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INTEGRAL TYPE CONTRACTION IN CONE METRIC SPACES

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Abstract: In this paper new concept of integral type contraction in cone metric space is introduced and a common fixed point theorem satisfying generalized contractive condition of integral type without requirement of continuity in cone metric spaces have been proved.

Key Words: Cone metric spaces, fixed points, compatible mapping, weak compatible mapping, integral type contraction

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1. Introduction

There exists a lot of generalizations of metric spaces and among which one is cone metric space. The notion of cones is introduced by Huang and Zhang [4] by substituting an ordered Banach space for the real numbers and proved some fixed point theorems in this space. Many authors studied this subject and many fixed point theorems have been proved. Branciari [2] initiated a study of contractive condition of integral type, giving an integral version of the Banach's contraction principle and that could be extended to more general contractive conditions. Khojasteh et al. [8] defined integral type contraction with respect to cone. Jungck [6, 7] discussed commuting mappings periodic and fixed points. In this paper, the concept of integral type contraction in cone metric space is introduced and some common fixed point theorems have been proved with different contractive type conditions.

Section 1 consists of the brief introduction of the content. In section 2 we have recorded preliminary definitions with examples which will be useful in the sequal. In section 3 we prove a common fixed point theorem satisfying generalized contractive condition of integral type without requirement of continuity in cone metric spaces.

2. Basic Definitions

We give the definition of cone metric space and some of their properties. The following notions will be used in order to prove the main result.

Definition 2.1: Let E be a real Banach space. A subset P of E is called a cone if and only if

(i) P is closed, nonempty and $P \neq \{0\}$

(ii) $a, b \in R$, $a, b \geq 0$, $x, y \in P$ imply $ax + by \in P$

(iii) $P \cap (-P) = \{0\}$ i.e $x \in P$ and $-x \in P$ imply that $x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2: A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$

The least positive number satisfying the above inequality is called the normal constant of P, while $x \leq y$ stands for $y - x \in int\, P$ (interior of P).

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that

$x_1 \leq x_2 \leq \cdots \leq x_n \cdot \leq \cdots \leq y$ for some $y \in E$,

then there is $x \in E$ such that $\|x_n - x\| \to 0$ (as $n \to \infty$). Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

In the following, we always suppose that E is a real Banach space, P is a cone in E with $int\, P \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Definition 2.3: Let X be a nonempty set. Suppose that a mapping $d: X \times X \to E$ satisfies:

(a) $\int_0^{d(x,y)} \varphi(t)dt \geq 0 \; \forall\, x, y \in X$

(b) $\int_0^{d(x,y)} \varphi(t)dt = 0$ if and only if $x = y$

(c) $\int_0^{d(x,y)} \varphi(t)dt = \int_0^{d(y,x)} \varphi(t)dt$

(d) $\int_0^{d(x,y)} \varphi(t)dt \leq \int_0^{d(x,z)} \varphi(t)dt + \int_0^{d(z,y)} \varphi(t)dt$

Then d is called a cone metric on X and (X, d) is called a cone metric space. It is obvious that cone metric spaces generalize metric spaces.

Definition 2.4: Two self mappings S and T of a cone metric space (X, d) are compatible if and only if $\lim_{n\to\infty} \int_0^{d(ASx_n, SAx_n,)} \varphi(t)dt = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Jungck-Rhoades [6] obtained the concept of weakly compatible as follows:

Definition 2.5: Let S and T be self maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Definition 2.6: Two self maps A and B are said to be weakly compatible if they commute at their coincidence point, that if $Ax = Bx$ for some x in X then $ABx = BAx$. It is easy to see that compatible mapping commute at their coincidence points.

Lemma 2.7 ([1]): Let A and B be weakly compatible self maps of a set X. If A and B have a unique point of coincidence $w = Ax = Bx$, then w is the unique common fixed point of A and B.

Definition 2.8: The function $\varphi : P \rightarrow E$ is called subadditive cone integrable function if and only if for all $a, b \in P$

$$\int_0^{a+b} \varphi(t)dt \leq \int_0^a \varphi(t)dt + \int_0^b \varphi(t)dt$$

Example 2.9: Let $E = X = R$, $d(x,y) = |x-y|$, $P = [0,\infty)$ and $\varphi(t) = \frac{1}{1+t}$ for all $t > 0$. Then for all $a, b \in P$,

$$\int_0^{a+b} \frac{dt}{t+1} = ln(a+b+1), \int_0^a \frac{dt}{t+1} = ln(a+1), \int_0^b \frac{dt}{t+1} = ln(b+1),$$

Since $ab \geq 0$, then $a+b+1 \leq a+b+1+ab = (a+1)(b+1)$.

