

## HOMOGENIZATION OF DIRICHLET OPTIMAL CONTROL PROBLEM FOR STOKES SYSTEM IN A PERFORATED DOMAIN

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**Abstract:** This article focuses on the asymptotic analysis of an optimal control problem (OCP) constrained by stationary Stokes equations in a periodically perforated domain. The interior region is influenced by distributive controls, and the Stokes operator involves oscillating coefficients for the state equations. The cost functional is of the Dirichlet type and includes an oscillating coefficient matrix. We characterize the optimal control and use the method of periodic unfolding to demonstrate the convergence of the OCP solutions to the solutions of the limit OCP governed by stationary Stokes equations in a non-perforated domain.

**Key Words:** Stokes equations; Homogenization; Optimal control; Perforated domain; Unfolding operator.

**AMS subject classifications:** 35B27, 35B40, 35Q93, 49J20, 76D07

### 1. Introduction

In this article, we consider the optimal control problem (OCP) governed by generalized stationary Stokes equations in a periodically perforated domain  $\mathcal{O}_\varepsilon^*$  (refer Section 2, for a detailed configuration of the domain). The size of holes in the perforated domain is of the same order as that of the period, and the holes are allowed to intersect the boundary of the domain. The control is applied in the interior region of the domain, and we wish to study the asymptotic analysis (homogenization) of an interior OCP subject to the constrained stationary Stokes equations with oscillating coefficients.

One can find several works in the literature regarding the homogenization of Stokes equations over a perforated domain. Using the multiple-scale expansion method, the authors in [17] studied the homogenization of Stokes equations in a porous medium with the Dirichlet boundary condition on the boundary of the holes. They obtained the Darcy's law as the limit law in the homogenized medium. In [10], the authors considered the Stokes system in a periodically perforated domain with non-homogeneous slip boundary conditions depending upon some parameter  $\gamma$ . Upon employing the Tartar's method of

oscillating test functions they obtained under homogenization, the limit laws, viz., Darcy's law (for  $\gamma < 1$ ), Brinkmann's law (for  $\gamma = 1$ ), and Stokes's type law (for  $\gamma > 1$ ). In [29], the author studied a similar problem using the method of periodic unfolding in perforated domains by [11]. Further, the type of behavior as seen in [10] was already observed in [13] by the authors while studying the homogeneous Fourier boundary conditions for the two-dimensional Stokes equation. Likewise, in [1,2], the author examined the Stokes equation in a perforated domain with holes of size much smaller than the small positive parameter  $\varepsilon$ , wherein they considered the boundary conditions on the holes to be of the Dirichlet type in [1] and the slip type in [2]. The domain geometry, more specifically, the size of the holes, determines the kind of limit law in these works. Also, the author in [7] employed the  $\Gamma$  – convergence techniques to get comparable results.

A few works concern the homogenization of the OCPs governed by the elliptic systems over the periodically perforated domains with different kinds of boundary conditions on the boundary of holes (of the size of the same order as that of the period). In this regard, with the use of different techniques, viz.,  $H_0$ -convergence in [21], two-scale convergence in [26], and unfolding methods in [8,24], the homogenized OCPs were thus obtained over the non-perforated domains. Further, the homogenization of OCPs subject to the boundary value problems concerning the steady Stokes equations mainly comprise the boundary conditions of the type: Dirichlet, Navier slip-friction, Neumann, Mixed, etc. The authors in [25] studied the homogenization of the OCPs subject to the Stokes equations with Dirichlet boundary conditions on the boundary of holes, where the size of the holes is of the same order as that of the period. Here, the authors could obtain the homogenized system, pertaining only to the case when the set of admissible controls was unconstrained. For more literature concerning the homogenization of optimal control problems in perforated domains, the reader is referred to [14-16,28] and the references therein. Also, over another type of oscillating domain, the authors in [19] studied the asymptotic analysis of the Stokes system with mixed boundary conditions of similar type on the thin oscillating domain. Furthermore, pertaining to the Navier-Stokes equations, the existence of the solutions to the mixed boundary value problem has been established by the authors in [4] for 2D bounded domain. For more literature related to the Stokes system with mixed boundary conditions, one may refer to [18,23,27] and the references therein.

The present article introduces an interior OCP subject to the generalized stationary Stokes equations in a periodically perforated domain  $\mathcal{O}_\varepsilon^*$ . We employ mixed boundary data on the boundary of the perforated domain, i.e., on the boundary of holes that do not intersect the outer boundary, the homogeneous Neumann boundary condition is prescribed, while on the rest part of the boundary, the homogeneous Dirichlet boundary condition is prescribed. The underlying objective of this article is to study the homogenization of this OCP. More specifically, we consider the minimization of the Dirichlet cost functional involving oscillating matrix (1), which is subject to the constrained generalized stationary Stokes equations (2).

The Stokes equations are generalized in the sense that we consider a second-order elliptic linear differential operator in divergence form with oscillating coefficients, i.e.,  $-\operatorname{div}(A_\varepsilon \nabla)$ , first studied for the fixed domain in [5, Chapter 1], instead of the classical Laplacian operator, which later on was studied by various authors for different types of  $\varepsilon$  – dependent varying domains. The main difficulty observed during the homogenization was identifying the limit pressure terms appearing in the state and the adjoint systems, which we overcame by introducing suitable corrector functions that solved some cell problems. We thus obtained the limit OCP associated with the stationary Stokes equation in a non-perforated domain.

The layout of this article is as follows: In the next section, we introduce the periodically Perforated domain  $\mathcal{O}_\varepsilon^*$  along with the notations that will be useful in the sequel. Section 3 is devoted to a detailed description of the considered OCP and the derivation of the optimality condition, followed by the characterization of the optimal control. In Section 4, we derive a priori estimates of the solutions to the considered OCP and its corresponding adjoint problem. In Section 5, we recall the definition of the method of periodic unfolding in perforated domains (see, [9,12]) and a few of its properties. Section 6, refers to the limit (homogenized) OCP. Finally, we derive the main convergence results in Section 7 followed by some important remarks.

## 2. Domain description and Notation

### 2.1 Domain description

Let  $\{b_1, \dots, b_n\}$  be a basis of  $\mathbb{R}^n$  ( $n \geq 2$ ), and  $W$  be the associated reference cell defined as

$$W = \left\{ w \in \mathbb{R}^n \mid w = \sum_{i=1}^n w_i b_i, (w_1, \dots, w_n) \in (0,1)^n \right\}.$$

Let us denote  $\mathcal{O}, W$ , and  $W^* = W \setminus Y$  by an open bounded subset of  $\mathbb{R}^n$ , a compact subset of  $\overline{W}$ , and the perforated reference cell, respectively. It is assumed that the boundary of  $Y$  is Lipschitz continuous and has a finite number of connected components.

Also, let  $\varepsilon > 0$  be a sequence that converges to zero and set

$$\mathcal{T} = \left\{ \zeta \in \mathbb{R}^n \mid \zeta = \sum_{t=1}^n z_t b_t, (z_1, \dots, z_n) \in \mathbb{Z}^n \right\}, Z_\varepsilon = \{ \zeta \in \mathcal{T} \mid \varepsilon(\zeta + W) \subset \mathcal{O} \}.$$

We take into account the perforated domain  $\mathcal{O}_\varepsilon^*$  (see Figure 1) given by  $\mathcal{O}_\varepsilon^* = \mathcal{O} \setminus Y_\varepsilon$ , where  $Y_\varepsilon = \cup_{\zeta \in \mathcal{T}} \varepsilon(\zeta + Y)$ . Now, let us denote  $\widehat{\mathcal{O}}_\varepsilon$  as the interior of the largest union of  $\varepsilon(\zeta + \overline{W})$

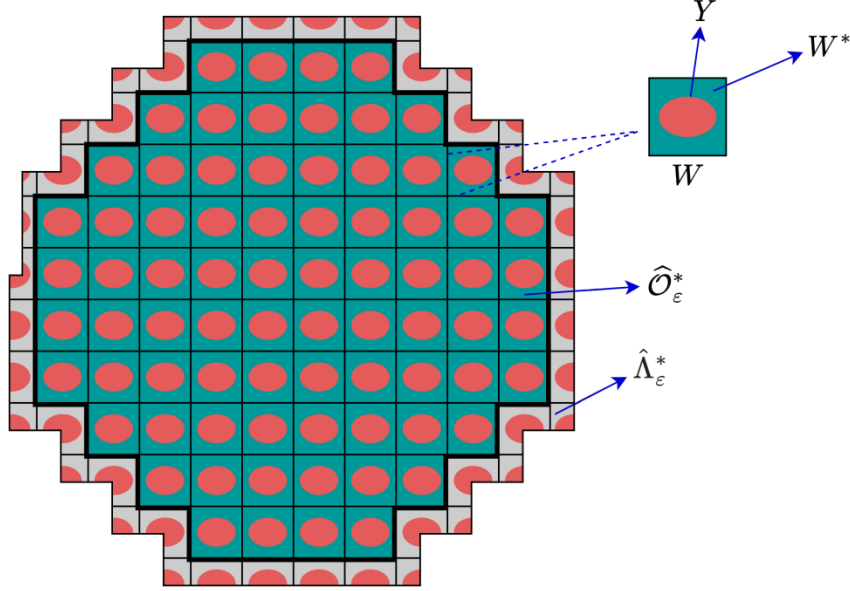


Figure 1: The Perforated domain  $\mathcal{O}_\varepsilon^*$  and the reference cell  $W$ .

Cells such that  $\varepsilon(\zeta + W) \subset \mathcal{O}$ , while  $\Lambda_\varepsilon \subset \mathcal{O}$  as containing the parts from  $\varepsilon(\zeta + \bar{W})$  cells intersecting the boundary  $\partial\mathcal{O}$ . More precisely, we write  $\Lambda_\varepsilon = \mathcal{O} \setminus \hat{\mathcal{O}}_\varepsilon$ , where

$$\hat{\mathcal{O}}_\varepsilon = \text{interior} \left\{ \bigcup_{\zeta \in Z_\varepsilon} \varepsilon(\zeta + \bar{W}) \right\}.$$

The associated perforated domains are defined as

$$\hat{\mathcal{O}}_\varepsilon^* = \hat{\mathcal{O}}_\varepsilon \setminus Y_\varepsilon, \quad \hat{\Lambda}_\varepsilon^* = \mathcal{O}_\varepsilon^* \setminus \hat{\mathcal{O}}_\varepsilon^*.$$

Also, we denote the boundary of the perforated domain  $\mathcal{O}_\varepsilon^*$  as

$$\partial\mathcal{O}_\varepsilon^* = \Gamma_1^\varepsilon \cup \Gamma_0^\varepsilon, \quad \text{where } \Gamma_1^\varepsilon = \partial\hat{\mathcal{O}}_\varepsilon \cap \partial Y_\varepsilon \text{ and } \Gamma_0^\varepsilon = \partial\mathcal{O}_\varepsilon^* \setminus \Gamma_1^\varepsilon,$$

which means that  $\Gamma_1^\varepsilon$  denotes the boundary of the set of holes contained in  $\hat{\mathcal{O}}_\varepsilon$ .

