Journal of Rajasthan Academy of Physical Sciences
ISSN: 0972-6306; URL: http://raops.org.in
Vol.23, No.3&4, July-December, 2024, 146-158

ON $\mathcal{W}_9$-CURVATURE TENSOR IN KENMOTSU MANIFOLDS

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Abstract: This paper aims to analyze the $\mathcal{W}_9$-curvature tensor on the Kenmotsu manifold, which arises from an almost contact Riemannian manifold under certain specific conditions that fulfill the following conditions: $\xi-\mathcal{W}_9$–flatness, $\mathcal{W}_9.\mathcal{R}=0$, $\phi-\mathcal{W}_9$–semi-symmetric, $\mathcal{W}_9.\mathcal{W}_9=0$, $\mathcal{W}_9.Q=0$, $\phi-\mathcal{W}_9$-flat and Pseudo–$\mathcal{W}_9$-flat to establish several notable results regarding a Kenmotsu manifold, and to further support our findings, we present an example of Kenmotsu manifolds.

Key Words: Kenmotsu Manifolds, $\mathcal{W}_9$-Curvature tensor, $\xi-\mathcal{W}_9$–flatness, $\phi-\mathcal{W}_9$-flat, $\mathcal{W}_9$–semi-symmetric, Einstein manifold, η-Einstein manifold.

2010 Mathematical Sciences Classification: 53C05, 53D10, 53E20

1. Introduction

Consider an almost contact Riemannian manifold $(\mathcal{M}^n, \mathrm{g})$ of odd dimension $(n=2m+1)$, associated having a contact form η, a corresponding vector field ξ, and a (1,1)-tensor field ϕ along with the corresponding Riemannian metric g. In 1971, K. Kenmotsu [8] explored a class of almost contact Riemannian manifolds that fulfill particular conditions. Such manifolds are referred to as Kenmotsu manifolds. Kenmotsu showed that a Kenmotsu manifold with condition $\mathcal{R}(\mathcal{K}_1,\mathcal{K}_2).\mathcal{R}=0$, must have a constant curvature of -1. Here, $\mathcal{R}$ represents the curvature tensor of type $(1,3)$. Kenmotsu demonstrated that a local Kenmotsu manifold can be expressed as a warped product $I\times_f M$, where I corresponds to an interval, $\mathcal{M}$ is a Kähler manifold, and the warping function is $f(t)=se^t$, with s being a non-zero constant. The properties of Kenmotsu manifolds have been studied by several researchers, including Yildiz and De [20], Haseeb [5], Haseeb and Prasad [6, 7], Wang [19], Özgür [9], Chaubey and Yadav[2], Tripathi and Gupta [16], De [3, 4] and many others. Atceken [1] has recently examined a note on P-Sasakian manifolds satisfying certain conditions. In addition, Ozturk, Aktan, and Murathan [10] studied on α-Kenmotsu manifolds.

In 2011 Tripathi and Gupta [15] introduced a new curvature tensor of type (1,3) named τ -curvature tensor. They define $\mathcal{W}_9$ curvature tensor for n-dimensional on a semi-Riemannian manifold. Later on, in 2022 $\mathcal{W}_9$ curvature tensor studies by Uygun et al. [17] on Kenmotsu metric spaces. In the Sasakian manifold Recently, Prakasha, Chavan[11] On the W5-curvature tensor of generalized Sasanian-space-form, Singh et al. [14], investigated the $\mathcal{W}_8$-curvature tensor on (ϵ)-Lorentzian para-Sasakian manifold, Uygun, and Atceken [18] explored On Kenmotsu metric spaces satisfying some conditions on the $\mathcal{W}_7$-curvature tensor and Sardar [13] explored the some results on(ϵ)-Kenmotsu manifolds.

Inspired by the preceding concepts, this study focuses on the $\mathcal{W}_9$ curvature tensor on the Kenmotsu manifold. Preciously, The structure of this paper is organized as follows: Section 1 covers the introduction. Necessary definitions are provided in Sections 2 and 3. In section 4, $\xi-\mathcal{W}_9$-flat in Kenmotsu manifolds has been discussed. In Section 5 satisfy the conditions $\mathcal{W}_9.\mathcal{R}=0$. Section 6 is dedicated to analyzing the $\phi-\mathcal{W}_9$–Semi-Symmetric condition. Furthermore, sections 7 and 8 derive important properties of the manifold that satisfy the conditions $\mathcal{W}_9.\mathcal{W}_9=0$, $\mathcal{W}_9.\mathcal{Q}=0$. Moreover, Sections 9 and 10 are dedicated to the study of $\phi-\mathcal{W}_9$-flat and Pseudo–$\mathcal{W}_9$-flat respectively. Finally, Section 11 provides an example of a 3-dimensional Kenmotsu manifold that confirms our findings.

2. Preliminaries

Consider $\mathcal{M}$ as an almost contact metric manifold of dimension n(where $n=2m+1$), possessing an almost contact metric structure $(\phi,\xi,\eta,\mathrm{g})$. This structure consists of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η along with a Riemannian metric g defined on $\mathcal{M}$ that fulfills the conditions listed below:

$$\phi^2(\mathcal{K}_1)=-\mathcal{K}_1+\eta(\mathcal{K}_1)\xi, \qquad \phi\circ\xi=0, \qquad \eta\circ\phi=0 \tag{1}$$

$$\eta(\xi)=1, \tag{2}$$

$$\mathrm{g}(\phi\mathcal{K}_1,\phi\mathcal{K}_2)=\mathrm{g}(\mathcal{K}_1,\mathcal{K}_2)-\eta(\mathcal{K}_1)\eta(\mathcal{K}_2), \tag{3}$$

$$\mathrm{g}(\mathcal{K}_1,\phi\mathcal{K}_2)=-\mathrm{g}(\phi\mathcal{K}_1,\mathcal{K}_2), \tag{4}$$

$$\mathrm{g}(\mathcal{K}_1,\xi)=\eta(\mathcal{K}_1), \tag{5}$$

for any vector fields $\mathcal{K}_1,\mathcal{K}_2$ on $\mathcal{M}$. An almost contact metric manifold is referred to as a Kenmotsu manifold [8] if it satisfies

$$(\nabla_{\mathcal{K}_1}\phi)(\mathcal{K}_2)=-\eta(\mathcal{K}_2)\phi\mathcal{K}_1-\mathrm{g}(\mathcal{K}_1,\phi\mathcal{K}_2)\xi, \tag{6}$$

$$\nabla_{\mathcal{K}_1}\xi=\mathcal{K}_1-\eta(\mathcal{K}_1)\xi, \tag{7}$$

where ∇ represents the Levi-Civita connection on the manifold $\mathcal{M}$.