Therefore, $ln(a+b+1) \leq ln(a+1)(b+1) = ln(a+1) + ln(b+1)$ This shows that φ is an example of sub additive cone integrable function

3. Common Fixed Point Theorems of Integral Type Contraction in Cone Metric Space

In this section, we prove some common fixed point theorems of integral type contraction for rational and linear contractive conditions.

Let R^+ be the set of non-negative real numbers and let $F: R^+ \rightarrow R^+$ be a mapping such that $F(0) = 0$ and F is continuous at 0.

The following Lemma is the key in proving our result. Its proof is similar to that of Jungck [5**Lemma 3.1:** Let $\{y_n\}$ be a sequence in a complete cone metric space (X, d). If there exists a $k \in (0,1)$ such that

$$\left\| \int_0^{d(y_n, y_{n+1})} \varphi(t)dt \right\| \leq k \left\| \int_0^{d(y_{n-1}, y_n)} \varphi(t)dt \right\|$$

for all n, then $\{y_n\}$ converges to a point in X.

Han and Xu [3] obtained some coincident point result for four mappings which do not satisfy continuity and commutativity on non-normal cone metric spaces. We improve this result for three pairs of weakly compatible mappings satisfying general contractive condition of integral type in normal cone metric spaces.

Theorem 3.2: Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. Suppose that $\varphi: P \rightarrow P$ is a non-vanishing map and a subadditive cone integrable on each $[a, b] \subset P$ such that for each $\in > 0$, $\int_0^{\in} \varphi(t)dt > 0$.

Let A, B, S, T, I and J be self mappings of (X, d) satisfying

(i) $AB(X) \subset J(X), ST(X) \subset I(X)$ (1)

(ii) $\int_0^{d(ABx,STy)} \varphi(t)dt \leq \int_0^{M_1(x,y)} \varphi(t)dt + \int_0^{M_2(x,y)} \varphi(t)dt$

where, $M_1(x,y) = \beta_1 d(Ix,Jy) + \beta_2 d(Ix,ABx) + \beta_3 d(Jy,STy)$

$+ \beta_4 d(Ix,STy) + \beta_5 d(Jy,ABx)$

and $M_2(x,y) = min\{d(Ix,Jy), d(Ix,ABx), d(Jy,STy), d(Ix,STy),$

$d(Jy,ABx)\}$

for all $x, y \in X$ where $\beta_i \geq 0$ $(i = 1,2,3,4,5)$ satisfying

$\beta_1 + \beta_2 + \beta_3 + 2max\{\beta_4, \beta_5\} < 1$

or, $\beta_1 + 2max\{\beta_2, \beta_3\} + \beta_4 + \beta_5 < 1$ (2)

(iii) One of $AB(X), I(X), ST(X), J(X)$ is a complete subspace of X then the mappings AB, ST, I and J have a unique point of coincidence in X. (3)

Moreover, if the pairs $\{AB, I\}$ and $\{ST, J\}$ are weakly compatible, then the mappings AB, ST, I and J have unique common fixed point in X.

Furthermore, if the pairs $(A,B), (A,I), (B,I), (S,T), (S,J)$ and (T,J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary point in X. Since $AB(X) \subseteq J(X)$, we can choose a point x_1 in X such that $ABx_0 = Jx_1$. Also, since $ST(X) \subseteq I(X)$, we can find a point x_2 with $STx_1 = Ix_2$. Using this process continuously, one can construct a sequence $\{x_n\}$ such that

$Jx_{2n+1} = ABx_{2n}, \quad Ix_{2n+2} = STx_{2n+1}, \quad n = 0,1,2, ...$

Denote $z_{2n} = ABx_{2n} = Jx_{2n+1}$

and $z_{2n+1} = STx_{2n+1} = Ix_{2n+2}$ for $n = 0,1,2, ...$

Now, we shall show that $\{z_n\}$ is a Cauchy sequence.