In Figure 1,  $\hat{\mathcal{O}}_\varepsilon^*$  and  $\hat{\Lambda}_\varepsilon^*$  respectively represent the dark perforated part and the remaining part of the perforated domain  $\hat{\mathcal{O}}_\varepsilon^*$ . While,  $\Gamma_1^\varepsilon$  and  $\Gamma_0^\varepsilon$  respectively represent the boundary of holes contained in  $\hat{\mathcal{O}}_\varepsilon^*$  and the boundary of holes contained in  $\hat{\Lambda}_\varepsilon^*$  along with the outer boundary  $\partial\mathcal{O}$ . In the following, we introduce a few notations that we shall use throughout this article.

## 2.2 Notation

- $A_\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$  a.e. in  $\mathcal{O}$ , for all  $\varepsilon > 0$ .
- $\mathbf{v}_\varepsilon = (v_{\varepsilon 1}, \dots, v_{\varepsilon n})$ , for any bold symbol vector function  $\mathbf{v}_\varepsilon$ .
- $\mathbf{v} = (v_1, \dots, v_n)$ , for any bold symbol vector function  $\mathbf{v}$ .

- $\mathcal{T} > 0$  is a given regularization parameter.
- $\boldsymbol{\eta}_\varepsilon$  denotes the outward normal unit vector to  $\Gamma_1^\varepsilon$ .
- $\boldsymbol{\eta}$  denotes the outward normal unit vector to  $\partial\mathcal{O}$ .
- $R : S$  denotes the element-wise product followed by summation between any matrices  $R$  and  $S$ .
- $\boldsymbol{v} \cdot \boldsymbol{w}$  denotes the standard dot product between any vector functions  $\boldsymbol{v}$  and  $\boldsymbol{w}$ .
- $M^t$  denotes the transpose of any matrix  $M$ .
- $\tilde{\boldsymbol{\psi}}$  is the zero extension of any function  $\boldsymbol{\psi}$  outside  $\mathcal{O}_\varepsilon^*$  to the whole of  $\mathcal{O}$ .
- $\tilde{\boldsymbol{\psi}} = (\tilde{\psi}_1, \dots, \tilde{\psi}_n)$ , for any vector function  $\boldsymbol{\psi}$ .
- $|F|$  is the Lebesgue measure of the measurable set  $F$ .
- The symbol  $C$  represents a generic constant that is positive and independent of  $\varepsilon$ .
- $\Theta = \frac{|W^*|}{|W|}$ , the proportion of the perforated reference cell  $W^*$  in the reference cell  $W$ .
- $\mathcal{M}_{W^*}(\phi)$  is the mean value of  $\phi$  on the perforated reference cell  $W^*$ .
- $\mathcal{M}_{W^*}(\boldsymbol{\phi}) = (\mathcal{M}_{W^*}(\phi_1), \mathcal{M}_{W^*}(\phi_n))$ , for vector function  $\boldsymbol{\phi}$ .
- $\{D \rightarrow \mathbb{R}\}$ , the set of all real valued functions defined on domain  $D$ .
- $D(\Omega)$ , is the space of infinitely many times differentiable functions with compact support in  $\Omega$ , for any open set  $\Omega \in \mathbb{R}^n$ .

### 3. Problem Description and Optimality Condition

Let us consider the following OCP associated with the Stokes system:

$$\boldsymbol{\theta}_\varepsilon \in (L^2(\mathcal{O}_\varepsilon^*))^n \left\{ J_\varepsilon(\boldsymbol{\theta}_\varepsilon) = \frac{1}{2} \int_{\mathcal{O}_\varepsilon^*} B_\varepsilon \nabla \boldsymbol{u}_\varepsilon(\boldsymbol{\theta}_\varepsilon) : \nabla \boldsymbol{u}_\varepsilon(\boldsymbol{\theta}_\varepsilon) + \frac{\mathcal{T}}{2} \int_{\mathcal{O}_\varepsilon^*} |\boldsymbol{\theta}_\varepsilon|^2 \right\} \quad (1)$$

subject to

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla \boldsymbol{u}_\varepsilon) + \nabla p_\varepsilon = \boldsymbol{f} + \boldsymbol{\theta}_\varepsilon \text{ in } \mathcal{O}_\varepsilon^*, \\ \operatorname{div}(\boldsymbol{u}_\varepsilon) = 0 \text{ in } \mathcal{O}_\varepsilon^*, \\ \boldsymbol{\eta}_\varepsilon \cdot A_\varepsilon \nabla \boldsymbol{u}_\varepsilon - p_\varepsilon \boldsymbol{\eta}_\varepsilon = \mathbf{0} \text{ on } \Gamma_1^\varepsilon, \\ \boldsymbol{u}_\varepsilon = \mathbf{0} \text{ on } \Gamma_0^\varepsilon, \end{cases} \quad (2)$$

where the target state  $\boldsymbol{v}_d = (v_{d1}, \dots, v_{dn})$  and the source function  $\boldsymbol{f}$  are defined on the space  $(L^2(\mathcal{O}))^n$ . The control function  $\boldsymbol{\theta}_\varepsilon$  is defined on the space  $(L^2(\mathcal{O}_\varepsilon^*))^n$ . Here, the matrix  $A_\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$ , where  $A(x) = (a_{ij}(x))_{1 \leq i, j \leq n}$  defined on the space  $(L^\infty(\mathcal{O}))^{n \times n}$  is assumed to obey the uniform ellipticity condition: there exist real constants  $m_1, m_2 > 0$  such that  $m_1 \|\lambda\|^2 \leq \sum_{i, j=1}^n a_{ij}(x) \lambda_i \lambda_j \leq m_2 \|\lambda\|^2$  for all  $\lambda \in \mathbb{R}^n$ , which is endowed

with Euclidian norm denoted by  $\|\cdot\|$ . Further, the coefficient matrix  $B_\varepsilon$ , not necessarily equal to  $A_\varepsilon$ , is symmetric, elliptic and is set to oscillate in  $x_1$  - direction, i.e.,  $B(x_1, x_2) = B\left(x_1, x_2, \frac{x_1}{\varepsilon}\right)$ . Also, we understand the action of the scalar boundary operator  $\boldsymbol{\eta}_\varepsilon \cdot A_\varepsilon \nabla$  on the vector  $\mathbf{u}_\varepsilon|_{\Gamma_1^\varepsilon}$  in a ‘‘diagonal’’ manner:  $(\boldsymbol{\eta}_\varepsilon \cdot A_\varepsilon \nabla \mathbf{u}_\varepsilon)_i = \boldsymbol{\eta}_\varepsilon \cdot A_\varepsilon \nabla u_{\varepsilon i}$ , for  $i \leq n$ .

We introduce the function space  $\left(H_{\Gamma_0^\varepsilon}^1(\mathcal{O}_\varepsilon^*)\right)^n := \{\boldsymbol{\phi} \in (H^1(\mathcal{O}_\varepsilon^*))^n \mid \boldsymbol{\phi}|_{\Gamma_0^\varepsilon} = \mathbf{0}\}$ . This is a Banach space endowed with the norm

$$\|\boldsymbol{\phi}\|_{\left(H_{\Gamma_0^\varepsilon}^1(\mathcal{O}_\varepsilon^*)\right)^n} := \|\nabla \boldsymbol{\phi}\|_{(L^2(\mathcal{O}_\varepsilon^*))^{n \times n}}, \forall \boldsymbol{\phi} \in \left(H_{\Gamma_0^\varepsilon}^1(\mathcal{O}_\varepsilon^*)\right)^n.$$

**Definition 2.1** We say a pair  $(\mathbf{u}_\varepsilon, p_\varepsilon) \in \left(H_{\Gamma_0^\varepsilon}^1(\mathcal{O}_\varepsilon^*)\right)^n \times L^2(\mathcal{O}_\varepsilon^*)$  is a weak solution to (2) if, for all  $\boldsymbol{\phi} \in \left(H_{\Gamma_0^\varepsilon}^1(\mathcal{O}_\varepsilon^*)\right)^n$ ,

$$\int_{\mathcal{O}_\varepsilon^*} A_\varepsilon \nabla \mathbf{u}_\varepsilon : \nabla \boldsymbol{\phi} \, dx - \int_{\mathcal{O}_\varepsilon^*} p_\varepsilon \operatorname{div}(\boldsymbol{\phi}) \, dx = \int_{\mathcal{O}_\varepsilon^*} (\mathbf{f} + \boldsymbol{\theta}_\varepsilon) \cdot \boldsymbol{\phi} \, dx, \quad (3)$$

and, for all  $w \in L^2(\mathcal{O}_\varepsilon^*)$ ,

$$\int_{\mathcal{O}_\varepsilon^*} \operatorname{div}(\mathbf{u}_\varepsilon) w \, dx = 0. \quad (4)$$

The existence of a unique weak solution  $(\mathbf{u}_\varepsilon(\boldsymbol{\theta}_\varepsilon), p_\varepsilon) \in \left(H_{\Gamma_0^\varepsilon}^1(\mathcal{O}_\varepsilon^*)\right)^n \times L^2(\mathcal{O}_\varepsilon^*)$  of the system (2) follows analogous to [6, Theorem IV.7.1]. Also, for each  $\varepsilon > 0$ , there exist a unique solution to the problem (1) that can be proved along the same line as in [22, Chapter 2, Theorem 1.2]. We call the optimal solution to (1) by the triplet  $(\bar{\mathbf{u}}_\varepsilon, \bar{p}_\varepsilon, \bar{\boldsymbol{\theta}}_\varepsilon)$ , with  $\bar{\mathbf{u}}_\varepsilon, \bar{p}_\varepsilon$  and  $\bar{\boldsymbol{\theta}}_\varepsilon$  as optimal state, pressure and control, respectively.