In a Kenmotsu manifold [8], the following equations are satisfied:

$$(\nabla_{\mathcal{K}_1}\eta)(\mathcal{K}_2)=\mathrm{g}(\mathcal{K}_1,\mathcal{K}_2)-\eta(\mathcal{K}_1)\eta(\mathcal{K}_2), \tag{8}$$

$$\mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)\xi=\eta(\mathcal{K}_1)\mathcal{K}_2-\eta(\mathcal{K}_2)\mathcal{K}_1, \tag{9}$$

$$\mathcal{R}(\xi,\mathcal{K}_1)\mathcal{K}_2=\eta(\mathcal{K}_2)\mathcal{K}_1-\mathrm{g}(\mathcal{K}_1,\mathcal{K}_2)\xi, \tag{10}$$

$$\mathcal{R}(\xi,\mathcal{K}_1)\xi = \mathcal{K}_1 - \eta(\mathcal{K}_1)\xi, \tag{11}$$

$$\mathcal{S}(\mathcal{K}_1,\xi) = -(n-1)\eta(\mathcal{K}_1), \tag{12}$$

$$Q\xi = -(n-1)\xi, \tag{13}$$

$$\mathcal{S}(\xi,\xi) = -(n-1), \tag{14}$$

where $\mathcal{R}$ represents the Riemannian curvature tensor and $\mathcal{S}$ denotes the Ricci tensor, defined by $\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2) = \mathfrak{g}(Q\mathcal{K}_1,\mathcal{K}_2)$, with Q being the Ricci operator. It yields to

$$\mathcal{S}(\phi\mathcal{K}_1,\phi\mathcal{K}_2) = \mathcal{S}(\mathcal{K}_1,\mathcal{K}_2) + (n-1)\eta(\mathcal{K}_1)\eta(\mathcal{K}_2) \tag{15}$$

where $\mathcal{R}$ is the Riemannian curvature tensor and $\mathcal{S}$ is the Ricci tensor.

Let $\{e_1, e_2, \ldots\ldots e_{2n}, e_{2n+1}\}$ represent a collection of orthonormal basis vector fields in $\mathcal{M}$. Consequently, the Ricci tensor $\mathcal{S}$ and the scalar curvature $\mathfrak{r}$ of the manifold are defined as follows:

$$\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2) = \sum_{i=1}^{2n+1} \epsilon_i \mathfrak{g}(\mathcal{R}(e_i,\mathcal{K}_1)\mathcal{K}_2, e_i)$$

$$\mathfrak{r} = \sum_{i=1}^{2n+1} \epsilon_i \mathcal{S}(e_i, e_i)$$

Additionally, we have

$$\mathfrak{g}(\mathcal{K}_1,\mathcal{K}_2) = \sum_{i=1}^{2n+1} \epsilon_i \mathfrak{g}(\mathcal{K}_1, e_i)\mathfrak{g}(\mathcal{K}_2, e_i)$$

for all $\mathcal{K}_1, \mathcal{K}_2, \in \chi(M)$ and ϵ_i=$\mathfrak{g}(e_i, e_i)$=± 1

Definition 2.1. *A Kenmotsu manifold $\mathcal{M}$ is classified as an η–Einstein manifold if its Ricci tensor $\mathcal{S}$, which is of type* $(0,2)$, *takes the following form[8, 12]*

$$\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2) = \mathfrak{a}\mathfrak{g}(\mathcal{K}_1,\mathcal{K}_2) + \mathfrak{b}\eta(\mathcal{K}_1)\eta(\mathcal{K}_2), \tag{16}$$

for arbitrary vector fields $\mathcal{K}_1, \mathcal{K}_2$; where $\mathfrak{a}$ and $\mathfrak{b}$ are scalar functions on $\mathcal{M}$. When $\mathfrak{b}$=0, the manifold $\mathcal{M}$ transforms into an Einstein manifold.

3. $\mathcal{W}_9$-Curvature tensor

The concept of $\mathcal{W}_9$-curvature tensor was defined by Tripathi and Gupta [15] of an n(where n=2m+1)-dimensional Riemannian manifold are, respectively, defined as

$$\mathcal{W}_9(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = \mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 + \frac{1}{n-1}[\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 - \mathfrak{g}(\mathcal{K}_2,\mathcal{K}_3)Q\mathcal{K}_1], \tag{17}$$

for all $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \in \chi(M)$. Where R represents the curvature tensor and S corresponds to the Ricci tensor of the manifold.

Choosing $\mathcal{K}_3=\xi$ in (17), we obtain

$$\mathcal{W}_9(\mathcal{K}_1,\mathcal{K}_2)\xi = \eta(\mathcal{K}_1)\mathcal{K}_2 - \eta(\mathcal{K}_2)\mathcal{K}_1 + \frac{1}{n-1}[\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2)\xi - \eta(\mathcal{K}_2)\mathcal{Q}\mathcal{K}_1]. \tag{18}$$

Putting $\mathcal{K}_1=\xi$ in (18), we have

$$\mathcal{W}_9(\xi,\mathcal{K}_2)\xi = \mathcal{K}_2 - \eta(\mathcal{K}_2)\xi. \tag{19}$$

Putting $\mathcal{K}_1=\xi$, in (17), we get

$$\mathcal{W}_9(\xi,\mathcal{K}_2)\mathcal{K}_3 = \eta(\mathcal{K}_3)\mathcal{K}_2 - \eta(\mathcal{K}_2)\mathcal{K}_3. \tag{20}$$

4. ξ-$\mathcal{W}_9$-flat Kenmotsu Manifold

Here, we investigate the ξ-$\mathcal{W}_9$-flat Kenmotsu Manifold:

Definition 4.1. *A Kenmotsu manifold is defined as ξ-$\mathcal{W}_9$-flat if:*

$$\mathcal{W}_9(\mathcal{K}_1,\mathcal{K}_2)\xi = 0, \tag{21}$$

For each vector field $\mathcal{K}_1,\mathcal{K}_2$ defined on $\mathcal{M}$. The $\mathcal{W}_9$ curvature tensor is given as in equation (17)

$$\mathcal{W}_9(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = \mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 + \frac{1}{n-1}[\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 - \mathrm{g}(\mathcal{K}_2,\mathcal{K}_3)\mathcal{Q}\mathcal{K}_1]. \tag{22}$$

Substituting $\mathcal{K}_3=\xi$ in equation (22), we obtain:

$$\mathcal{W}_9(\mathcal{K}_1,\mathcal{K}_2)\xi = \mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)\xi + \frac{1}{n-1}[\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2)\xi - \mathrm{g}(\mathcal{K}_2,\xi)\mathcal{Q}\mathcal{K}_1]. \tag{23}$$

By using (21) in (23), we get

$$\mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)\xi + \frac{1}{n-1}[\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2)\xi - \mathrm{g}(\mathcal{K}_2,\xi)\mathcal{Q}\mathcal{K}_1] = 0. \tag{24}$$

Using (5) and (9) in (24), we have

$$\eta(\mathcal{K}_1)\mathcal{K}_2 - \eta(\mathcal{K}_2)\mathcal{K}_1 + \frac{1}{n-1}[\mathcal{S}(\mathcal{K}_1,\mathcal{K}_2)\xi - \eta(\mathcal{K}_2)\mathcal{Q}\mathcal{K}_1] = 0. \tag{25}$$

Taking inner product with $\mathcal{K}_4 \in \chi(M)$ in (25) and, if we choose $\mathcal{K}_2=\xi$, we derive

$$\mathcal{S}(\mathcal{K}_1,\mathcal{K}_4) = -(n-1)\mathrm{g}(\mathcal{K}_1,\mathcal{K}_4). \tag{26}$$

Therefore, we state the following:

Theorem 4.1. *A Kenmotsu manifold $\mathcal{M}$ is $\xi - \mathcal{W}_9 - flat$ if and only if the manifold is an Einstein manifold.*

Contracting (26), we obtain

$$\mathfrak{r} = -n(n-1) \tag{27}$$

Corollary 4.1. *If a Kenmotsu manifold $\mathcal{M}$ is $\xi - \mathcal{W}_9 - flat$, then $\mathcal{M}$ is of negative constant scalar curvature.*

Furthermore, we focus on a Kenmotsu manifold meeting the condition $\mathcal{W}_9$=0, then we obtain from (17),

$$\mathcal{R}(\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_4) = \frac{1}{n-1}[\mathrm{g}(\mathcal{K}_2,\mathcal{K}_3)\mathrm{g}(Q\mathcal{K}_1,\mathcal{K}_4) - \mathcal{S}(\mathcal{K}_1,\mathcal{K}_2)\mathrm{g}(\mathcal{K}_3,\mathcal{K}_4)]. \quad (28)$$

After simplification, we get

$$\mathcal{R}(\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_4) = \frac{1}{n-1}[\mathrm{g}(\mathcal{K}_2,\mathcal{K}_3)\mathcal{S}(\mathcal{K}_1,\mathcal{K}_4) - \mathcal{S}(\mathcal{K}_1,\mathcal{K}_2)\mathrm{g}(\mathcal{K}_3,\mathcal{K}_4)]. \quad (29)$$

Using (26) in (29), we get

$$\mathcal{R}(\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_4) = -[\mathrm{g}(\mathcal{K}_2,\mathcal{K}_3)\mathrm{g}(\mathcal{K}_1,\mathcal{K}_4) - \mathrm{g}(\mathcal{K}_1,\mathcal{K}_2)\mathrm{g}(\mathcal{K}_3,\mathcal{K}_4)]. \quad (30)$$

Hence $\mathcal{M}$ is of constant curvature -1.

Corollary 4.2. *Let $\mathcal{M}$ be an n(where $n = 2m + 1$)-dimensional Kenmotsu manifold. If $\mathcal{M}$ satisfies the $\mathcal{W}_9$=0, then it is isomorphic to the hyperbolic space $\mathcal{H}^n(-1)$.*

5. Kenmotsu Manifold Satisfying $\mathcal{W}_9.\mathcal{R} = 0$ Condition

Here, we investigate the $\mathcal{W}_9.\mathcal{R} = 0$ condition. thus, we arrive at

$$\mathcal{W}_9(\xi,\mathcal{K}_4)\mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 - \mathcal{R}(\mathcal{W}_9(\xi,\mathcal{K}_4)\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3$$
$$-\mathcal{R}(\mathcal{K}_1,\mathcal{W}_9(\xi,\mathcal{K}_4)\mathcal{K}_2)\mathcal{K}_3$$
$$-\mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{W}_9(\xi,\mathcal{K}_4)\mathcal{K}_3 = 0. \quad (31)$$

By substituting $\mathcal{K}_3$=ξ in equation (31), we find

$$\mathcal{W}_9(\xi,\mathcal{K}_4)\mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)\xi - \mathcal{R}(\mathcal{W}_9(\xi,\mathcal{K}_4)\mathcal{K}_1,\mathcal{K}_2)\xi$$
$$-\mathcal{R}(\mathcal{K}_1,\mathcal{W}_9(\xi,\mathcal{K}_4)\mathcal{K}_2)\xi$$
$$-\mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{W}_9(\xi,\mathcal{K}_4)\xi = 0. \quad (32)$$

By using (9) in (32) and on simplification, we obtain

$$-\eta((\mathcal{W}_9(\xi,\mathcal{K}_4)\mathcal{K}_1)\mathcal{K}_2 + \eta((\mathcal{W}_9(\xi,\mathcal{K}_4)\mathcal{K}_2)\mathcal{K}_1 - \mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)(\mathcal{W}_9(\xi,\mathcal{K}_4)\xi = 0. \quad (33)$$