Using (2), we get

$$\int_0^{d(z_{2n}, z_{2n+1})} \varphi(t)dt = \int_0^{d(ABx_{2n}, STx_{2n+1})} \varphi(t)dt$$

$$\leq \int_0^{M_1(x_{2n}, x_{2n+1})} \varphi(t)dt + \int_0^{M_2(x_{2n}, x_{2n+1})} \varphi(t)dt$$

where $M_1(x_{2n}, x_{2n+1}) = \beta_1 d(Ix_{2n}, Jx_{2n+1}) + \beta_2 d(Ix_{2n}, ABx_{2n})$

$+ \beta_3 d(Jx_{2n+1}, STx_{2n+1}) + \beta_4 d(Ix_{2n}, STx_{2n+1})$

$+ \beta_5 d(Jx_{2n+1}, ABx_{2n})$

$= \beta_1 d(z_{2n-1}, z_{2n}) + \beta_2 d(z_{2n-1}, z_{2n}) + \beta_3 d(z_{2n}, z_{2n-1})$

$+ \beta_4 d(z_{2n-1}, z_{2n+1}) + \beta_5 d(z_{2n}, z_{2n})$

$= \beta_1 d(z_{2n-1}, z_{2n}) + \beta_2 d(z_{2n-1}, z_{2n}) + \beta_3 d(z_{2n}, z_{2n+1})$

$+ \beta_4 [d(z_{2n-1}, z_{2n}) + d(z_{2n}, z_{2n+1})]$

and $M_2(x_{2n}, x_{2n+1}) = min\{d(Ix_{2n}, Jx_{2n+1}), d(Ix_{2n}, ABx_{2n}), d(Jx_{2n+1}, STx_{2n+1}),$

$d(Ix_{2n}, STx_{2n+1}), d(Jx_{2n+1}, ABx_{2n})\}$

$= min\{d(z_{2n-1}, z_{2n}), d(z_{2n-1}, z_{2n}), d(z_{2n}, z_{2n+1}),$

$d(z_{2n-1}, z_{2n+1}), d(z_{2n}, z_{2n})\}$

$= min\{d(z_{2n-1}, z_{2n}), d(z_{2n-1}, z_{2n}), d(z_{2n}, z_{2n+1}),$

$d(z_{2n-1}, z_{2n+1}), 0\}$

$= 0$

Therefore,

$\int_0^{d(ABx_{2n}, STx_{2n+1})} \varphi(t)dt = \int_0^{d(z_{2n}, z_{2n+1})} \varphi(t)dt$

$\leq \int_0^{\{\beta_1 d(z_{2n-1}, z_{2n}) + \beta_2 d(z_{2n-1}, z_{2n}) + \beta_3 d(z_{2n}, z_{2n+1}) + \beta_4 [d(z_{2n-1}, z_{2n}) + d(z_{2n}, z_{2n+1})]\}} \varphi(t)dt$

or, $\int_0^{d(z_{2n+1}, z_{2n})} \varphi(t)dt \leq \left(\frac{\beta_1 + \beta_2 + \beta_4}{1 - \beta_3 - \beta_4}\right) \int_0^{d(z_{2n-1}, z_{2n})} \varphi(t)dt$ (4)

[by subadditivity of cone]

Similarly, $\int_0^{d(z_{2n+2}, z_{2n+1})} \varphi(t)dt = \int_0^{d(ABx_{2n+2}, STx_{2n+1})} \varphi(t)dt$

$\leq \int_0^{M_1(x_{2n+2}, x_{2n+1})} \varphi(t)dt + \int_0^{M_2(x_{2n+2}, x_{2n+1})} \varphi(t)dt$

where $M_1(x_{2n+2}, x_{2n+1}) = \beta_1 d(Ix_{2n+2}, Jx_{2n+1}) + \beta_2 d(Ix_{2n+2}, ABx_{2n+2})$

$+ \beta_3 d(Jx_{2n+1}, STx_{2n+1}) + \beta_4 d(Ix_{2n+2}, STx_{2n+1})$

$+ \beta_5 d(Jx_{2n+1}, ABx_{2n+2})$

$= \beta_1 d(z_{2n+1}, z_{2n}) + \beta_2 d(z_{2n+1}, z_{2n+2})$

$+ \beta_3 d(z_{2n}, z_{2n+1}) + \beta_4 d(z_{2n+1}, z_{2n+1})$

$+ \beta_5 d(z_{2n}, z_{2n+2})$

$= \beta_1 d(z_{2n+1}, z_{2n}) + \beta_2 d(z_{2n+1}, z_{2n+2})$

$+ \beta_3 d(z_{2n}, z_{2n+1}) + \beta_5 [d(z_{2n}, z_{2n+1})$

$+ d(z_{2n+1}, z_{2n+2})]$

and $M_2(x_{2n+2}, x_{2n+1}) = min\{d(Ix_{2n+2}, Jx_{2n+1}), d(Ix_{2n+2}, ABx_{2n+2}),$