**Optimality Condition:** The optimality condition is given by  $J'_\varepsilon(\boldsymbol{\theta}) \cdot (\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}_\varepsilon) \geq 0$ , for all  $\boldsymbol{\theta} \in (L^2(\mathcal{O}_\varepsilon^*))^n$  (see, [22, Chapter 2, Page 48]). One can obtain further simplification of this condition as  $\int_{\mathcal{O}_\varepsilon^*} (\bar{\mathbf{v}}_\varepsilon + \mathcal{J} \bar{\boldsymbol{\theta}}_\varepsilon) \cdot (\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}_\varepsilon) \geq 0$ , for all  $\boldsymbol{\theta} \in (L^2(\mathcal{O}_\varepsilon^*))^n$  (see, [22, Chapter 2]), where the pair  $(\bar{\mathbf{v}}_\varepsilon, \bar{q}_\varepsilon)$  is the solution to the following adjoint problem :

$$\begin{cases} -\operatorname{div}(A_\varepsilon^t \nabla \bar{\mathbf{v}}_\varepsilon) + \nabla \bar{q}_\varepsilon = -\operatorname{div}(B_\varepsilon \nabla \bar{\mathbf{u}}_\varepsilon) \text{ in } \mathcal{O}_\varepsilon^*, \\ \operatorname{div}(\bar{\mathbf{v}}_\varepsilon) = 0 \text{ in } \mathcal{O}_\varepsilon^*, \\ \boldsymbol{\eta}_\varepsilon \cdot A_\varepsilon^t \nabla \bar{\mathbf{v}}_\varepsilon - \bar{q}_\varepsilon \boldsymbol{\eta}_\varepsilon = \boldsymbol{\eta}_\varepsilon \cdot B_\varepsilon \nabla \bar{\mathbf{u}}_\varepsilon \text{ on } \Gamma_1^\varepsilon, \\ \bar{\mathbf{v}}_\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon. \end{cases} \quad (5)$$

We call  $\bar{\mathbf{v}}_\varepsilon$  and  $\bar{q}_\varepsilon$ , the adjoint state and pressure, respectively. The existence of unique weak solution  $(\bar{\mathbf{v}}_\varepsilon, \bar{q}_\varepsilon)$  to (5) can now be proved in a way similar to that system (2).

The following theorem characterizes the optimal control, the proof of which follows analogous to standard procedure laid in [22, Chapter 2, Theorem 1.4].

**Theorem 3.1.** Let  $(\bar{\mathbf{u}}_\varepsilon, \bar{p}_\varepsilon, \bar{\boldsymbol{\theta}}_\varepsilon)$  be the optimal solution of the problem (1) and  $(\bar{\mathbf{v}}_\varepsilon, \bar{q}_\varepsilon)$  solves (5), then the optimal control is characterized by

$$\bar{\boldsymbol{\theta}}_\varepsilon = -\frac{1}{\mathcal{J}} \bar{\mathbf{v}}_\varepsilon \text{ a. e. in } \mathcal{O}_\varepsilon^*. \quad (6)$$

Conversely, suppose that a triplet  $(\check{\mathbf{u}}_\varepsilon, \check{p}_\varepsilon, \check{\boldsymbol{\theta}}_\varepsilon) \in \left(H_{\Gamma_0^\varepsilon}^1(\mathcal{O}_\varepsilon^*)\right)^n \times L^2(\mathcal{O}_\varepsilon^*) \times \left(L^2(\mathcal{O}_\varepsilon^*)\right)^n$  and a pair  $(\check{\mathbf{v}}_\varepsilon, \bar{q}_\varepsilon) \in \left(H_{\Gamma_0^\varepsilon}^1(\mathcal{O}_\varepsilon^*)\right)^n \times L^2(\mathcal{O}_\varepsilon^*)$  solves the following system:

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla \check{\mathbf{u}}_\varepsilon) + \nabla \check{p}_\varepsilon = \mathbf{f} - \frac{1}{\mathcal{J}} \check{\mathbf{v}}_\varepsilon & \text{in } \mathcal{O}_\varepsilon^*, \\ -\operatorname{div}(A_\varepsilon^t \nabla \check{\mathbf{v}}_\varepsilon) + \nabla \check{q}_\varepsilon = -\operatorname{div}(B_\varepsilon \nabla \check{\mathbf{v}}_\varepsilon) & \text{in } \mathcal{O}_\varepsilon^*, \\ \operatorname{div}(\check{\mathbf{u}}_\varepsilon) = 0, \operatorname{div}(\check{\mathbf{v}}_\varepsilon) = 0 & \text{in } \mathcal{O}_\varepsilon^*, \\ \boldsymbol{\eta}_\varepsilon \cdot A_\varepsilon \nabla \check{\mathbf{u}}_\varepsilon - \check{p}_\varepsilon \boldsymbol{\eta}_\varepsilon = \mathbf{0} & \text{on } \Gamma_1^\varepsilon, \\ \boldsymbol{\eta}_\varepsilon \cdot A_\varepsilon^t \nabla \check{\mathbf{v}}_\varepsilon - \bar{q}_\varepsilon \boldsymbol{\eta}_\varepsilon = \boldsymbol{\eta}_\varepsilon \cdot B_\varepsilon \nabla \check{\mathbf{u}}_\varepsilon & \text{on } \Gamma_1^\varepsilon, \\ \check{\mathbf{u}}_\varepsilon = \mathbf{0}, \check{\mathbf{v}}_\varepsilon = \mathbf{0} & \text{on } \Gamma_0^\varepsilon. \end{cases}$$

Then the triplet  $(\check{\mathbf{u}}_\varepsilon, \check{p}_\varepsilon - \frac{1}{\mathcal{J}} \check{\mathbf{v}}_\varepsilon)$  is the optimal solution of (1).

#### 4. A Priori Estimates

This section lays down the estimates for the optimal solution to the problem (1) and the associated solution to the adjoint problem (5). These estimates are uniform and independent of the parameter  $\varepsilon$ . The proof is standard and employs the following two lemmas:

**Lemma 4.1** (Lemma A.4). [3] There exists a constant  $C \in \mathbb{R}^+$ , independent of  $\varepsilon$ , such that

$$\|\mathbf{v}\|_{\left(L^2(\mathcal{O}_\varepsilon^*)\right)^n} \leq C \|\nabla \mathbf{v}\|_{\left(L^2(\mathcal{O}_\varepsilon^*)\right)^{n \times n}}, \quad \forall \mathbf{v} \in \left(H_{\Gamma_0^\varepsilon}^1(\mathcal{O}_\varepsilon^*)\right)^n. \quad (7)$$

**Lemma 4.2** (Lemma 5.1). [13] for each  $\varepsilon > 0$  and  $q_\varepsilon \in L^2(\mathcal{O}_\varepsilon^*)$ , there exists  $\mathbf{g}_\varepsilon \in \left(H_{\Gamma_0^\varepsilon}^1(\mathcal{O}_\varepsilon^*)\right)^n$  and a constant  $C \in \mathbb{R}^+$ , independent of  $\varepsilon$ , such that

$$\operatorname{div}(\mathbf{g}_\varepsilon) = q_\varepsilon \text{ and } \|\nabla \mathbf{g}_\varepsilon\|_{\left(L^2(\mathcal{O}_\varepsilon^*)\right)^{n \times n}} \leq C(\mathcal{O}) \|q_\varepsilon\|_{L^2(\mathcal{O}_\varepsilon^*)}. \quad (8)$$

**Theorem 4.3** For each  $\varepsilon > 0$ , let  $(\bar{\mathbf{u}}_\varepsilon, \bar{p}_\varepsilon, \bar{\boldsymbol{\theta}}_\varepsilon)$  be the optimal solution of the problem (1) and  $(\bar{\mathbf{v}}_\varepsilon, \bar{q}_\varepsilon)$  solves the corresponding adjoint problem (5). Then, one has  $\bar{\boldsymbol{\theta}}_\varepsilon \in \left(H_{\Gamma_0^\varepsilon}^1(\mathcal{O}_\varepsilon^*)\right)^n$  and there exists a constant  $C \in \mathbb{R}^+$ , independent of  $\varepsilon$  such that

$$\|\bar{\boldsymbol{\theta}}_\varepsilon\|_{\left(L^2(\mathcal{O}_\varepsilon^*)\right)^n} \leq C, \quad (9)$$

$$\|\bar{u}_\varepsilon\|_{\left(H_{\Gamma_0^\varepsilon}^1(\mathcal{O}_\varepsilon^*)\right)^n} \leq C, \quad (10)$$

$$\|\bar{v}_\varepsilon\|_{\left(H_{\Gamma_0^\varepsilon}^1(\mathcal{O}_\varepsilon^*)\right)^n} \leq C, \quad (11)$$

$$\|\bar{p}_\varepsilon\|_{L^2(\mathcal{O}_\varepsilon^*)} \leq C, \quad (12)$$

$$\|\bar{q}_\varepsilon\|_{L^2(\mathcal{O}_\varepsilon^*)} \leq C. \quad (13)$$

### 5. The Method of Periodic Unfolding for Perforated Domains

We evoke the definition of the periodic unfolding operator and few of its properties as stated in [9,12]. Given  $x \in \mathbb{R}^n$ , we denote the greatest integer and the fractional parts of  $x$  respectively by  $[x]_w$  and  $\{x\}_w$ . That is,  $[x]_w = \sum_{j=1}^n k_j b_j$  be the unique integer combination of periods and  $\{x\}_w = x - [x]_w$ . In particular, we have for  $\varepsilon > 0$ ,

$$x = \varepsilon \left( \left[ \frac{x}{\varepsilon} \right]_w + \left\{ \frac{x}{\varepsilon} \right\}_w \right), \quad \forall x \in \mathbb{R}^n.$$

**Definition 5.1** The unfolding operator  $T_\varepsilon^* : \{\mathcal{O}_\varepsilon^* \rightarrow \mathbb{R}\} \rightarrow \{\mathcal{O} \times W^* \rightarrow \mathbb{R}\}$  is defined as

$$T_\varepsilon^*(u)(x, y) = \begin{cases} u \left( \varepsilon \left[ \frac{x}{\varepsilon} \right]_w + \varepsilon y \right) & \text{a.e. for } (x, y) \in \hat{\mathcal{O}}_\varepsilon \times W^*, \\ 0 & \text{a.e. for } (x, y) \in \hat{\Lambda}_\varepsilon^* \times W^*. \end{cases}$$