By using (19), (20) in (33) and on simplification, we get

$$\eta(\mathcal{K}_1)\eta(\mathcal{K}_4)\mathcal{K}_2 - \eta(\mathcal{K}_2)\eta(\mathcal{K}_4)\mathcal{K}_1 + 2 + \mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4 = 0 \quad (34)$$

By substituting $\mathcal{K}_1$=ξ in equation (34), we obtain

$$\mathrm{g}(\mathcal{K}_4,\mathcal{K}_2) = \eta(\mathcal{K}_4)\eta(\mathcal{K}_2). \quad (35)$$

Replacing $\mathcal{K}_4$ by $Q\mathcal{K}_4$ and using (12), we arrive at

$$\mathcal{S}(\mathcal{K}_4,\mathcal{K}_2) = -(n-1)\eta(\mathcal{K}_4)\eta(\mathcal{K}_2). \quad (36)$$

Therefore, we can state the following theorem:

Theorem 5.1 *Let $\mathcal{M}$ be a n(where $n = 2m + 1$) dimensional Kenmotsu manifold satisfying the condition $W_9.\mathcal{R} = 0$, then the manifold is a special type of η-Einstein manifold defined by*

$\mathcal{S}(\mathcal{K}_4, \mathcal{K}_2) = -(n-1)\eta(\mathcal{K}_4)\eta(\mathcal{K}_2)$.

6. ϕ–$\mathcal{W}_9$-Semi-Symmetric Condition in Kenmotsu Manifold

This section focuses on the study of the $\phi - \mathcal{W}_9$-Semi-Symmetric condition within Kenmotsu manifolds:

Definition 6.1. *A Kenmotsu manifold of dimension n(where $n = 2m + 1$) is referred to as ϕ–$\mathcal{W}_9$–semi-symmetric if it meets the following condition:*

$$\mathcal{W}_9(\mathcal{K}_1, \mathcal{K}_2).\phi = 0, \tag{37}$$

for arbitrary vector fields $\mathcal{K}_1, \mathcal{K}_2$ on $\mathcal{M}$.

Now (37) turns into

$$(\mathcal{W}_9(\mathcal{K}_1, \mathcal{K}_2).\phi)\mathcal{K}_3 = \mathcal{W}_9(\mathcal{K}_1, \mathcal{K}_2)\phi\mathcal{K}_3 - \phi\mathcal{W}_9(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3. \tag{38}$$

Making use of (17) in (38), we get

$$\mathcal{R}(\mathcal{K}_1, \mathcal{K}_2)\phi\mathcal{K}_3 - \phi\mathcal{R}(\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3$$
$$+\frac{1}{n-1}[\mathrm{g}(\mathcal{K}_2, \mathcal{K}_3)\phi(\mathcal{Q}\mathcal{K}_1) - \mathrm{g}(\mathcal{K}_2, \phi\mathcal{K}_3)(\mathcal{Q}\mathcal{K}_1)] = 0. \tag{39}$$

Putting $\mathcal{K}_3$=ξ in (39), we obtain

$$\eta(\mathcal{K}_2)\phi(\mathcal{Q}\mathcal{K}_1) = (n-1)\phi(\mathcal{R}(\mathcal{K}_1, \mathcal{K}_2)\xi). \tag{40}$$

By using (1), (9) and operating ϕ in (40), we get

$$-\eta(\mathcal{K}_2)\mathcal{Q}\mathcal{K}_1 - (n-1)\eta(\mathcal{K}_1)\eta(\mathcal{K}_2)\xi = (n-1)(\eta(\mathcal{K}_2)\mathcal{K}_1 - \eta(\mathcal{K}_1)\mathcal{K}_2). \tag{41}$$

By taking the inner product with $\mathcal{K}_4$ in (41), we get

$$-\eta(\mathcal{K}_2)\mathrm{g}(\mathcal{Q}\mathcal{K}_1, \mathcal{K}_4) - (n-1)\eta(\mathcal{K}_1)\eta(\mathcal{K}_2)\eta(\mathcal{K}_4) = (n-1)(\eta(\mathcal{K}_2)\mathrm{g}(\mathcal{K}_1, \mathcal{K}_4)$$
$$-\eta(\mathcal{K}_1)\mathrm{g}(\mathcal{K}_2, \mathcal{K}_4)). \tag{42}$$

Putting $\mathcal{K}_2$=ξ in (42) and use (2),(5) we get

$$\mathcal{S}(\mathcal{K}_1, \mathcal{K}_4) = -(n-1)\mathrm{g}(\mathcal{K}_1, \mathcal{K}_4). \tag{43}$$

Therefore, we state the following:

Theorem 6.1. *Let $\mathcal{M}$ be an n(where $n = 2m + 1$)-dimensional Kenmotsu manifold. If $\mathcal{M}$ satisfies the ϕ–$\mathcal{W}_9$–semi-symmetric condition, then it is an Einstein manifold.*

Contracting (6.7), we obtain

$$\mathfrak{r} = -n(n-1) \tag{44}$$

Corollary 6.1. *If a Kenmotsu manifold $\mathcal{M}$ is $\phi-\mathcal{W}_9$-Semi-Symmetric, then $\mathcal{M}$ is of negative constant scalar curvature.*

7. Kenmotsu Manifold Satisfying $\mathcal{W}_9.\mathcal{W}_9 = 0$ Condition

Here, we investigate the Kenmotsu manifold that fulfills the $\mathcal{W}_9.\mathcal{W}_9$=0. Therefore, we have

$$(\mathcal{W}_9(\mathcal{K}_1,\mathcal{K}_2)\mathcal{W}_9)(\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5 = \mathcal{W}_9(\mathcal{K}_1,\mathcal{K}_2)\mathcal{W}_9(\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5$$
$$-\mathcal{W}_9(\mathcal{W}_9(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3,\mathcal{K}_4)\mathcal{K}_5$$
$$-\mathcal{W}_9(\mathcal{K}_3,\mathcal{W}_9(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_4)\mathcal{K}_5$$
$$-\mathcal{W}_9(\mathcal{K}_3,\mathcal{K}_4)\mathcal{W}_9(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_5 = 0, \quad (45)$$

for any $\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_4,\mathcal{K}_5 \in \chi(M)$. Setting $\mathcal{K}_1 = \mathcal{K}_5 = \xi$ in (45), we get

$$(\mathcal{W}_9(\xi,\mathcal{K}_2)\mathcal{W}_9)(\mathcal{K}_3,\mathcal{K}_4)\xi = \mathcal{W}_9(\xi,\mathcal{K}_2)\mathcal{W}_9(\mathcal{K}_3,\mathcal{K}_4)\xi$$
$$-\mathcal{W}_9(\mathcal{W}_9(\xi,\mathcal{K}_2)\mathcal{K}_3,\mathcal{K}_4)\xi$$
$$-\mathcal{W}_9(\mathcal{K}_3,\mathcal{W}_9(\xi,\mathcal{K}_2)\mathcal{K}_4)\xi$$
$$-\mathcal{W}_9(\mathcal{K}_3,\mathcal{K}_4)\mathcal{W}_9(\xi,\mathcal{K}_2)\xi = 0. \quad (46)$$

