quad (51)$$

Using (19) in (51), we have

$$(1-\varepsilon)\mathfrak{N}(\widetilde{\mathcal{W}}_8(\xi,U)X)Y - (1-\varepsilon)\mathfrak{N}(\widetilde{\mathcal{W}}_8(\xi,U)Y)X - \widetilde{\mathfrak{R}}(X,Y)\widetilde{\mathcal{W}}_8(\xi,U)\xi = 0. \quad (52)$$

By virtue of equation (24) in (52), we have

$$(1-\varepsilon)\mathfrak{N}[\widetilde{\mathfrak{R}}(\xi,U)X + \frac{1}{n-1}\{\tilde{S}(\xi,U)X - \tilde{S}(U,X)\xi\}]Y$$
$$-(1-\varepsilon)\mathfrak{N}[\widetilde{\mathfrak{R}}(\xi,U)Y + \frac{1}{n-1}\{\tilde{S}(\xi,U)Y - \tilde{S}(U,Y)\xi\}]X$$
$$-\widetilde{\mathfrak{R}}(X,Y)[\widetilde{\mathfrak{R}}(\xi,U)\xi + \frac{1}{n-1}\{\tilde{S}(\xi,U)\xi - \tilde{S}(U,\xi)\xi\}] = 0. \quad (53)$$

Using equations (17), (18) in (53), we have

$$(1-\varepsilon)[\mathrm{g}(U,X)Y-\mathrm{g}(U,Y)X]+\frac{1}{n-1}\{\tilde{S}(U,X)Y-\tilde{S}(U,Y)X\}$$
$$+\varepsilon(1-\varepsilon)\mathfrak{N}(U)[\mathfrak{N}(Y)X-\mathfrak{N}(X)Y]-\widetilde{\mathfrak{R}}(X,Y)U=0. \quad (54)$$

Replacing Y by ξ in (54)

$$(1-\varepsilon)[\mathrm{g}(U,X)\xi-\mathrm{g}(U,\xi)X]+\frac{1}{n-1}\{\tilde{S}(U,X)\xi-\tilde{S}(U,\xi)X\}$$
$$+\varepsilon(1-\varepsilon)\mathfrak{N}(U)[\mathfrak{N}(\xi)X-\mathfrak{N}(X)\xi]-\widetilde{\mathfrak{R}}(X,\xi)U=0. \quad (55)$$

Now using equations (17), (18) and (9) in (55), we have

$$\frac{1}{n-1}\tilde{S}(U,X)\xi=\varepsilon(1-\varepsilon)\mathfrak{N}(X)\mathfrak{N}(U)\xi. \quad (56)$$

Taking inner product with ξ in (56), we have

$$\tilde{S}(U,X)=\varepsilon(1-\varepsilon)(n-1)\mathfrak{N}(X)\mathfrak{N}(U). \quad (57)$$

Hence we have the following:

Theorem 7.1: An (ε)-LP-Sasakian manifold with quarter-symmetric metric connection satisfying $\widetilde{\mathcal{W}}_8.\widetilde{\mathfrak{R}}=0$, is a special type of $\mathfrak{N}$-Einstein manifold given by

$\tilde{S}(U,X)=\varepsilon(1-\varepsilon)(n-1)\mathfrak{N}(X)\mathfrak{N}(U)$.

8. (ε)-LP-Sasakian manifold with quarter-symmetric metric connection satisfying $\widetilde{\mathcal{W}}_6.\widetilde{\mathfrak{R}}=0$

In this section, we introduced an (ε)-LP -Sasakian manifold with quarter-symmetric metric connection satisfying $\widetilde{\mathcal{W}}_6.\widetilde{\mathfrak{R}}=0$. In this case, we have

$$\widetilde{\mathcal{W}}_6(\xi,U)\widetilde{\mathfrak{R}}(X,Y)Z-\widetilde{\mathfrak{R}}(\widetilde{\mathcal{W}}_6(\xi,U)X,Y)Z-\widetilde{\mathfrak{R}}(X,\widetilde{\mathcal{W}}_6(\xi,U)Y)Z-$$
$$\widetilde{\mathfrak{R}}(X,Y)\widetilde{\mathcal{W}}_6(\xi,U)Z=0. \quad (58)$$

