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ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY GALUE-TYPE FUNCTION

[1]**Aditya Lagad,** [2]**R.N. Ingle** and [3]**P. Thirupathi Reddy**
[1]Department of Mathematics, N.E.S. Science College,
Nanded - 431 605, Maharashtra, India.
Email: lagadac@gmail.com
[2]Department of Mathematics, Bahirji Smarak Mahavidyalay,
Bashmathnagar - 431 512, Maharashtra, India.
Email: ingleraju11@gmail.com
[3]Department of Mathematics, DRK Institute of Science and Technology,
Bowarmpet - 500 043, Hyderabad, Telangana, India.
Email: reddypt2@gmail.com

Abstract: The study of the geometric properties of analytic functions and their numerous applications in a variety of mathematical fields, including fractional calculus, probability distributions, and special functions, has drawn significant and impressive attention to Geometric Function Theory (GFT), one of the most prominent branches of complex analysis, in recent years. In this work, we introduce and investigate a new subclass of analytic functions in the open unit disc E with negative coefficients defined by Galue-type Struve function. The object of the present paper is to determine the coefficient inequality, Integral representation, extreme points and subordination results for this class.

Key Words and Phrases: analytic, starlike coefficient estimate, error function, subordination.

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1. Introduction

Let A denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $E = \{z \in \mathbb{C}: |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $u(z)$ which are all univalent in E. A function $u \in A$ is a starlike function of the order $\upsilon, 0 \leq \upsilon < 1$, if it satisfy

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \upsilon, (z \in E). \tag{2}$$

We denote this class with $S^*(\upsilon)$.

A function $u \in A$ is a convex function of the order $\upsilon, 0 \leq \upsilon < 1$, if it satisfy

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \upsilon, (z \in E). \tag{3}$$

We denote this class with $K(\upsilon)$.

Let T denote the class of functions analytic in E that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \ (z \in E) \tag{4}$$

and let $T^*(\upsilon) = T \cap S^*(\upsilon),\ C(\upsilon) = T \cap K(\upsilon)$. The class $T^*(\upsilon)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [11] and others. For $u \in A$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{5}$$

their convolution (or Hadamard product), denoted by $(u * g)$, is defined as

$$(u * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * u)(z) \ \ (z \in E). \tag{6}$$

Note that $u * g \in A$.

The Galue-type Struve function (GTSF) was introduced in [4,5,9] and defined by

$$\alpha\mathcal{W}_{p,b,c,\xi}^{\lambda,\mu}(z) = z + \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(\lambda n+\mu)\Gamma\left(\alpha n+\frac{p}{\xi}+\frac{b+2}{2}\right)} \left(\frac{z}{2}\right)^{2n+p+1} \quad (z \in E). \tag{7}$$

where $\alpha \in \mathbb{N},\ z, p, b, c \in \mathbb{C},\ \lambda > 0,\ \xi > 0$ and μ is an arbitrary parameter. It is evident that when $\lambda = \alpha = 1, \mu = \frac{3}{2}$ and $\xi = 1$ in (7) then we have the generalized Struve function (see [6,7]) defined by

$$\mathcal{H}_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(n+3/2)\Gamma\left(n+p+\frac{b+2}{2}\right)} \left(\frac{z}{2}\right)^{2n+p+1} \quad (z \in E) \tag{8}$$

where $z, p, b, c \in \mathbb{C}$. Using (7), consider the function

$$\mathcal{U}_{p,b,c,\xi}(z) = 2^p \sqrt{\pi}\, \Gamma\left(\frac{p}{\xi} + \frac{b+c}{2}\right) z^{\frac{-(p+1)}{2}} \alpha\mathcal{W}_{p,b,c,\xi}^{\lambda,\mu}\left(\sqrt{z}\right) \quad (z \in E) \tag{9}$$

Using the Pochhammer symbol defined in terms of Euler's gamma function, Oyekan [8] presented the relation

$$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \gamma(\gamma+1)\cdots(\gamma+n-1)$$

so that from (9), we have

$$\mathcal{V}_{p,b,c,\xi}(z) = z\mathcal{U}_{p,b,c,\xi}(z) = z + \sum_{n=2}^{\infty} \left(\frac{\left(\frac{-c}{4}\right)^n}{(\mu)_{\lambda(n-1)}(\gamma)_{\alpha(n-1)}}\right) z^n \quad (z \in E). \tag{10}$$