$d(Jx_{2n+1}, STx_{2n+1}), d(Ix_{2n+2}, STx_{2n+1}),$

$d(Jx_{2n+1}, ABx_{2n+2})\}$
$= min\,\{d(z_{2n+1}, z_{2n}), d(z_{2n+1}, z_{2n+2}), d(z_{2n}, z_{2n+1}),$
$d(z_{2n+1}, z_{2n+1}), d(z_{2n}, z_{2n+2})\}$
$= min\,\{d(z_{2n+1}, z_{2n}), d(z_{2n+1}, z_{2n+2}), d(z_{2n}, z_{2n+1}), 0,$
$d(z_{2n}, z_{2n+2})\}$
$= 0$

Therefore,

$$\int_0^{d(z_{2n+2},z_{2n+1})} \varphi(t)dt$$
$$\leq \int_0^{\{\beta_1 d(z_{2n+1},z_{2n})+\beta_2 d(z_{2n+1},z_{2n+2})+\beta_3 d(z_{2n},z_{2n+1})+\beta_5[d(z_{2n},z_{2n+1})+d(z_{2n+1},z_{2n+2})]\}} \varphi(t)dt$$
$$\leq \beta_1 \int_0^{d(z_{2n+1},z_{2n})} \varphi(t)dt + \beta_2 \int_0^{d(z_{2n+1},z_{2n+2})} \varphi(t)dt + \beta_3 \int_0^{d(z_{2n},z_{2n+1})} \varphi(t)dt$$
$$+ \beta_5 \int_0^{d(z_{2n},z_{2n+1})} \varphi(t)dt + \beta_5 \int_0^{d(z_{2n+1},z_{2n+2})} \varphi(t)dt$$

[by subadditivity of $'\varphi'$]

or, $\int_0^{d(z_{2n+2},z_{2n+1})} \varphi(t)dt \leq \left(\frac{\beta_1+\beta_3+\beta_5}{1-\beta_2-\beta_5}\right)\int_0^{d(z_{2n+1},z_{2n})} \varphi(t)dt$ (5)

Using (4) in (5), we get

$$\int_0^{d(z_{2n+2},z_{2n+1})} \varphi(t)dt \leq \left(\frac{\beta_1+\beta_3+\beta_5}{1-\beta_2-\beta_5}\right)\left(\frac{\beta_1+\beta_2+\beta_4}{1-\beta_3-\beta_4}\right)\int_0^{d(z_{2n-1},z_{2n})} \varphi(t)dt$$
$$\leq \left(\frac{\beta_1+\beta_3+\beta_5}{1-\beta_2-\beta_5}\right)\left(\frac{\beta_1+\beta_2+\beta_4}{1-\beta_3-\beta_4}\right)\int_0^{d(z_{2n-1},z_{2n})} \varphi(t)dt$$
$$\leq \left(\frac{\beta_1+\beta_3+\beta_5}{1-\beta_2-\beta_5}\right)\left(\frac{\beta_1+\beta_2+\beta_4}{1-\beta_3-\beta_4}\right)\left(\frac{\beta_1+\beta_3+\beta_5}{1-\beta_2-\beta_5}\right)\int_0^{d(z_{2n-2},z_{2n-1})} \varphi(t)dt$$
$$\leq \cdots \leq \left(\frac{\beta_1+\beta_3+\beta_5}{1-\beta_2-\beta_5}\right)\left(\frac{\beta_1+\beta_2+\beta_4}{1-\beta_3-\beta_4}.\frac{\beta_1+\beta_3+\beta_5}{1-\beta_2-\beta_5}\right)^n \int_0^{d(z_1,z_0)} \varphi(t)dt$$

and $\int_0^{d(z_{2n+3},z_{2n+2})} \varphi(t)dt \leq \left(\frac{\beta_1+\beta_2+\beta_4}{1-\beta_3-\beta_4}\right)\int_0^{d(z_{2n+2},z_{2n+1})} \varphi(t)dt$

$$\leq \cdots \leq \left(\frac{\beta_1+\beta_2+\beta_4}{1-\beta_3-\beta_4}.\frac{\beta_1+\beta_3+\beta_5}{1-\beta_2-\beta_5}\right)^{n+1} \int_0^{d(z_1,z_0)} \varphi(t)dt$$