**Proposition 5.2** In the following there are the properties of the unfolding operator:

- (i)  $T_\varepsilon^*$  is linear and continuous from  $L^2(\mathcal{O}_\varepsilon^*)$  to  $L^2(\mathcal{O} \times W^*)$ .
- (ii) Let  $u, v \in L^2(\mathcal{O}_\varepsilon^*)$ . Then  $T_\varepsilon^*(uw) = T_\varepsilon^*(u)T_\varepsilon^*(v)$ .
- (iii) Let  $u \in L^2(\mathcal{O})$ . Then  $T_\varepsilon^*(u) \rightarrow u$  strongly in  $L^2(\mathcal{O} \times W^*)$ .
- (iv) Let  $u \in L^1(\mathcal{O}_\varepsilon^*)$ . Then

$$\int_{\hat{\mathcal{O}}_\varepsilon^*} u(x) dx = \int_{\mathcal{O}_\varepsilon^*} u(x) dx - \int_{\hat{\Lambda}_\varepsilon^*} u(x) dx = \frac{1}{|W^*|} \int_{\mathcal{O} \times W^*} T_\varepsilon^*(u)(x, y) dx dy.$$

- (v) For each  $\varepsilon > 0$ , let  $\{u_\varepsilon\} \in L^2(\mathcal{O})$  and  $u_\varepsilon \rightarrow u$  strongly in  $L^2(\mathcal{O})$ .

Then  $T_\varepsilon^*(u_\varepsilon) \rightarrow u$  strongly in  $L^2(\mathcal{O} \times W^*)$ .

- (vi) Let  $u \in L^2(W^*)$  be a  $W$  – periodic function and  $u_\varepsilon(x) = u\left(\frac{x}{\varepsilon}\right)$ . Then,

$$T_\varepsilon^*(u_\varepsilon)(x, y) = \begin{cases} u(y) & \text{a.e. for } (x, y) \in \hat{\mathcal{O}}_\varepsilon \times W^*, \\ 0 & \text{a.e. for } (x, y) \in \hat{\Lambda}_\varepsilon \times W^*. \end{cases}$$

- (vii) Let  $f_\varepsilon \in L^2(\mathcal{O}_\varepsilon^*)$  be uniformly bounded. Then, there exists  $f \in L^2(\mathcal{O} \times W^*)$  such that  $T_\varepsilon^*(f_\varepsilon) \rightarrow f$  weakly in  $L^2(\mathcal{O} \times W^*)$ , and



$$\tilde{f}_\varepsilon \rightharpoonup \frac{1}{|W|} \int_{W^*} f(\cdot, y) dy \text{ weakly in } L^2(\mathcal{O}).$$

**Proposition 5.3** Let  $\mathcal{O} \subset \mathbb{R}^n$  be bounded with Lipschitz boundary. Let  $f_\varepsilon \in H^1(\mathcal{O}_\varepsilon^*)$  be such that  $f_\varepsilon = 0$  on  $\partial\mathcal{O} \cap \partial\mathcal{O}_\varepsilon^*$  and satisfy,  $\|\nabla f_\varepsilon\|_{(L^2(\mathcal{O}_\varepsilon^*))^n} \leq C$ .

Then, there exist  $f \in H_0^1(\mathcal{O})$  and  $\hat{f} \in L^2(\mathcal{O}; H_{per}^1(W^*))$  with  $\mathcal{M}_{W^*}(\hat{f}) = 0$ , such that up to a subsequence,

$$\begin{cases} T_\varepsilon^*(\nabla f_\varepsilon) \rightharpoonup \nabla f + \nabla_y \hat{f} & \text{weakly in } (L^2(\mathcal{O} \times W^*))^n, \\ T_\varepsilon^*(f_\varepsilon) \rightharpoonup f & \text{strongly in } L^2(\mathcal{O}; H^1(W^*)). \end{cases}$$

## 6. Limit Optimal Control Problem

In this section, we present the limit optimal control problem. To do so, we first present the following cell problems.

For  $1 \leq j, \beta \leq n$ , and  $\mathbf{P}_j^\beta(y) = y_j e_\beta$ , let the correctors  $(\chi_j^\beta, \Pi_j^\beta) \in (H^1(W^*))^n \times L^2(W^*)$  solve the cell problem

$$\begin{cases} -\operatorname{div}_y (A(y)\nabla_y (\mathbf{P}_j^\beta - \chi_j^\beta)) + \nabla_y \Pi_j^\beta = \mathbf{0} \text{ in } W^*, \\ \boldsymbol{\eta} \cdot A(y)\nabla_y (\mathbf{P}_j^\beta - \chi_j^\beta) - \Pi_j^\beta \boldsymbol{\eta} = \mathbf{0} \text{ on } \partial W^* \setminus \partial W, \\ \operatorname{div}_y (\mathbf{P}_j^\beta - \chi_j^\beta) = 0 \text{ in } W^*, \\ (\chi_j^\beta, \Pi_j^\beta) \text{ is } W^* \text{ - periodic,} \\ \mathcal{M}_{W^*}(\chi_j^\beta) = \mathbf{0}, \end{cases} \quad (14)$$

the correctors  $(\mathbf{H}_j^\beta, Q_j^\beta) \in (H^1(W^*))^n \times L^2(W^*)$  solves the cell problem

$$\begin{cases} -\operatorname{div}_y (A^t(y)\nabla_y (\mathbf{P}_j^\beta - \mathbf{H}_j^\beta)) + \nabla_y Q_j^\beta = \mathbf{0} \text{ in } W^*, \\ \boldsymbol{\eta} \cdot A(y)\nabla_y (\mathbf{P}_j^\beta - \mathbf{H}_j^\beta) - Q_j^\beta \boldsymbol{\eta} = \mathbf{0} \text{ on } \partial W^* \setminus \partial W, \\ \operatorname{div}_y (\mathbf{P}_j^\beta - \mathbf{H}_j^\beta) = 0 \text{ in } W^*, \\ (\mathbf{H}_j^\beta, Q_j^\beta) \text{ is } W^* \text{ - periodic,} \\ \mathcal{M}_{W^*}(\mathbf{H}_j^\beta) = \mathbf{0}, \end{cases} \quad (15)$$

and the correctors  $(\mathbf{T}_j^\beta, R_j^\beta) \in (H^1(W^*))^n \times L^2(W^*)$  solves the cell problem

$$\left\{ \begin{array}{l} -\operatorname{div}_y (B(y)\nabla_y (\mathbf{P}_j^\beta - \boldsymbol{\chi}_j^\beta) - A^t(y)\nabla_y \mathbf{T}_j^\beta) + \nabla_y R_j^\beta = \mathbf{0} \text{ in } W^*, \\ \boldsymbol{\eta} \cdot (B(y)\nabla_y (\mathbf{P}_j^\beta - \boldsymbol{\chi}_j^\beta) - A^t(y)\nabla_y \mathbf{T}_j^\beta) - R_j^\beta \boldsymbol{\eta} = \mathbf{0} \text{ on } \partial W^* \setminus \partial W, \\ \operatorname{div}_y (\mathbf{P}_j^\beta - \mathbf{T}_j^\beta) = 0 \text{ in } W^*, \\ (\mathbf{T}_j^\beta, R_j^\beta) \text{ is } W^* \text{-periodic,} \\ \mathcal{M}_{W^*} (\mathbf{T}_j^\beta) = \mathbf{0}. \end{array} \right. \quad (16)$$

Over  $\mathcal{O}$ , we define the elliptic and constant tensors  $D = (d_{ij}^{\alpha\beta})_{1 \leq i, j, \alpha, \beta \leq n}$ , its transpose  $D^t = (d_{ji}^{\alpha\beta})_{1 \leq i, j, \alpha, \beta \leq n}$  and the perturbed  $B^\# = (b_{ij}^{\#\alpha\beta})_{1 \leq i, j, \alpha, \beta \leq n}$  as

$$d_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta} - \frac{1}{|W^*|} \int_{W^*} A(y)\nabla_y (\mathbf{P}_j^\beta - \boldsymbol{\chi}_j^\beta) : \nabla_y \boldsymbol{\chi}_i^\alpha dy,$$

$$d_{ji}^{\beta\alpha} = a_{ji}^{\beta\alpha} - \frac{1}{|W^*|} \int_{W^*} A^t(y)\nabla_y (\mathbf{P}_j^\beta - \mathbf{H}_j^\beta) : \nabla_y \mathbf{H}_i^\alpha dy,$$

$$b_{ij}^{\#\alpha\beta} = b_{0ij}^{\alpha\beta} - \frac{1}{|W^*|} \int_{W^*} (B(y)\nabla_y (\mathbf{P}_j^\beta - \boldsymbol{\chi}_j^\beta) - A^t(y)\nabla_y \mathbf{T}_j^\beta) : \nabla_y \mathbf{T}_j^\beta dy,$$

with  $a_{ij}^{\alpha\beta}$ ,  $a_{ji}^{\beta\alpha}$ , and  $b_{0ij}^{\alpha\beta}$  forms the respective entires of the tensors  $A_0$ ,  $A_0^t$ , and  $B_0$  as

$$a_{ij}^{\alpha\beta} = \frac{1}{|W^*|} \int_{W^*} A(y)\nabla_y (\mathbf{P}_j^\beta - \boldsymbol{\chi}_j^\beta) : \nabla_y \mathbf{P}_i^\alpha dy,$$

$$a_{ji}^{\beta\alpha} = \frac{1}{|W^*|} \int_{W^*} A^t(y)\nabla_y (\mathbf{P}_j^\beta - \mathbf{H}_j^\beta) : \nabla_y \mathbf{P}_i^\alpha dy,$$

$$b_{0ij}^{\alpha\beta} = \frac{1}{|W^*|} \int_{W^*} (B(y)\nabla_y (\mathbf{P}_j^\beta - \boldsymbol{\chi}_j^\beta) - A^t(y)\nabla_y \mathbf{T}_j^\beta) : \nabla_y (\mathbf{P}_i^\alpha) dy.$$