Using the convolution principle, Oyekan [8], defined the function

$$\mathcal{L}_{p,b,c}^{\lambda,\mu,\xi}(z) = (f * \mathcal{V}_{p,b,c,\xi})(z) = z + \sum_{n=2}^{\infty}\left(\frac{\left(\frac{-c}{4}\right)^{n}}{(\mu)_{\lambda(n-1)}(\gamma)_{\alpha(n-1)}}\right) a_n z^n \quad (z \in E). \tag{11}$$

$p, b, c \in \mathbb{C}, \gamma = \frac{p}{\xi} + \frac{b+2}{2} \neq 0, -1, -2, \cdots, \alpha \in \mathbb{N}, \lambda, \xi > 0$ and μ is an arbitrary parameter. Function $\mathcal{V}$ in (10) is the normalized form of Galue-type Struve function and is analytic in $\mathbb{C}$, while (10) is the simplied version.

A special function that occurs in probability, statistics, material science and partial differential equation is the error function. The error function is use in quantum mechanics to eliminate the probability of observing a particle in a specified region. The error function

$$eru(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^{n-1}z^{n+1}}{(2n+1)!} \tag{12}$$

was reported in [9] and for additional information see [1, 2]. In particular, Ramchandran et al. [10] made a slight modification to (12) and came up with the function

$$Eru(z) = z + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(2n-1)(n-1)!}z^n \quad (z \in E). \tag{13}$$

where the function $Eru(z)$ was used to define a class of analytic functions and solved some coefficient problems.

Using the convolution concept, and in view of (11) and (13), we can deduce the function

$$\mathcal{G}(z) = \left(\mathcal{L}_{p,b,c}^{\lambda,\mu,\xi} * Eru\right)(z) = z + \sum_{n=2}^{\infty}\phi_n(\mu, \lambda, \gamma, c)a_n z^n \tag{14}$$

where

$$\phi_n = \phi_n(\mu, \lambda, \gamma, c) = \left(\frac{\left(\frac{-c}{4}\right)^{n-1}}{(2n-1)(n-1)!\,(\mu)_{\lambda(n-1)}(\gamma)_{\alpha(n-1)}}\right)$$

Definition 1.1. For $\hbar \geq 0, 0 \leq \ell < 1$, we set $S(\hbar, \ell)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1) and satisfy

$$Re\left(\frac{\mathcal{G}(z)}{z}\right) \geq \hbar\left|(\mathcal{G}(z))' - \frac{\mathcal{G}(z)}{z}\right| + \ell \tag{15}$$

where $\mathcal{G}(z)$ *is given by* (14).

We further let $TS(\hbar, \ell) = S(\hbar, \ell) \cap T$.

In this paper, we obtain coefficient inequalities, extreme points, integral means inequalities for the functions in the class $TS(\hbar, \ell)$ and also subordination results for the class of function $S(\hbar, \ell)$.

2. Coefficient Estimates

Theorem 2.1. The function v defined by (1) is in the class $S(\hbar, \ell)$ if

$\sum_{n=2}^{\infty}[1+\hbar(n-1)]\,\phi_n|a_n| \leq 1-\ell,$ (16)

where $\hbar \geq 0, 0 \leq \ell < 1$ *and* ϕ_n is given by (14).

Proof. It suffices to show that

$$\hbar\left|\left(\mathcal{G}(z)\right)' - \frac{\mathcal{G}(z)}{z}\right| - Re\left\{\frac{\mathcal{G}(z)}{z} - 1\right\} \leq 1-\ell.$$

We have the next inequality

$$\begin{aligned} &\hbar\left|\left(\mathcal{G}(z)\right)' - \frac{\mathcal{G}(z)}{z}\right| - Re\left\{\frac{\mathcal{G}(z)}{z} - 1\right\} \\ \leq\ & \hbar\left|\frac{\sum_{n=2}^{\infty}(n-1)\phi_n a_n z^n}{z}\right| + \left|\frac{\sum_{n=2}^{\infty}\phi_n a_n z^n}{z}\right| \\ \leq\ & \hbar\sum_{n=2}^{\infty}(n-1)\,\phi_n|a_n| + \sum_{n=2}^{\infty}\phi_n\,|a_n| \\ =\ & \sum_{n=2}^{\infty}[1+\hbar(n-1)]\,\phi_n|a_n|. \end{aligned} \tag{17}$$

The last expression is bounded above by $(1-\ell)$ if

$\sum_{n=2}^{\infty}[1+\hbar(n-1)]\,\phi_n|a_n| \leq 1-\ell$ (18)

and the proof of theorem is completed.