By using (19) and (20) in (46), we obtain

$$2\eta(\mathcal{K}_2)\eta(\mathcal{K}_4)\xi - \mathcal{W}_9(\mathcal{K}_2,\mathcal{K}_4)\xi + \eta(\mathcal{K}_2)\mathcal{K}_4 - 3\eta(\mathcal{K}_2)\eta(\mathcal{K}_4)\xi = 0. \quad (47)$$

Using (18) in (47), we get

$$-\eta(\mathcal{K}_2)\eta(\mathcal{K}_4)\xi + \eta(\mathcal{K}_4)\mathcal{K}_2 - \frac{1}{n-1}\{\mathcal{S}(\mathcal{K}_2,\mathcal{K}_4)\xi - \eta(\mathcal{K}_4)\mathcal{Q}\mathcal{K}_2\} = 0. \quad (48)$$

Taking inner product with $W \in \chi(M)$, we have

$$-\eta(\mathcal{K}_2)\eta(\mathcal{K}_4)\eta(W) + \eta(\mathcal{K}_4)g(\mathcal{K}_2,W)$$
$$-\frac{1}{n-1}\{\mathcal{S}(\mathcal{K}_2,\mathcal{K}_4)\eta(W) - \eta(\mathcal{K}_4)g(\mathcal{Q}\mathcal{K}_2,W)\} = 0. \quad (49)$$

Putting $\mathcal{K}_4$=ξ in (49) and on simplification, we get

$$\mathcal{S}(\mathcal{K}_2,W) = -(n-1)g(\mathcal{K}_2,W) \quad (50)$$

Therefore, we state the following:

Theorem 7.1. *Let $\mathcal{M}$ be an n(where $n = 2m + 1$)-dimensional Kenmotsu manifold. If $\mathcal{M}$ satisfies the $\mathcal{W}_9.\mathcal{W}_9 = 0$ condition, then it is an Einstein manifold.*

8. Kenmotsu Manifold Satisfying $\mathcal{W}_9.\mathcal{Q} = 0$ Condition

Here, we investigate the Kenmotsu manifold that fulfills the $\mathcal{W}_9.\mathcal{Q} = 0$.

Hence, we arrive at:

$$\mathcal{W}_9(\mathcal{K}_1,\mathcal{K}_2)\mathcal{Q}\mathcal{K}_3 - \mathcal{Q}(\mathcal{W}_9(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3) = 0. \quad (51)$$

Putting $\mathcal{K}_2=\xi$ in (51), we obtain

$$\mathcal{W}_9(\mathcal{K}_1,\xi)\mathcal{Q}\mathcal{K}_3 - \mathcal{Q}(\mathcal{W}_9(\mathcal{K}_1,\xi)\mathcal{K}_3) = 0. \quad (52)$$

By virtue of (17) in (52), we obtain

$$\mathcal{R}(\mathcal{K}_1,\xi)\mathcal{Q}\mathcal{K}_3 + \frac{1}{n-1}\{\mathcal{S}(\mathcal{K}_1,\xi)\mathcal{Q}\mathcal{K}_3 - \mathrm{g}(\xi,\mathcal{Q}\mathcal{K}_3)\mathcal{Q}\mathcal{K}_1\}$$
$$-\mathcal{Q}[\mathcal{R}(\mathcal{K}_1,\xi)\mathcal{K}_3 + \frac{1}{n-1}\{\mathcal{S}(\mathcal{K}_1,\xi)\mathcal{K}_3$$
$$-\mathrm{g}(\xi,\mathcal{K}_3)\mathcal{Q}\mathcal{K}_1\}] = 0. \quad (53)$$

By using (5), (10) and (12) in (53), we obtain

$$\eta(\mathcal{Q}\mathcal{K}_3)\mathcal{K}_1 - \eta(\mathcal{K}_3)\mathcal{Q}\mathcal{K}_1 = 0. \quad (54)$$

By using (5) in (54), we get

$$\mathcal{S}(\mathcal{K}_3,\xi)\mathcal{K}_1 - \eta(\mathcal{K}_3)\mathcal{Q}\mathcal{K}_1 = 0. \quad (55)$$

Putting $\mathcal{K}_3 = \xi$ and using (5) and (14) in (55), we obtain

$$(n-1)\mathcal{K}_1 + \mathcal{Q}\mathcal{K}_1 = 0. \quad (56)$$

By applying the inner product with $\mathcal{K}_4$ in (56) and after simplification, we obtain

$$\mathcal{S}(\mathcal{K}_1,\mathcal{K}_4) = -(n-1)\mathrm{g}(\mathcal{K}_1,\mathcal{K}_4). \quad (57)$$

Therefore, we state the following:

Theorem 8.1. *Let $\mathcal{M}$ be an $n(where n = 2m+1)$-dimensional Kenmotsu manifold. If $\mathcal{M}$ satisfies the $W_9.\mathcal{Q} = 0$ condition, then it is an Einstein manifold.*

9. ϕ–$\mathcal{W}_9$–flat Kenmotsu Manifold

Definition 9.1. *$\mathcal{M}$ is termed a ϕ-$\mathcal{W}_9$-flat Kenmotsu manifold if*

$$\phi^2\mathcal{W}_9(\phi\mathcal{K}_1,\phi\mathcal{K}_2)\phi\mathcal{K}_3 = 0, \quad (58)$$

for any vector fields $\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3 \in \chi(M)$.