Now, we present in the following the limit OCP:

$$\boldsymbol{\theta} \in (L^2(\mathcal{O}))^n \left\{ J(\boldsymbol{\theta}) = \frac{1}{2} \int_{\mathcal{O}} B^\# \nabla \mathbf{u} : \nabla \mathbf{u} dx + \frac{\mathcal{T}}{2} \int_{\mathcal{O}} |\boldsymbol{\theta}|^2 dx \right\} \quad (17)$$

subject to

$$\left\{ \begin{array}{l} -\sum_{j, \alpha, \beta=1}^n \frac{\partial}{\partial x_\alpha} \left( d_{ij}^{\alpha\beta} \frac{\partial u_j}{\partial x_\beta} \right) + \nabla p = \mathbf{f} + \boldsymbol{\theta} \text{ in } \mathcal{O}, \\ \operatorname{div} (\mathbf{u}) = 0 \text{ in } \mathcal{O}, \\ \mathbf{u} = 0 \text{ on } \partial \mathcal{O}. \end{array} \right. \quad (18)$$

The existence of this unique pair  $(\mathbf{u}, p) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O})$  can be found in [5, Chapter 1]. Further, the problem (17) is a standard one and there exists a unique weak solution to it, one can follow the arguments introduced in [22, Chapter 2, Theorem 1.2]. We call the triplet  $(\bar{\mathbf{u}}, \bar{p}, \bar{\theta}) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O}) \times (L^2(\mathcal{O}))^n$ , the optimal solution to (17), with  $\bar{\mathbf{u}}, \bar{p}$  and  $\bar{\theta}$  as the optimal state, pressure, and control, respectively.

Now, we introduce the limit adjoint system associated with (18): Find a pair  $(\bar{\mathbf{v}}, \bar{q}) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O})$  which solves the system

$$\begin{cases} -\sum_{j,\alpha,\beta=1}^n \frac{\partial}{\partial x_\alpha} \left( a_{ji}^{\alpha\beta} \frac{\partial \bar{v}_j}{\partial x_\beta} \right) + \nabla \bar{q} = -\sum_{j,\alpha,\beta=1}^n \frac{\partial}{\partial x_\alpha} \left( b_{ij}^{\#\alpha\beta} \frac{\partial \bar{u}_j}{\partial x_\beta} \right) & \text{in } \mathcal{O}, \\ \operatorname{div}(\bar{\mathbf{v}}) = 0 & \text{in } \mathcal{O}. \end{cases} \quad (19)$$

A unique weak solution  $(\bar{\mathbf{v}}, \bar{q}) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O})$  to (19) exists, which is easy to establish analogous to the limit state equation (6.5) by employing the ellipticity of tensors  $D^t$  and  $B^\#$ . Next, in the below-mentioned result, we state a result similar to Theorem 3.1 that characterizes the optimal control  $\bar{\theta}$  in terms of the adjoint state  $\bar{\mathbf{v}}$  and the proof of which follows analogous to the standard procedure laid in [22, Chapter 2, Theorem 1.4].

**Theorem 6.1.** Let  $(\bar{\mathbf{u}}, \bar{p}, \bar{\theta})$  be the optimal solution to (6.4) and  $(\bar{\mathbf{u}}, \bar{q})$  be the corresponding adjoint solution to (6.6), then the optimal control is characterized by

$$\bar{\theta} = -\frac{1}{\mathcal{J}} \bar{\mathbf{v}} \text{ a. e. in } \mathcal{O}. \quad (20)$$

Conversely, suppose that a triplet  $(\check{\mathbf{u}}, \check{p}, \check{\theta}) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O}) \times (L^2(\mathcal{O}))^n$  and a pair  $(\check{\mathbf{v}}, \check{q}) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O})$ , respectively, satisfy the following systems:

$$\begin{cases} -\sum_{j,\alpha,\beta=1}^n \frac{\partial}{\partial x_\alpha} \left( a_{ji}^{\alpha\beta} \frac{\partial \check{u}_j}{\partial x_\beta} \right) + \nabla \check{q} = f - \frac{1}{\mathcal{J}} \check{\mathbf{v}} & \text{in } \mathcal{O}, \\ \operatorname{div}(\check{\mathbf{u}}) = 0 & \text{in } \mathcal{O}, \end{cases}$$

and

$$\begin{cases} -\sum_{j,\alpha,\beta=1}^n \frac{\partial}{\partial x_\alpha} \left( a_{ji}^{\alpha\beta} \frac{\partial \check{v}_j}{\partial x_\beta} \right) + \nabla \check{q} = -\sum_{j,\alpha,\beta=1}^n \frac{\partial}{\partial x_\alpha} \left( b_{ij}^{\#\alpha\beta} \frac{\partial \check{u}_j}{\partial x_\beta} \right) & \text{in } \mathcal{O}, \\ \operatorname{div}(\check{\mathbf{v}}) = 0 & \text{in } \mathcal{O}. \end{cases}$$

Then, the triplet  $(\check{\mathbf{u}}, \check{p}, \frac{1}{\mathcal{J}} \check{\mathbf{v}})$  is the optimal solution to (17).

## 7. Convergence Results

We present here the key findings on the convergence analysis of the optimal solutions to the problem (1) and its corresponding adjoint system (5) by using the method of periodic unfolding for perforated domains described in Section 5.

**Theorem 7.1** For given  $\varepsilon > 0$ , let the triplets  $(\bar{\mathbf{u}}_\varepsilon, \bar{p}_\varepsilon, \bar{\boldsymbol{\theta}}_\varepsilon)$  and  $(\bar{\mathbf{u}}, \bar{p}, \bar{\boldsymbol{\theta}})$ , respectively, be the optimal solutions of the problems (1) and (17). Then

$$T_\varepsilon^*(A_\varepsilon) \rightarrow A \text{ strongly in } (L^2(\mathcal{O} \times W^*))^{n \times n} \quad (21)$$

$$\tilde{\boldsymbol{\theta}}_\varepsilon \rightharpoonup \Theta \bar{\boldsymbol{\theta}} \text{ weakly in } (L^2(\mathcal{O}))^n, \quad (22)$$

$$\tilde{\bar{\mathbf{u}}}_\varepsilon \rightharpoonup \Theta \bar{\mathbf{u}} \text{ weakly in } (H_0^1(\mathcal{O}))^n, \quad (23)$$

$$\tilde{\bar{\mathbf{v}}}_\varepsilon \rightharpoonup \Theta \bar{\mathbf{v}} \text{ weakly in } (H_0^1(\mathcal{O}))^n, \quad (24)$$

$$\tilde{p}_\varepsilon \rightharpoonup \frac{\Theta}{n} A_0 \nabla \bar{\mathbf{u}} : I + \Theta \bar{p} \text{ weakly in } L^2(\mathcal{O}), \quad (25)$$

$$\tilde{q}_\varepsilon \rightharpoonup \frac{\Theta}{n} (A_0^t \nabla \bar{\mathbf{v}} : I - B_0^t \nabla \bar{\mathbf{u}} : I) + \Theta \bar{q} \text{ weakly in } L^2(\mathcal{O}), \quad (26)$$

where tensors  $A_0, B_0^t$  are defined in Section 6,  $I$  is the  $n \times n$  identity matrix,  $\bar{\boldsymbol{\theta}}$  is characterized through (20) and the pairs  $(\bar{\mathbf{u}}_\varepsilon, \bar{p}_\varepsilon)$  and  $(\bar{\mathbf{v}}_\varepsilon, \bar{q}_\varepsilon)$  solve respectively the systems (5) and (19).

*Proof.* First, upon using Proposition 5.2 (vi) on the entries of the matrix  $A_\varepsilon$ , we obtain (21(a)) under the passage of limit  $\varepsilon \rightarrow 0$ . Similarly, one can prove the convergence for the matrix  $A_\varepsilon^t$  under unfolding. Next, in view of Theorem 4.3 and the fact that the triplet  $(\bar{\mathbf{u}}_\varepsilon, \bar{p}_\varepsilon, \bar{\boldsymbol{\theta}}_\varepsilon)$  is an optimal solution to problem (1), one gets uniform estimates for the sequences  $\{\bar{\boldsymbol{\theta}}_\varepsilon\}, \{\bar{\mathbf{u}}_\varepsilon\}, \{\bar{p}_\varepsilon\}, \{\bar{\mathbf{v}}_\varepsilon\}$  and  $\{\bar{q}_\varepsilon\}$  in the spaces  $(L^2(\mathcal{O}_\varepsilon^*))^n, (H_{\Gamma_0^*}^1(\mathcal{O}_\varepsilon^*))^n, L^2(\mathcal{O}_\varepsilon^*), (H_{\Gamma_\varepsilon^*}^1(\mathcal{O}_\varepsilon^*))^n$ , and  $L^2(\mathcal{O}_\varepsilon^*)$ , respectively.

Using the uniform estimate of the sequence  $\{\bar{\boldsymbol{\theta}}_\varepsilon\}$  in the space  $(L^2(\mathcal{O}_\varepsilon^*))^n$  and Proposition 5.2 (i), we have the sequence  $\{T_\varepsilon^* \bar{\boldsymbol{\theta}}_\varepsilon\}$  to be uniformly bounded in the space  $(L^2(\mathcal{O} \times W^*))^n$ . Thus, by weak compactness, there exists a subsequence not relabelled and a function  $\hat{\boldsymbol{\theta}}$  in  $(L^2(\mathcal{O} \times W^*))^n$ , such that

$$T_\varepsilon^*(\bar{\boldsymbol{\theta}}_\varepsilon) \rightharpoonup \hat{\boldsymbol{\theta}} \text{ weakly in } (L^2(\mathcal{O} \times W^*))^n \quad (27)$$

Now, using Proposition 5.2 (vii) in (7.2) gives

$$\tilde{\boldsymbol{\theta}}_\varepsilon \rightharpoonup \frac{1}{|W|} \int_{W^*} \hat{\boldsymbol{\theta}}(x, y) dy = \Theta \boldsymbol{\theta}_0 \text{ weakly in } (L^2(\mathcal{O}))^n, \quad (28)$$

where,  $\boldsymbol{\theta}_0 = \mathcal{M}_{W^*}(\hat{\boldsymbol{\theta}})$ .