In the following theorem, we obtain necessary and sufficient conditions for functions in $TS(\hbar,\ell)$.

Theorem 2.2. For $\hbar \geq 0, 0 \leq \ell < 1$, a function v of the form (4) to be in the class $TS(\hbar,\ell)$ if and only if

$\sum_{n=2}^{\infty}[1+\hbar(n-1)]\,\phi_n|a_n| \leq 1-\ell.$

Proof. Suppose $v(z)$ of the form (2) is in the class $TS(\hbar,\ell)$. Then

$$Re\left(\frac{\mathcal{G}(z)}{z}\right) \geq \hbar\left|\left(\mathcal{G}(z)\right)' - \frac{\mathcal{G}(z)}{z}\right| + \ell.$$

Equivalently,

$$Re\left[1 - \sum_{n=2}^{\infty}\phi_n\,|a_n|z^{n-1}\right] - \hbar\left[\sum_{n=2}^{\infty}(n-1)\,\phi_n a_n z^{n-1}\right] \geq \ell.$$

Letting z to be real values and as $|z| \to 1$, we have

$$1 - \sum_{n=2}^{\infty}\phi_n\,|a_n| - \hbar\sum_{n=2}^{\infty}(n-1)\,\phi_n|a_n| \geq \ell$$

which implies

$\sum_{n=2}^{\infty}[1+\hbar(n-1)]\,\phi_n|a_n| \leq 1-\ell,$ (19)

where $\hbar \geq 0, 0 \leq \ell < 1, \phi_n$ is given by (14) and the sufficiency follows from Theorem 2.1.

Corollary 2.1. If $v \in TS(\hbar,\ell)$ then $a_n \leq \frac{1-\ell}{[1+\hbar(n-1)]\phi_n}$.

Equality holds for the function

$v(z) = z - \frac{1-\ell}{[1+\hbar(n-1)]\phi_n} z^n, \hbar \geq 0,\ 0 \leq \ell < 1,\ \phi_n$ is given by (14).

3. Extreme Points

Theorem 3.1. *Let* $v_1(z) = z$ *and* $v_n(z) = z - \frac{1-\ell}{[1+\hbar(n-1)]\phi_n} z^n, n \geq 2$ *for* $\hbar \geq 0, 0 \leq \ell < 1, \phi_n$ *is* given by (14). Then $v(z)$ is in the class ϕ_n if and only if it can be expressed in the form $v(z) = \sum_{n=1}^{\infty} \lambda_n v_n(z)$, where λ_n and $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. If $v(z) = \sum_{n=1}^{\infty} \lambda_n v_n(z)$ with $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$. Then

$$\begin{aligned} v(z) &= \sum_{n=1}^{\infty} \lambda_n v_n(z) \\ &= \lambda_1 v_1(z) + \sum_{n=2}^{\infty} \lambda_n v_n(z) \\ &= (1 - \sum_{n=2}^{\infty} \lambda_n) z + \sum_{n=2}^{\infty} \left[\lambda_n \left(z - \frac{1-\ell}{[1+\hbar(n-1)]\phi_n} z^n\right)\right] \\ &= z - \sum_{n=2}^{\infty} \frac{1-\ell}{[1+\hbar(n-1)]\phi_n} z^n. \end{aligned} \tag{20}$$

Now $\sum_{n=2}^{\infty} \frac{[1+\hbar(n-1)]\phi_n}{1-\ell} \frac{1-\ell}{[1+\hbar(n-1)]\phi_n} \lambda^n$