Then from (58), it follows that

$$\mathrm{g}(\mathcal{W}_9(\phi\mathcal{K}_1,\phi\mathcal{K}_2)\phi\mathcal{K}_3,\phi\mathcal{K}_4) = 0. \quad (59)$$

Using (17) in (59), we get

$$\mathrm{g}(\mathcal{R}(\phi\mathcal{K}_1,\phi\mathcal{K}_2)\phi\mathcal{K}_3,\phi\mathcal{K}_4) = -\frac{1}{n-1}\{\mathcal{S}(\phi\mathcal{K}_1,\phi\mathcal{K}_2)\mathrm{g}(\phi\mathcal{K}_3,\phi\mathcal{K}_4)$$
$$-\mathcal{S}(\phi\mathcal{K}_1,\phi\mathcal{K}_4)\mathrm{g}(\phi\mathcal{K}_2,\phi\mathcal{K}_3)\}. \quad (60)$$

Where $\mathcal{K}_4 \in \chi(M)$.

Let $\{e_1, e_2, \ldots\ldots e_{n-1}, \xi\}$ represent a collection of orthonormal basis vector fields in $\mathcal{M}$. Using that $\{\phi e_1, \phi e_2, \ldots\ldots \phi e_{n-1}, \xi\}$ is also local orthonormal basis in $\mathcal{M}$. If we take $\mathcal{K}_1 = \mathcal{K}_4 = e_i$ and taking sum over i, where 1≤i≤n-1, then we get

$$\sum_{i=1}^{n-1} \epsilon_i \mathrm{g}(\mathcal{R}(\phi e_i, \phi\mathcal{K}_2)\phi\mathcal{K}_3, \phi e_i) = -\frac{1}{n-1}\sum_{i=1}^{n-1} \epsilon_i \mathcal{S}(\phi e_i, \phi\mathcal{K}_2)\mathrm{g}(\phi\mathcal{K}_3, \phi e_i)$$

$$+\frac{1}{n-1}\sum_{i=1}^{n-1} \epsilon_i \mathcal{S}(\phi e_i, \phi e_i)\mathrm{g}(\phi\mathcal{K}_2, \phi\mathcal{K}_3). \quad (61)$$

It is easy to see that

$$\sum_{i=1}^{n-1} \epsilon_i \mathrm{g}(\mathcal{R}(\phi e_i, \phi\mathcal{K}_2)\phi\mathcal{K}_3, \phi e_i) = \mathcal{S}(\phi\mathcal{K}_2, \phi\mathcal{K}_3) + \mathrm{g}(\phi\mathcal{K}_2, \phi\mathcal{K}_3). \quad (62)$$

and

$$\sum_{i=1}^{n-1} \epsilon_i \mathcal{S}(\phi e_i, \phi\mathcal{K}_2)\mathrm{g}(\phi\mathcal{K}_3, \phi e_i) = \mathcal{S}(\phi\mathcal{K}_2, \phi\mathcal{K}_3). \quad (63)$$

$$\sum_{i=1}^{n-1} \epsilon_i \mathcal{S}(\phi e_i, \phi e_i) = \mathfrak{r} + (n-1). \quad (64)$$

Using (62), (63) and (64) in (61), we get

$$n\mathcal{S}(\phi\mathcal{K}_2, \phi\mathcal{K}_3) - r(\mathrm{g}(\phi\mathcal{K}_2, \phi\mathcal{K}_3)) = 0. \quad (65)$$

By using (3), (15) in (65), we get

$$\mathcal{S}(\mathcal{K}_2, \mathcal{K}_3) = \frac{\mathfrak{r}}{n}\mathrm{g}(\mathcal{K}_2, \mathcal{K}_3) - \frac{(\mathfrak{r}+n^2-n)}{n}\eta(\mathcal{K}_2)\eta(\mathcal{K}_3). \quad (66)$$

Thus, we can say the following:

Theorem 9.1. *Let $\mathcal{M}$ be an n(where $n = 2m + 1$)-dimensional Kenmotsu manifold. If $\mathcal{M}$ satisfies the ϕ–$\mathcal{W}_9$-flat, then it is an η-Einstein manifold.*

Contracting (66), we obtain

$$\mathfrak{r} = -n(n-1) \quad (67)$$

Corollary 9.1. *If a Kenmotsu manifold $\mathcal{M}$ is ϕ–$\mathcal{W}_9$-flat, then $\mathcal{M}$ is of negative constant scalar curvature.*

10. Pseudo–$\mathcal{W}_9$-flat hyperbolic Kenmotsu manifold

We explore Pseudo-$\mathcal{W}_9$-flat Kenmotsu manifold in this section,

Definition 10.1. *A hyperbolic Kenmotsu manifold is termed pseudo-$\mathcal{W}_9$-flat if it fulfills*

$$\mathrm{g}(\mathcal{W}_9(\phi\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3, \phi\mathcal{K}_4) = 0, \quad (68)$$

for any vector fields $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4 \in \chi(M)$.

Applying (17) and (68), we get

$$\mathrm{g}(\mathcal{R}(\phi\mathcal{K}_1, \mathcal{K}_2)\mathcal{K}_3, \phi\mathcal{K}_4) = -\frac{1}{n-1}\{\mathcal{S}(\phi\mathcal{K}_1, \mathcal{K}_2)\mathrm{g}(\mathcal{K}_3, \phi\mathcal{K}_4)$$

$-\mathcal{S}(\phi\mathcal{K}_1,\phi\mathcal{K}_4)\mathrm{g}(\mathcal{K}_2,\mathcal{K}_3)\}.$ (69)

Let $\{e_1,e_2,\ldots..e_{n-1},\xi\}$ represent a collection of orthonormal basis vector fields in $\mathcal{M}$. Utilizing that $\{\phi e_1,\phi e_2,\ldots..\phi e_{n-1},\xi\}$ is also local orthonormal basis in $\mathcal{M}$. If we consider $\mathcal{K}_1=\mathcal{K}_4=e_i$ and taking sum over i, where 1≤i≤n-1, then we obtain

$$\sum_{i=1}^{n-1}\epsilon_i\mathrm{g}(\mathcal{R}(\phi e_i,\mathcal{K}_2)\mathcal{K}_3,\phi e_i)=-\frac{1}{n-1}\sum_{i=1}^{n-1}\epsilon_i\mathcal{S}(\phi e_i,\mathcal{K}_2)\mathrm{g}(\mathcal{K}_3,\phi e_i)$$