Employing Proposition 5.2 (i), we have the uniform boundedness of the sequences  $\{T^\varepsilon(\bar{\mathbf{u}}_\varepsilon)\}, \{T^\varepsilon(\nabla \bar{\mathbf{u}}_\varepsilon)\}$ , and  $\{T^\varepsilon(p_\varepsilon)\}$  in the respective spaces  $(L^2(\mathcal{O}; H^1(W^*)))^n, (L^2(\mathcal{O} \times W^*))^{n \times n}$ , and  $L^2(\mathcal{O} \times W^*)$ . Further, upon employing Proposition 5.3 and Proposition 5.2 (vii), there exist subsequences not relabelled and function  $\hat{\mathbf{u}}$  with  $\mathcal{M}_{W^*}(\hat{\mathbf{u}}) = \mathbf{0}, \mathbf{u}_0$  and  $\hat{p}$  in spaces  $(L^2(\mathcal{O}; H_{per}^1(W^*)))^n, (H_0^1(\mathcal{O}))^n$ , and  $L^2(\mathcal{O} \times W^*)$ , respectively, such that

$$T_\varepsilon^*(\bar{\mathbf{u}}_\varepsilon) \rightarrow \mathbf{u}_0 \text{ strongly in } \left( L^2(\mathcal{O}; H^1(W^*)) \right)^n, \quad (29a)$$

$$T_\varepsilon^*(\nabla \bar{\mathbf{u}}_\varepsilon) \rightarrow \nabla \mathbf{u}_0 + \nabla_y \hat{\mathbf{u}} \text{ weakly in } (L^2(\mathcal{O} \times W^*))^{n \times n} \quad (29b)$$

$$\tilde{\bar{\mathbf{u}}}_\varepsilon \rightarrow \Theta \mathbf{u}_0 \text{ weakly in } (H_0^1(\mathcal{O}))^n, \quad (29c)$$

$$T_\varepsilon^*(\bar{p}_\varepsilon) \rightarrow \hat{p} \text{ weakly in } L^2(\mathcal{O} \times W^*), \quad (29d)$$

$$\tilde{\bar{p}}_\varepsilon \rightarrow \Theta \mathcal{M}_{W^*}(\hat{p}) \text{ weakly in } L^2(\mathcal{O}). \quad (29e)$$

Likewise, for the associated adjoint counterparts, viz.,  $\bar{\mathbf{v}}_\varepsilon$ , and  $\bar{q}_\varepsilon$ , one obtains that there exist subsequences not relabelled and functions  $\hat{\mathbf{v}}$  with  $\mathcal{M}_{W^*}(\hat{\mathbf{v}}) = \mathbf{0}, \mathbf{v}_0$ , and  $\hat{q}$  in spaces  $\left( L^2(\mathcal{O}; H_{per}^1(W^*)) \right)^n, (H_0^1(\mathcal{O}))^n$ , and  $L^2(\mathcal{O} \times W^*)$ , respectively, such that

$$T_\varepsilon^*(\bar{\mathbf{v}}_\varepsilon) \rightarrow \mathbf{v}_0 \text{ strongly in } \left( L^2(\mathcal{O}; H^1(W^*)) \right)^n, \quad (30a)$$

$$T_\varepsilon^*(\nabla \bar{\mathbf{v}}_\varepsilon) \rightarrow \nabla \mathbf{v}_0 + \nabla_y \hat{\mathbf{v}} \text{ weakly in } (L^2(\mathcal{O} \times W^*))^{n \times n} \quad (30b)$$

$$\tilde{\bar{\mathbf{v}}}_\varepsilon \rightarrow \Theta \mathbf{v}_0 \text{ weakly in } (H_0^1(\mathcal{O}))^n, \quad (30c)$$

$$T_\varepsilon^*(\bar{q}_\varepsilon) \rightarrow \hat{q} \text{ weakly in } L^2(\mathcal{O} \times W^*), \quad (30d)$$

$$\tilde{\bar{q}}_\varepsilon \rightarrow \Theta \mathcal{M}_{W^*}(\hat{q}) \text{ weakly in } L^2(\mathcal{O}). \quad (30e)$$

The identification of the limit functions  $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{p}, \hat{q}, \mathcal{M}_{W^*}(\hat{p})$  and  $\mathcal{M}_{W^*}(\hat{q})$  is carried out in subsequent steps.

**Step 1: (Claim)** : For all  $\boldsymbol{\varphi} \in (H_0^1(\mathcal{O}))^n, \boldsymbol{\psi} \in \left( L^2(\mathcal{O}; H_{per}^1(W^*)) \right)^n$ , and  $w \in L^2(\mathcal{O})$ , we claim that the ordered quadruplet  $(\mathbf{u}_0, \hat{\mathbf{u}}, \hat{p}, \boldsymbol{\theta}_0) \in (H_0^1(\mathcal{O}))^n \times \left( L^2(\mathcal{O}; H_{per}^1(W^*)) \right)^n \times L^2(\mathcal{O} \times W^*) \times (L^2(\mathcal{O}))^n$  is a unique solution to the following limit system:

$$\begin{cases} \int_{\mathcal{O} \times W^*} A(y) (\nabla \mathbf{u}_0 + \nabla_y \hat{\mathbf{u}}(x, y)) : (\nabla \boldsymbol{\varphi} + \nabla_y \boldsymbol{\psi}) dx dy \\ - \int_{\mathcal{O} \times W^*} \hat{p}(x, y) (\operatorname{div}(\boldsymbol{\varphi}) + \operatorname{div}_y(\boldsymbol{\psi})) dx dy = |W^*| \int_{\mathcal{O}} (\mathbf{f} + \boldsymbol{\theta}_0) \cdot \boldsymbol{\varphi} dx, \\ \text{and, } \int_{\mathcal{O}} \operatorname{div}(\mathbf{u}_0) w dx = 0, \end{cases} \quad (31)$$

and the ordered triplet  $(\mathbf{v}_0, \hat{\mathbf{v}}, \hat{q}) \in (H_0^1(\mathcal{O}))^n \times \left( L^2(\mathcal{O}; H_{per}^1(W^*)) \right)^n \times L^2(\mathcal{O} \times W^*)$  is a unique solution to the following limit adjoint system:

$$\begin{cases} \int_{\mathcal{O}_{xW^*}} A^t(y) (\nabla \mathbf{v}_0 + \nabla_y \widehat{\mathbf{v}}(x, y)) : (\nabla \boldsymbol{\varphi} + \nabla_y \boldsymbol{\psi}) dx dy \\ - \int_{\mathcal{O}_{xW^*}} \widehat{q}(x, y) (\operatorname{div}(\boldsymbol{\varphi}) + \operatorname{div}_y(\boldsymbol{\psi})) dx dy \\ = \int_{\mathcal{O}_{xW^*}} B(y) (\nabla \mathbf{u}_0 + \nabla_y \widehat{\mathbf{u}}(x, y)) : (\nabla \boldsymbol{\varphi} + \nabla_y \boldsymbol{\psi}) dx dy, \\ \text{and, } \int_{\mathcal{O}} \operatorname{div}(\mathbf{v}_0) \omega dx = 0. \end{cases} \quad (32)$$

Proof of the Claim: Here, we will furnish the proof (31). Analogously, one can easily establish (32). Towards the proof of (31), let us consider a test function  $\boldsymbol{\varphi} \in (D(\mathcal{O}))^n$  in (3) and use properties (i), (ii), and (iv) of Proposition 5.2 to get

$$\begin{aligned} & \frac{1}{|\mathcal{W}|} \int_{\mathcal{O}_{xW^*}} T_\varepsilon^*(A_\varepsilon) T_\varepsilon^*(\nabla \bar{\mathbf{u}}_\varepsilon) : T_\varepsilon^*(\nabla \boldsymbol{\varphi}) dx dy + \int_{\widehat{\Lambda}_\varepsilon^*} A_\varepsilon \nabla \mathbf{u}_\varepsilon : \nabla \boldsymbol{\varphi} dx - \int_{\widehat{\Lambda}_\varepsilon^*} p_\varepsilon \operatorname{div}(\boldsymbol{\varphi}) dx \\ & - \frac{1}{|\mathcal{W}|} \int_{\mathcal{O}_{xW^*}} T_\varepsilon^*(\bar{p}_\varepsilon) T_\varepsilon^*(\operatorname{div}(\boldsymbol{\varphi})) dx dy = \frac{1}{|\mathcal{W}|} \int_{\mathcal{O}_{xW^*}} \left( (T_\varepsilon^*(\mathbf{f}) + T_\varepsilon^*(\bar{\boldsymbol{\theta}}_\varepsilon)) \cdot T_\varepsilon^*(\boldsymbol{\varphi}_\varepsilon) \right) dx dy \\ & + \int_{\widehat{\Lambda}_\varepsilon^*} (\mathbf{f} + \bar{\boldsymbol{\theta}}_\varepsilon) \cdot \boldsymbol{\varphi} dx. \end{aligned} \quad (33)$$

Using Proposition 5.2 (iii), the fact that  $\lim_{\varepsilon \rightarrow 0} |\widehat{\Lambda}_\varepsilon^*| = 0$ , and convergences (27), (21), (29b), (29d), we have under the passage of limit  $\varepsilon \rightarrow 0$  in (33)

$$\begin{aligned} & \int_{\mathcal{O}_{xW^*}} A(y) (\nabla \mathbf{u}_0 + \nabla_y \widehat{\mathbf{u}}(x, y)) : \nabla \boldsymbol{\varphi} dx dy \\ & - \int_{\mathcal{O}_{xW^*}} \widehat{p}(x, y) \operatorname{div}(\boldsymbol{\varphi}) dx dy = |\mathcal{W}^*| \int_{\mathcal{O}} (\mathbf{f} + \boldsymbol{\theta}_0) \cdot \boldsymbol{\varphi} dx \end{aligned} \quad (34)$$

which remains valid for every  $\boldsymbol{\varphi} \in (H_0^1(\mathcal{O}))^n$ , by density.

Now, consider the function  $\boldsymbol{\varphi}_\varepsilon(x) = \varepsilon \phi(x) \boldsymbol{\xi} \left( \frac{x}{\varepsilon} \right)$ , where  $\phi \in D(\mathcal{O})$  and  $\boldsymbol{\xi} \in \left( H_{per}^1(W^*) \right)^n$ .