Replacing Z by ξ in (58), we have

$$\widetilde{\mathcal{W}}_6(\xi,U)\widetilde{\mathfrak{R}}(X,Y)\xi-\widetilde{\mathfrak{R}}(\widetilde{\mathcal{W}}_6(\xi,U)X,Y)\xi-\widetilde{\mathfrak{R}}(X,\widetilde{\mathcal{W}}_6(\xi,U)Y)\xi-$$
$$\widetilde{\mathfrak{R}}(X,Y)\widetilde{\mathcal{W}}_6(\xi,U)\xi=0. \quad (59)$$

Using (19) in (59), we have

$$(1-\varepsilon)\mathfrak{N}(\widetilde{\mathcal{W}}_6(\xi,U)X)Y-(1-\varepsilon)\mathfrak{N}(\widetilde{\mathcal{W}}_6(\xi,U)Y)X-\widetilde{\mathfrak{R}}(X,Y)\widetilde{\mathcal{W}}_6(\xi,U)\xi=0. \quad (60)$$

By using equation (22) in (60), we have

$$(1-\varepsilon)\mathfrak{N}[\widetilde{\mathfrak{R}}(\xi,U)X+\frac{1}{n-1}\{\mathrm{g}(\xi,U)\bar{Q}X-\tilde{S}(U,X)\xi\}]Y$$
$$-(1-\varepsilon)\mathfrak{N}[\widetilde{\mathfrak{R}}(\xi,U)Y+\frac{1}{n-1}\{\mathrm{g}(\xi,U)\bar{Q}Y-\tilde{S}(U,Y)\xi\}]X$$

$$-\widetilde{\mathfrak{R}}(X,Y)[\widetilde{\mathfrak{R}}(\xi,U)\xi + \tfrac{1}{n-1}\{\mathrm{g}(\xi,U)\bar{Q}\xi - \tilde{S}(U,\xi)\xi\}] = 0. \quad (61)$$

Using equations (17), (18) in (61) and using $Y = \xi$, we have

$$(1-\varepsilon)[\mathrm{g}(U,X)\xi - \mathrm{g}(U,\xi)X] + \frac{1}{n-1}\{\tilde{S}(U,X)\xi - \tilde{S}(U,\xi)X\}$$
$$+\varepsilon(1-\varepsilon)\mathfrak{N}(U)[\mathfrak{N}(\xi)X - \mathfrak{N}(X)\xi] - \widetilde{\mathfrak{R}}(X,\xi)U = 0. \quad (62)$$

$$\frac{1}{n-1}\tilde{S}(U,X)\xi = 4(1-\varepsilon)\mathfrak{N}(U)X - (1-\varepsilon)[\mathrm{g}(U,X)\xi - \mathfrak{N}(U)X]$$
$$+2(1-\varepsilon)\mathfrak{N}(X)\mathfrak{N}(U)\xi + \widetilde{\mathfrak{R}}(X,\xi)\mathrm{U} \quad (63)$$

Now using equations (17), (18) in (63), we have

$$\frac{1}{n-1}\tilde{S}(U,X)\xi = 3(1-\varepsilon)\mathfrak{N}(U)X + 2(1-\varepsilon)\mathfrak{N}(X)\mathfrak{N}(U)\xi. \quad (64)$$

Taking inner product with ξ in (64), we have

$$\tilde{S}(U,X) = (1-\varepsilon)(n-1)\mathfrak{N}(X)\mathfrak{N}(U). \quad (65)$$

Hence we have the following:

Theorem 8.1: *An (ε)-LP-Sasakian manifold with quarter-symmetric metric connection satisfying $\widetilde{W}_6.\widetilde{\mathfrak{R}} = 0$, is a special type of $\mathfrak{N}$-Einstein manifold given as*

$$\tilde{S}(U,X) = (1-\varepsilon)(n-1)\mathfrak{N}(X)\mathfrak{N}(U).$$

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