$+\frac{1}{n-1}\sum_{i=1}^{n-1}\epsilon_i\mathcal{S}(\phi e_i,\phi e_i)\mathrm{g}(\mathcal{K}_2,\mathcal{K}_3).$ (70)

It can be easily confirmed that

$\sum_{i=1}^{n-1}\epsilon_i\mathrm{g}(\mathcal{R}(\phi e_i,\mathcal{K}_2)\mathcal{K}_3,\phi e_i)=\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3)+\mathrm{g}(\mathcal{K}_2,\mathcal{K}_3).$ (71)

and

$\sum_{i=1}^{n-1}\epsilon_i\mathcal{S}(\phi e_i,\mathcal{K}_2)\mathrm{g}(\mathcal{K}_3,\phi e_i)=\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3).$ (72)

Using (64), (71) and (72) in (70), we get

$\mathcal{S}(\mathcal{K}_2,\mathcal{K}_3)=\frac{\mathrm{r}}{n}\mathrm{g}(\mathcal{K}_2,\mathcal{K}_3).$ (73)

Thus we can say the following:

Theorem 10.1. *Let $\mathcal{M}$ be an n(where $n=2m+1$)-dimensional Kenmotsu manifold. If $\mathcal{M}$ satisfies the Pseudo–$\mathcal{W}_9$-flat, then it is an Einstein manifold.*

11. Example

Example 11.1. *Let us evaluate the* $3-$*dimensional manifold*

$\mathcal{M}$={(k_1,k_2,k_3) ∈R^3: $z\neq 0$}, with (k_1,k_2,k_3), representing the standard coordinates in $\mathbb{R}^3$.The vector fields

$$e_1=e^{k_3}\frac{\partial}{\partial k_1},\quad e_2=e^{k_3}\frac{\partial}{\partial k_2},\quad e_3=-\frac{\partial}{\partial k_3}$$

Let g represent the Riemannian metric specified by

$g(e_i,e_i)=1,\quad where\quad 1\leq i\leq 3$

$g(e_i,e_j)=0,\quad where\quad i\neq j\quad 1\leq i\leq 3$

Let ϕ represent the (1,1)-tensor field expressed by

$\phi(e_1)=-e_2,\quad \phi(e_2)=e_1,\quad \phi(e_3)=0.$

Let η represent the 1-form expressed by $\eta(\mathcal{K}_1)=\mathrm{g}(\mathcal{K}_1,e_3)$ for any $\mathcal{K}_1\in\chi\,(M)$.

for any vector field $\mathcal{K}_1$=$\lambda_1\frac{\partial}{\partial k_1}$+$\lambda_2\frac{\partial}{\partial k_2}$+$\lambda_3\frac{\partial}{\partial k_3}\in\chi(\mathbb{T}\mathbb{R}^3)$, then we have

$g(\mathcal{K}_1,\mathcal{K}_1) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad g(\phi\mathcal{K}_1,\phi\mathcal{K}_1) = \lambda_1^2 + \lambda_2^2$

$\eta(e_3) = 1, \phi^2(\mathcal{K}_1) = -\mathcal{K}_1 + \eta(\mathcal{K}_1)e_3, g(\phi\mathcal{K}_1,\phi\mathcal{K}_1) = g(\mathcal{K}_1,\mathcal{K}_1) - \eta(\mathcal{K}_1)\eta(\mathcal{K}_1),$

where $\mathcal{K}_1 \in \chi\ (M)$.

Then, for $e_3 = \xi$, the structure (ϕ,ξ,η,g) introduces an almost contact metric structure on $\mathcal{M}$. Let ∇ represent the Levi-Civita connection related to the metric tensor g. Thus, we obtain

$[e_1,e_2] = 0, \quad [e_2,e_3] = e_2, \quad [e_3,e_1] = -e_1,$

The Levi-Civita connection ∇ associated with the metric g is expressed by Koszul's formula

$2g(\nabla_{\mathcal{K}_1}\mathcal{K}_2,\mathcal{K}_3) = \mathcal{K}_1 g(\mathcal{K}_2,\mathcal{K}_3) + \mathcal{K}_2 g(\mathcal{K}_3,\mathcal{K}_1) - \mathcal{K}_3 g(\mathcal{K}_1,\mathcal{K}_2)$

$+g([\mathcal{K}_1,\mathcal{K}_2],\mathcal{K}_3) - g([\mathcal{K}_2,\mathcal{K}_3],\mathcal{K}_1) + g([\mathcal{K}_3,\mathcal{K}_1],\mathcal{K}_2).$

By using the above formula, we obtain

$\nabla_{e_1}e_1 = -e_1, \quad ,\nabla_{e_2}e_1 = 0, \quad \nabla_{e_3}e_1 = 0,$

$\nabla_{e_1}e_2 = 0, \quad \nabla_{e_2}e_2 = -e_3, \quad \nabla_{e_3}e_2 = 0,$

$\nabla_{e_1}e_3 = e_1, \quad \nabla_{e_2}e_3 = e_2 \quad \nabla_{e_3}e_3 = 0,$

From the above properties, the manifold satisfies $\nabla_{\mathcal{K}_1}\xi = \mathcal{K}_1 - \eta(\mathcal{K}_1)\xi$, for $e_3 = \xi$. Therefore, the manifold qualifies as a Kenmotsu manifold. Using the formula

$\mathcal{R}(\mathcal{K}_1,\mathcal{K}_2)\mathcal{K}_3 = \nabla_{\mathcal{K}_1}\nabla_{\mathcal{K}_2}\mathcal{K}_3 - \nabla_{\mathcal{K}_2}\nabla_{\mathcal{K}_1}\mathcal{K}_3 - \nabla_{[\mathcal{K}_1,\mathcal{K}_2]}\mathcal{K}_3$

we calculate the following expressions:

$\mathcal{R}(e_1,e_2)e_3 = 0, \quad \mathcal{R}(e_2,e_3)e_3 = -e_2, \quad \mathcal{R}(e_1,e_3)e_3 = -e_1$

$\mathcal{R}(e_1,e_2)e_2 = -e_1, \quad \mathcal{R}(e_2,e_3)e_2 = -e_3, \quad \mathcal{R}(e_1,e_3)e_2 = 0$

$\mathcal{R}(e_1,e_2)e_1 = 0, \quad \mathcal{R}(e_2,e_3)e_1 = 0, \quad \mathcal{R}(e_1,e_3)e_1 = e_1$

According to the expressions for the curvature tensor $\mathcal{R}$, presented above, The Ricci tensors are calculated as follows:

$\mathcal{S}(e_1,e_1) = \mathcal{S}(e_2,e_2) = \mathcal{S}(e_3,e_3) = -2.$

Consequently

$$\sum_{i=1}^{3} \epsilon_i \mathcal{S}(e_i,e_i) = \mathfrak{r} = -6.$$

Hence, from the above discussion, it is clear that theorems (4.1, 6.1, 7.1, 8.1, 9.1,10.1,11.1) are effectively satisfied by considering of an example. Therefore, my example completely corresponds with our findings.

12. Conclusions

In this paper, we put forward the idea that ξ-$\mathcal{W}_9$-flat Kenmotsu manifold is an Einstein manifold and under the condition $\mathcal{W}_9 = 0$, is isomorphic to the hyperbolic space $\mathcal{H}^n(-1)$. Next, we examined $W_9.\mathcal{R} = 0$ in the Kenmotsu manifold and concluded that it is a special type of η-Einstein manifold. Again, we explored the Kenmotsu manifold with the ϕ–$\mathcal{W}_9$–semi-symmetric condition and verified that it is an Einstein manifold. Furthermore, we explored the Kenmotsu manifold under the $\mathcal{W}_9.\mathcal{W}_9 = 0$ condition and established that it is an Einstein manifold. Additionally, we investigated the Kenmotsu manifold under the condition $\mathcal{W}_9.\mathcal{Q} = 0$ and verified that it is an Einstein manifold and $\phi-\mathcal{W}_9$-flat Kenmotsu Manifold is an η-Einstein manifold. In the final section, we examined the Pseudo–$\mathcal{W}_9$-flat Kenmotsu manifold and determined that it is an Einstein manifold.

Acknowledgments: The authors appreciate the referees for their valuable comments and constructive suggestions which have significantly enhanced the quality of this paper.

References

[1] Atceken, M. (2024). A note on P-Sasakian manifolds satisfying certain conditions, Proyecciones (Antofagasta, Online), **43**(4), 899-910.

[2] Chaubey, S. and Yadav, S.K. (2018). Study of Kenmotsu manifolds with semi-symmetric. metric connection, Univ. J. Math. Appl., **1**(2), 89-97

[3] De, U.C. and Pathak, G. (2004). On 3-dimensional Kenmotsu manifolds, Indian J. pure Appl. Math., **35**, 159-165.

[4] De, U.C. (2008). On ϕ-symmetric Kenmotsu manifolds, International Electronic Journal of Geometry, **1**(1), 33-38.

[5] Haseeb, A. (2017). Some results on projective on projective curvature tensor in an ϵ-Kenmotsu manifold, Plestine Journal of Math., **6**, 196-203.

[6] Haseeb, A. and Prasad, R. (2021). Certain results on Lorentzian para-Kenmotsu manifolds, Bol. Soc. Paran. Mat, **39**(3), 201-220.

[7] Haseeb, A. and Prasad, R. (2017). Certain curvature conditions in Kenmotsu manifolds with respect to the semi-symmetric metric connection, Communications of the Korean Mathematical Society, **32**(4), 1033-1045.

[8] Kenmotsu, K. (1972). A class of almost contact Riemannian manifolds, Tohoku Math. J., **24**, 93-103.

[9] Özgür, C. and De, U.C. (2006). On the quasi-conformal curvature tensor of a Kenmotsu manifold, Mathematica Pannonica, **17**(2), 221-228.

[10] Ozturk, H. and Aktan, N. and Murathan, C. (2010). On α-Kenmotsu manifolds satisfying certain conditions, APPS. Applied Sciences, **12**, 115-126.

[11] Prakash, D.G., Chavan, V. and Mırjı, K. (2016). On the W5-curvature tensor of generalized Sasakian-space-form, Konuralp J. Math., **4**(1), 45-53.

[12] Sasaki, S. (1965, 1966, 1967). Almost Contact Manifolds I, II, III, Mathematical Institute Tohoku University,

[13] Sardar, A. (2020). Some results on (ϵ)-Kenmotsu manifolds, Facta Universitatis Series: Mathematics and Informatics, **35** (1), 273-282.

[14] Singh, A., Kumar, L. and Yadav, S.K. (2023). W8-curvature tensor on (ϵ)-Lorentzian para-Sasakian manifold, Journal of Rajasthan Academy of Physical Sciences, **22**, 279-288.

[15] Tripathi, M.M. and Gupta, P. (2011). T-curvature tensor on a semi-Riemannian manifold, J. Adv. Math. Studies, **4**(1), 117-129.

[16] Tripathi, M.M. and Gupta, P. (2012). On $(N(k), \xi)$-semi-Riemannian manifolds: Semisymmetries, arXiv preprint arXiv, **1202**(6138).

[17] Uygun, P., Dirik, S. and Atceken, M. (2022). Some curvature characterizations on Kenmotsu metric spaces, Gulf Journal of Mathematics, **13**(2), 78-86.

[18] Uygun, P. and Atceken, M. (2023). On Kenmotsu metric spaces satisfying some conditions on the W_7-curvature tensor, Malaya J. Mat. **11**(04), 447-456.

[19] Wang, Y. and Wang, W. (2017). An Einstein-like Metric on Almost Kenmotsu Manifolds, Filomat, **31**(15), 4695-4702.

[20] Yíldíz, A. and De, U.C. (2010). On a type of Kenmotsu manifolds, Differential Geometry-Dynamical Systems, **12**, 289-298.