Employing properties (ii), (iii), and (vi) of Proposition 5.2, one can easily obtain

$$T_\varepsilon^*(\boldsymbol{\varphi}_\varepsilon)(x, y) \rightarrow \mathbf{0} \text{ stongly in } (L^2(\mathcal{O} \times W^*))^n, \quad (35a)$$

$$T_\varepsilon^*(\nabla \boldsymbol{\varphi}_\varepsilon)(x, y) \rightarrow \phi(x) \nabla_y \boldsymbol{\xi}(y) \text{ stongly in } (L^2(\mathcal{O} \times W^*))^{n \times n}. \quad (35b)$$

Let us use the test function  $\boldsymbol{\varphi}_\varepsilon$  in (3) and employ properties (i), (ii), and (iv) of Proposition 5.2 to get

$$\begin{aligned} & \frac{1}{|\mathcal{W}|} \int_{\mathcal{O}_{xW^*}} T_\varepsilon^*(A_\varepsilon) T_\varepsilon^*(\nabla \bar{\mathbf{u}}_\varepsilon) : T_\varepsilon^*(\nabla \boldsymbol{\varphi}_\varepsilon) dx dy + \int_{\widehat{\Lambda}_\varepsilon^*} A_\varepsilon \nabla \mathbf{u}_\varepsilon : \nabla \boldsymbol{\varphi}_\varepsilon dx - \int_{\widehat{\Lambda}_\varepsilon^*} p_\varepsilon \operatorname{div}(\boldsymbol{\varphi}_\varepsilon) dx \\ & - \frac{1}{|\mathcal{W}|} \int_{\mathcal{O}_{xW^*}} T_\varepsilon^*(\bar{p}_\varepsilon) T_\varepsilon^*(\operatorname{div}(\boldsymbol{\varphi}_\varepsilon)) dx dy = \frac{1}{|\mathcal{W}|} \int_{\mathcal{O}_{xW^*}} \left( T_\varepsilon^*(\mathbf{f}) + T_\varepsilon^*(\bar{\boldsymbol{\theta}}_\varepsilon) \right) \cdot T_\varepsilon^*(\boldsymbol{\varphi}_\varepsilon) dx dy \\ & + \int_{\widehat{\Lambda}_\varepsilon^*} \bar{\boldsymbol{\theta}}_\varepsilon \cdot \boldsymbol{\varphi}_\varepsilon dx. \end{aligned} \quad (36)$$

In (36), the absolute value of each integral over  $\widehat{\Lambda}_\varepsilon^*$  is bounded above with a bound of order  $\varepsilon |\widehat{\Lambda}_\varepsilon^*|$  or  $|\widehat{\Lambda}_\varepsilon^*|$ . This with the fact that  $\lim_{\varepsilon \rightarrow 0} |\widehat{\Lambda}_\varepsilon^*| = 0$ , and convergences (27), (21), (29b), (29d), and (35), gives under the passage of limit  $\varepsilon \rightarrow 0$

$$\frac{1}{|\mathbb{W}|} \int_{\mathcal{O} \times W^*} A(y) (\nabla \mathbf{u}_0 + \nabla_y \widehat{\mathbf{u}}(x, y)) : \nabla_y \boldsymbol{\psi} \, dx \, dy - \frac{1}{|\mathbb{W}|} \int_{\mathcal{O} \times W^*} \widehat{p}(x, y) \operatorname{div}_y(\boldsymbol{\psi}) \, dx \, dy = 0, \quad (37)$$

which remains valid for every  $\phi \boldsymbol{\xi} = \boldsymbol{\psi} \in \left( L^2(\mathcal{O}; H_{per}^1(W^*)) \right)^n$ , by density.

Further, for all  $w \in L^2(\mathcal{O})$ , we have  $\int_{\mathcal{O}^*} \operatorname{div}(\overline{\mathbf{u}}_\varepsilon) w \, dx = 0$ .

Now, upon applying unfolding on this and using properties (i), (ii), and (iii) of Proposition 5.2 along with convergence (22), we get under the passage of limit  $\varepsilon \rightarrow 0$

$$\frac{1}{|\mathbb{W}|} \int_{\mathcal{O} \times W^*} \left( \operatorname{div}(\mathbf{u}_0) + \operatorname{div}_y(\widehat{\mathbf{u}}) \right) w \, dx \, dy = 0,$$

which eventually gives upon using the fact that  $\widehat{\mathbf{u}}$  is  $W^*$ -periodic, for all  $w \in L^2(\mathcal{O})$ :

$$\int_{\mathcal{O}} \operatorname{div}(\mathbf{u}_0) w \, dx = 0. \quad (38)$$

Finally, upon adding (34) with (37) and considering (38), we establish (31). This settles the proof of the claim.

**Step 2:** First, we are going to identify the limit functions  $\widehat{\mathbf{u}}$ ,  $\widehat{v}$ ,  $\widehat{p}$ , and  $\widehat{q}$ . Next, using these identifications, we will identify  $\mathcal{M}_{W^*}(\widehat{p})$  and  $\mathcal{M}_{W^*}(\widehat{q})$ .

**Identification of  $\widehat{\mathbf{u}}$ ,  $\widehat{v}$ ,  $\widehat{p}$ ,  $\widehat{q}$ :** Taking successively  $\boldsymbol{\varphi} \equiv 0$  and  $\boldsymbol{\psi} \equiv 0$  in (31), yields

$$\begin{cases} -\operatorname{div}_y \left( A(y) \nabla_y \widehat{\mathbf{u}}(x, y) \right) + \nabla_y \widehat{p}(x, y) = \operatorname{div}_y(A(y)) \nabla \mathbf{u}_0(x) & \text{in } \mathcal{O} \times W^*, \\ -\operatorname{div}_x \left( \int_{W^*} A(y) \left( \nabla \mathbf{u}_0(x) + \nabla_y \widehat{\mathbf{u}}(x, y) \right) dy \right) + \left( \int_{W^*} \widehat{p}(x, y) dy \right) = |\mathbb{W}^*| \boldsymbol{\theta}_0 & \text{in } \mathcal{O}, \\ \operatorname{div}(\mathbf{u}_0) = 0 & \text{in } \mathcal{O}, \\ \widehat{\mathbf{u}}(x, \cdot) \text{ is } W^* \text{-periodic.} \end{cases} \quad (39)$$

In the first line of (39), we have the  $y$ -independence of  $\nabla \mathbf{u}_0(x)$  and the linearity of operators, viz., divergence and gradient, which suggests  $\widehat{\mathbf{u}}(x, y)$  and  $\widehat{p}(x, y)$  to be of the following form (see, for e.g., [20, Page 15]):

$$\begin{cases} \widehat{\mathbf{u}}(x, y) = -\sum_{j, \beta=1}^n \boldsymbol{\chi}_j^\beta(y) \frac{\partial \mathbf{u}_{0j}}{\partial x_\beta} + \mathbf{u}_1(x), \\ \widehat{p}(x, y) = \sum_{j, \beta=1}^n \Pi_j^\beta(y) \frac{\partial \mathbf{u}_{0j}}{\partial x_\beta} + p_0(x). \end{cases} \quad (39(i))$$

where the ordered pair  $(\mathbf{u}_1, p_0) \in (H^1(\mathcal{O}))^n \times L^2(\mathcal{O})$ , and for  $1 \leq j, \beta \leq n$ , the pair  $(\chi_j^\beta, \Pi_j^\beta)$  satisfy the cell problem (14). Likewise, we obtain the corresponding adjoint weak formulation (32):

$$\left\{ \begin{array}{l} -\operatorname{div}_y (A(y) \nabla_y \hat{\mathbf{v}}(x, y)) + \nabla_y \hat{q}(x, y) = \operatorname{div}_y (A(y)) \nabla \mathbf{u}_0(x) \\ \quad - \operatorname{div}_y (B(y) (\nabla \mathbf{u}_0(x) + \nabla_y \hat{\mathbf{u}}(x, y))) \quad \text{in } \mathcal{O} \times W^*, \\ -\operatorname{div}_y \left( \int_{W^*} A(y) (\nabla \mathbf{u}_0(x) + \nabla_y \hat{\mathbf{v}}(x, y)) dy \right) + \nabla \left( \int_{W^*} \hat{q}(x, y) dy \right) \\ \quad = -\operatorname{div}_x \left( \int_{W^*} B(y) (\nabla \mathbf{u}_0(x) + \nabla_y \hat{\mathbf{u}}(x, y)) dy \right) \quad \text{in } \mathcal{O}, \\ \quad \operatorname{div}(\mathbf{v}_0) = 0 \text{ in } \mathcal{O}, \\ \quad \hat{\mathbf{v}}(x, \cdot) \text{ is } W^* \text{-periodic,} \end{array} \right. \quad (40)$$

and,

$$\left\{ \begin{array}{l} \hat{\mathbf{v}}(x, y) = -\sum_{j, \beta=1}^n \mathbf{H}_j^\beta(y) \frac{\partial v_{0j}}{\partial x_\beta} + \sum_{j, \beta=1}^n \mathbf{T}_j^\beta(y) \frac{\partial u_{0j}}{\partial x_\beta} + \mathbf{v}_1(x), \\ \hat{q}(x, y) = \sum_{j, \beta=1}^n Q_j^\beta(y) \frac{\partial v_{0j}}{\partial x_\beta} - \sum_{j, \beta=1}^n R_j^\beta(y) \frac{\partial u_{0j}}{\partial x_\beta} + q_0(x), \end{array} \right. \quad (41)$$

where the ordered pair  $(\mathbf{v}_1, p_0) \in (H^1(\mathcal{O}))^n \times L^2(\mathcal{O})$ , and for  $1 \leq j, \beta \leq n$ , the pair  $(\mathbf{H}_j^\beta, Q_j^\beta)$  and  $(\mathbf{T}_j^\beta, R_j^\beta)$  respectively satisfy the cell problems (15) and (16).