$= \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1.$

Then $v \in TS(\hbar, \ell)$. Conversely suppose that $v \in TS(\hbar, \ell)$. Then Corollary (16) gives

$$a_n \leq \frac{1-\ell}{[1+\hbar(n-1)]\phi_n},\ n \geq 2$$

set $$\lambda_n = \frac{[1+\hbar(n-1)]\phi_n}{1-\ell} a_n,\ n \geq 2$$

where $$\lambda_n = 1 - \sum_{n=2}^{\infty} \lambda_n .$$

Then

$$\begin{aligned} v(z) &= z - \sum_{n=2}^{\infty} a_n z^n \\ &= z - \sum_{n=2}^{\infty} \lambda_n \frac{1-\ell}{[1+\hbar(n-1)]\phi_n} \\ &= z - [1 - \sum_{n=2}^{\infty} \lambda_n] + \sum_{n=2}^{\infty} \lambda_n v_n(z) \\ &= \lambda_1 v_1(z) + \sum_{n=2}^{\infty} \lambda_n v_n(z) \\ &= \sum_{n=1}^{\infty} \lambda_n v_n(z). \end{aligned} \tag{21}$$

The poof of theorem is completed □

4. Integral Means Inequalities

Definition 4.1. [12] (Subordination principle) for analytic function g and h with $g(0) = h(0)$, g is said to be subordinate to h, denoted by $g \prec h$ if there exists an analytic function ω such that $\omega(0) = 0, |\omega(z)| < 1$ and $g(z) = h(\omega(z))$, for all $z \in U$.

Lemma 4.1. If the function $v(z)$ and $g(z)$ are analytic in U with $g(z) \prec v(z)$ then $\int_0^{2\pi} \left|g(re^{i\theta})\right|^p d\theta \leq \int_0^{2\pi} \left|v(re^{i\theta})\right|^p d\theta \; (0 \leq r < 1, p > 0 \; and \; z = re^{i\theta})$.

Theorem 4.1. Suppose $v \in TS_{\vartheta}^{a,c}(\hbar, \ell), p > 0, \hbar \geq 0, 0 \leq \ell < 1$ and $v(z)$ is defined by

$v_2(z) = z - \frac{1-\ell}{(1+\hbar)\phi_2} z^2.$

Then for $z = re^{i\theta}$, $0 \leq r < 1$,

$$\int_0^{2\pi} |v(z)|^p \, d\theta \leq \int_0^{2\pi} |v_2(z)|^p \, d\theta \tag{22}$$

Proof. For $v(z) = z - \sum_{n=2}^{\infty} |a_n| \, z^n$, (22) is equivalent to proving that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} |a_n| \, z^{n-1}\right|^p d\theta \leq \int_0^{2\pi} \left|1 - \frac{1-\ell}{(1+\hbar)\phi_2} z\right|^p d\theta, \; (p > 0).$$

By applying Littlewood's subordination theorem (Lemma 4.1), it would be sufficient to show that

$$1 - \sum_{n=2}^{\infty} |a_n| \, z^{n-1} \prec 1 - \frac{1-\ell}{[1+\hbar(n-1)]\phi_2} z \tag{23}$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| \, z^{n-1} \prec 1 - \frac{1-\ell}{[1+\hbar(n-1)]\phi_2} \omega(z).$$

We have $\omega(z) = \frac{[1+\hbar(n-1)]\phi_n}{1-\ell} \sum_{n=2}^{\infty} a_n \, z^{n-1}$ and and $\omega(z)$ is analytic in U with $\omega(0) = 0$. Moreover it suffices to prove that $\omega(z)$ satisfies $|\omega(z)| < 1, z \in U$. Now

$$\begin{aligned} |\omega(z)| &= \left|\sum_{n=2}^{\infty} \frac{[(1+\hbar(n-1)\phi_n}{1-\ell} a_n z^{n-1}\right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{[1+\hbar(n-1)]\phi_n}{1-\ell} |a_n| \\ &\leq |z| < 1. \end{aligned} \tag{24}$$

Thus is view of the inequality (24) the subordination (23) follows, which proves the Theorem.