**Identification of  $\mathcal{M}_{W^*}(\bar{p})$  and  $\mathcal{M}_{W^*}(\bar{q})$ :** Choosing the test function  $\mathbf{y} = (y_1, \dots, y_n)$  in the weak formulation of (14), we get

$$\sum_{i, l, k, \alpha=1}^n \int_{W^*} a_{lk} \frac{\partial}{\partial y_k} (\mathbf{P}_j^\beta - \chi_j^\beta) \cdot \frac{\partial \mathbf{P}_i^\alpha}{\partial y_l} \frac{\partial y_i}{\partial y_\alpha} dy = n \int_{W^*} \Pi_j^\beta dy. \quad (42)$$

In view of (29(c)), 39(i), and (42), we observe that

$$\mathcal{M}_{W^*}(\hat{p}) = \frac{1}{n |W^*|} \sum_{i, j, l, k, \alpha, \beta=1}^n \int_{W^*} a_{lk} \frac{\partial}{\partial y_k} (\mathbf{P}_j^\beta - \chi_j^\beta) \cdot \frac{\partial \mathbf{P}_i^\alpha}{\partial y_l} \frac{\partial y_i}{\partial y_\alpha} \frac{\partial u_{0j}}{\partial x_\beta} dy + p_0,$$

which upon using the definition of  $a_{ij}^{\alpha\beta}$ , gives

$$\mathcal{M}_{W^*}(\hat{p}) = \frac{1}{n} \sum_{i, j, \alpha, \beta=1}^n a_{ij}^{\alpha\beta} \frac{\partial u_{0j}}{\partial x_\beta} \frac{\partial y_i}{\partial y_\alpha} + p_0. \quad (43)$$

Also, we re-write the equation (43) to get the identification of  $\mathcal{M}_{W^*}(\hat{p})$  as

$$\mathcal{M}_{W^*}(\hat{p}) = \frac{1}{n} A_0 \nabla \mathbf{u}_0 : I + p_0. \quad (44)$$

Likewise, one can obtain the identification of  $\mathcal{M}_{W^*}(\hat{q})$  as

$$\mathcal{M}_{W^*}(\hat{q}) = \frac{1}{n} (A_0^t \nabla \mathbf{v}_0 : I - B_0^t \nabla \mathbf{u}_0 : I) + q_0. \quad (44(i))$$

Thus, from (29(c)) and (44); (18) and (44(i)), we have the following weak convergences:



$$\tilde{p}_\varepsilon \rightharpoonup \frac{\Theta}{n} A_0 \nabla \mathbf{u}_0 : I + \Theta p_0 \text{ weakly in } L^2(\mathcal{O}), \quad (45a)$$

$$\tilde{q}_\varepsilon \rightharpoonup \frac{\Theta}{n} (A_0^t \nabla \mathbf{v}_0 : I - B_0^t \nabla \mathbf{u}_0 : I) + \Theta q_0 \text{ weakly in } L^2(\mathcal{O}). \quad (45b)$$

**Step 3: (Claim):** The pairs  $(\mathbf{u}_0, p_0)$  and  $(\mathbf{v}_0, q_0)$  solve the systems (18) and (19), respectively.

Proof of the Claim: We now prove that the pair  $(\mathbf{u}_0, p_0)$  solves the system (18). The proof that the pair  $(\mathbf{v}_0, q_0)$  solves the system (19) follows analogously. Substituting the values of  $\hat{\mathbf{u}}(x, y)$  and  $\hat{p}(x, y)$  from expression (39(i)) into equation (34), we get

$$\begin{aligned} & \frac{1}{|\mathcal{W}|} \sum_{l,k=1}^n \int_{\mathcal{O} \times \mathcal{W}^*} a_{lk} \left( \frac{\partial \mathbf{u}_0}{\partial x_k} - \sum_{j,\beta=1}^n \frac{\partial \chi_j^\beta}{\partial y_k} \frac{\partial \mathbf{u}_{0j}}{\partial x_\beta} \right) \frac{\partial \varphi}{\partial x_l} dx dy - \\ & \frac{1}{|\mathcal{W}|} \sum_{j,\beta=1}^n \int_{\mathcal{O} \times \mathcal{W}^*} \Pi_j^\beta \frac{\partial \mathbf{u}_{0j}}{\partial x_\beta} \operatorname{div}(\boldsymbol{\varphi}) dx dy \\ & - \Theta \int_{\mathcal{O}} p_0 \operatorname{div}(\boldsymbol{\varphi}) dx = \Theta \int_{\mathcal{O}} \boldsymbol{\theta}_0 \cdot \boldsymbol{\varphi} dx. \end{aligned} \quad (46)$$

Considering  $\mathbf{P}_j^\beta = (0, \dots, y_j, \dots, 0)$  with  $y_j$  at the  $\beta$ -th position, we can express the terms  $\frac{\partial \mathbf{u}_0}{\partial x_k}$ ,  $\frac{\partial \varphi}{\partial x_l}$ , and  $\operatorname{div}(\boldsymbol{\varphi})$  as

$$\frac{\partial \mathbf{u}_0}{\partial x_k} = \sum_{j,\beta=1}^n \frac{\partial \mathbf{P}_j^\beta}{\partial y_k} \frac{\partial \mathbf{u}_{0j}}{\partial x_\beta}, \quad \frac{\partial \varphi}{\partial x_l} = \sum_{i,\alpha=1}^n \frac{\partial \mathbf{P}_i^\alpha}{\partial y_l} \frac{\partial \varphi_i}{\partial x_\alpha}, \quad \operatorname{div}(\boldsymbol{\varphi}) = \sum_{i,\alpha=1}^n \operatorname{div}_y(\mathbf{P}_i^\alpha) \frac{\partial \varphi_i}{\partial x_\alpha}.$$

Substituting these expressions in (46), we obtain

$$\begin{aligned} & \sum_{i,j,\alpha,\beta=1}^n \int_{\mathcal{O}} \left( \frac{1}{|\mathcal{W}^*|} \sum_{l,k=1}^n \int_{\mathcal{W}^*} a_{lk} \frac{\partial}{\partial y_k} (\mathbf{P}_j^\beta - \chi_j^\beta) \frac{\partial \mathbf{P}_i^\alpha}{\partial y_l} dy \right) \frac{\partial \mathbf{u}_{0j}}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} dx \\ & - \sum_{i,j,\alpha,\beta=1}^n \int_{\mathcal{O}} \left( \frac{1}{|\mathcal{W}^*|} \int_{\mathcal{W}^*} \Pi_j^\beta \operatorname{div}_y(\mathbf{P}_i^\alpha) dy \right) \frac{\partial \mathbf{u}_{0j}}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} dx \\ & - \int_{\mathcal{O}} p_0 \operatorname{div}(\boldsymbol{\varphi}) dx = \int_{\mathcal{O}} \boldsymbol{\theta}_0 \cdot \boldsymbol{\varphi} dx. \end{aligned} \quad (47)$$

Now, choosing the test function  $\chi_i^\alpha$  in the weak formulation of (6.1), we get upon using the fact that  $\operatorname{div}_y(\chi_i^\alpha) = \operatorname{div}_y(\mathbf{P}_i^\alpha) = \delta_{i\alpha}$ , where  $\delta$  denotes the Kronecker delta function, the following:

$$\int_{\mathcal{W}^*} A(y) \nabla_y (\mathbf{P}_j^\beta - \chi_j^\beta) : \nabla_y \chi_i^\alpha dy = \int_{\mathcal{W}^*} \Pi_j^\beta \delta_{i\alpha} dy. \quad (48)$$

Further, substituting (48) in (47), we obtain

$$\begin{aligned} & \sum_{i,j,\alpha,\beta=1}^n \int_{\mathcal{O}} \left( \frac{1}{|\mathcal{W}^*|} \sum_{l,k=1}^n \int_{\mathcal{W}^*} a_{lk} \frac{\partial}{\partial y_k} (\mathbf{P}_j^\beta - \chi_j^\beta) \frac{\partial}{\partial y_l} (\mathbf{P}_i^\alpha - \chi_i^\alpha) dy \right) \frac{\partial \mathbf{u}_{0j}}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} dx \\ & - \int_{\mathcal{O}} p_0 \operatorname{div}(\boldsymbol{\varphi}) dx = \int_{\mathcal{O}} \boldsymbol{\theta}_0 \cdot \boldsymbol{\varphi} dx. \end{aligned} \quad (49)$$

Also, we can write equation (49) as

$$\sum_{i,j,\alpha,\beta=1}^n b_{ij}^{\alpha\beta} \frac{\partial u_{0j}}{\partial x_\beta} \frac{\partial \varphi_i}{\partial x_\alpha} dx - \int_{\mathcal{O}} p_0 \operatorname{div}(\boldsymbol{\varphi}) dx = \int_{\mathcal{O}} \boldsymbol{\theta}_0 \cdot \boldsymbol{\varphi} dx, \quad (50)$$

which holds true for all  $\boldsymbol{\varphi} \in (H_0^1(\mathcal{O}))^n$ . Also, from equation (38), we have  $\int_{\mathcal{O}} \operatorname{div}(\mathbf{u}_0) w dx = 0$ , for every  $w \in L^2(\mathcal{O})$ . This together with equation (50) implies that, for  $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ , the pair  $(\mathbf{u}_0, p_0) \in (H_0^1(\mathcal{O}))^n \times L^2(\mathcal{O})$  satisfies the variational formulation of the system (18).

Therefore, we obtain the optimality system for the minimization problem (17). Also, in view of Theorem 6.1, we conclude that the triple  $(\mathbf{u}_0, p_0, \boldsymbol{\theta}_0)$  is indeed an optimal solution to the problem (17). Finally, upon considering the optimal solution's uniqueness, we establish that the subsequent pair of triplets are equal:

$$(\bar{\mathbf{u}}, \bar{p}, \bar{\boldsymbol{\theta}}) = (\mathbf{u}_0, p_0, \boldsymbol{\theta}_0). \quad (51)$$

This completes the proof of Theorem 7.1.

## 8. Conclusions

We have addressed the limiting behavior of an interior OCP corresponding to Stokes equations in an  $nD$  ( $n \geq 2$ ) periodically perforated domain  $\mathcal{O}_\varepsilon^*$  via the technique of periodic unfolding in perforated domains (see, [9-12]). We employed the Neumann boundary condition on the part of the boundary of the perforated domain. Firstly, we characterized the unique minimizer of the problem (1) in terms of the adjoint state. Secondly, we deduced the a priori optimal bounds for control, state, pressure, and their corresponding adjoint state and pressure functions. Thereafter, the limiting analysis for the considered OCP is carried out upon employing the periodic unfolding method in perforated domains. We observed the convergence between the optimal solution to the problem (3.1) posed on the perforated domain  $\mathcal{O}_\varepsilon^*$  and the optimal solution to that of the limit problem (6.4) governed by stationary Stokes equation posed on a non-perforated domain  $\mathcal{O}$ .

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