5. Subordination Results

Definition 5.1. (Subordination factor sequence) A sequence $\{ b_n \}_{n=2}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $v(z) = \sum_{n=2}^{\infty} a_n \, z^n, a_1 = 1$ is regular, univalent and convex *in* U, we have

$$\sum_{n=1}^{\infty} b_n \, a_n z^n \prec v(z), \; z \in U. \tag{25}$$

Theorem 5.1 [12]. The sequence $\{ b_n \}_{n=2}^{\infty}$ is a subordinating factor sequence if and only if

$Re\{1 + 2\sum_{n=1}^{\infty} b_n \, z^n\} > 0, z \in U.$

Theorem 5.2. *Let* $v \in \mathcal{G}(z)$ and $g(z)$ any function in the usual class of convex function $\mathbb{C}$. Then $\frac{(1+\hbar)\phi_n}{2(1-\ell)+(1+\hbar)\phi_n}(v * g)(z) \prec g(z)$ *where* $\hbar \geq 0, 0 \leq \ell < 1$ *with* ϕ_n is given by (14)

$$Re\{v(z)\} > -\frac{(1-\ell)+(1+\hbar)\phi_n}{(1+\hbar)\phi_n},\ z \in E. \quad (26)$$

The constant $\frac{(1+\hbar)\phi_n}{2(1-\ell)+(1+\hbar)\phi_n}$ is the best estimate.

Proof. Let $v \in \mathcal{G}(z)$ and $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathbb{C}$.

Then

$$\frac{(1+\hbar)\phi_n}{2(1-\ell)+(1+\hbar)\phi_n}(v * g)(z) = \frac{(1+\hbar)\phi_n}{2(1-\ell)+(1+\hbar)\phi_n}\left(z + \sum_{n=2}^{\infty} c_n a_n z^n\right).$$

Then by Definition 5.1, the subordination result holds true if $\left\{\frac{(1+\hbar)\phi_n}{2(1-\ell)+(1+\hbar)\phi_n}\right\}_{n=1}^{\infty}$ is a subordinating factor sequence with $a_1 = 1$. In view of Theorem 5.1, this is equivalent to the following inequality.

$$Re\left\{1 + \sum_{n=1}^{\infty} \frac{(1+\hbar)\phi_n}{(1-\ell)+(1+\hbar)\phi_n} a_n z^n\right\} > 0,\ z \in U. \quad (27)$$

Now for $|z| = r < 1$, we have

$$\begin{aligned} & Re\left\{1 + \sum_{n=1}^{\infty} \frac{(1+\hbar)\phi_n}{2(1-\ell)+(1+\hbar)\phi_n} a_n z^n\right\} \\ = \ & Re\left\{1 + \frac{(1+\hbar)\phi_n}{(1-\ell)+(1+\hbar)\phi_n} z + \frac{\sum_{n=2}^{\infty}(1+\hbar)\phi_n a_n z^n}{(1-\ell)+(1+\hbar)\phi_n}\right\} \\ \geq \ & 1 - \frac{(1+\hbar)\phi_n}{(1-\ell)+(1+\hbar)\phi_n} r - \frac{\sum_{n=2}^{\infty}(1+\hbar)\phi_n a_n r^n}{(1-\ell)+(1+\hbar)\phi_n} \\ \geq \ & 1 - \frac{(1+\hbar)\phi_n}{(1-\ell)+(1+\hbar)\phi_n} r - \frac{1-\ell}{(1-\ell)+(1+\hbar)\phi_n} r \\ > \ & 0. \end{aligned} \quad (28)$$

Using (16) and the fact that $[1 + \hbar(n-1)]\phi_n$ is increasing function for $n \geq 2$.

This proves the inequality (27) and hence also the subordination result (25) asserted by Theorem 5.3.

The inequality (26) follows from (25) by taking

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in \mathbb{C}.$$

Now we consider the function $v(z) = z - \frac{1-\ell}{(1+\hbar)\phi_n} z^2$, where $\hbar \geq 0, 0 \leq \ell < 1$. Clearly $F \in \mathcal{G}(z)$. For the function (25) becomes

$$\frac{(1+\hbar)\phi_n}{2(1-\ell)+(1+\hbar)\phi_n}v(z) \prec \frac{z}{1-z}.$$

It is easily verified that

$$\min Re\left\{\frac{(1+\hbar)\phi_n}{2(1-\ell)+(1+\hbar)\phi_n}v(z)\right\} = \frac{-1}{2}, z \in U.$$

This shows that the constant $\frac{(1+\hbar)\phi_n}{2(1-\ell)+(1+\hbar)\phi_n}v(z) \prec \frac{z}{1-z}$ is best possible.

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