Let $A = \frac{\beta_1+\beta_2+\beta_4}{1-\beta_3-\beta_4}$, $B = \frac{\beta_1+\beta_3+\beta_5}{1-\beta_2-\beta_5}$

If $\beta_1 + \beta_2 + \beta_3 + 2max\{\beta_4, \beta_5\} < 1$, then

$$AB = \left(\frac{\beta_1+\beta_2+\beta_4}{1-\beta_3-\beta_4}\right)\left(\frac{\beta_1+\beta_3+\beta_5}{1-\beta_2-\beta_5}\right) < 1.1 = 1$$

If $\beta_1 + 2max\{\beta_2, \beta_3\}+\beta_4 + \beta_5 < 1$, then

$$AB = \left(\frac{\beta_1+\beta_2+\beta_4}{1-\beta_3-\beta_4}\right)\left(\frac{\beta_1+\beta_3+\beta_5}{1-\beta_2-\beta_5}\right)$$
$$= \left(\frac{\beta_1+\beta_2+\beta_4}{1-\beta_2-\beta_5}\right)\left(\frac{\beta_1+\beta_3+\beta_5}{1-\beta_3-\beta_4}\right) < 1.1 = 1$$

Now, for any $n > m$, we have

$$\int_0^{d(z_{2n+1},z_{2m+1})} \varphi(t)dt \leq \int_0^{\substack{[d(z_{2n+1},z_{2n})+d(z_{2n},z_{2n-1})+\cdots+d(z_{2m+3},z_{2m+2}) \\ +d(z_{2m+2},z_{2m+1})]}} \varphi(t)dt$$
$$\leq \int_0^{d(z_{2n+1},z_{2m})} \varphi(t)dt + \int_0^{d(z_{2n},z_{2n-1})} \varphi(t)dt + \cdots$$
$$+ \int_0^{d(z_{2m+3},z_{2m+2})} \varphi(t)dt + \int_0^{d(z_{2m+2},z_{2m+1})} \varphi(t)dt$$
$$\leq [(AB)^{m+1} + (AB)^{m+2} + \cdots + (AB)^n] \int_0^{d(z_0,z_1)} \varphi(t)dt$$
$$+B[(AB)^m + (AB)^{m+1} + \cdots + (AB)^{n-1}] \int_0^{d(z_0,z_1)} \varphi(t)dt$$
$$\leq \left(\sum_{i=m+1}^{n}(AB)^i + B\sum_{i=m}^{n-1}(AB)^i\right) \int_0^{d(z_0,z_1)} \varphi(t)dt$$
$$\leq \left(\frac{(AB)^{m+1}}{1-AB} + B\frac{(AB)^m}{1-AB}\right) \int_0^{d(z_0,z_1)} \varphi(t)dt$$
$$= (A+1)B\frac{(AB)^m}{1-AB} \int_0^{d(z_0,z_1)} \varphi(t)dt$$

In an analogous way, we get

$$\int_0^{d(z_{2n+1},z_{2m+1})} \varphi(t)dt \leq (A+1)B\frac{(AB)^m}{1-AB} \int_0^{d(z_1,z_0)} \varphi(t)dt$$
$$\int_0^{d(z_{2n},z_{2m+1})} \varphi(t)dt \leq (A+1)B\frac{(AB)^m}{1-AB} \int_0^{d(z_1,z_0)} \varphi(t)dt$$
$$\int_0^{d(z_{2n},z_{2m})} \varphi(t)dt \leq (B+1)\frac{(AB)^{m+1}}{1-AB} \int_0^{d(z_1,z_0)} \varphi(t)dt$$

and $$\int_0^{d(z_{2n+1},z_{2m})} \varphi(t)dt \leq (B+1)\frac{(AB)^{m+1}}{1-AB} \int_0^{d(z_1,z_0)} \varphi(t)dt$$

Thus, for $n > m > 0$, we have

$$\int_0^{d(z_n,z_m)} \varphi(t)dt \leq max\left\{(B+1)\frac{(AB)^{m+1}}{1-AB}, (A+1)B\frac{(AB)^m}{1-AB}\right\} \int_0^{d(z_1,z_0)} \varphi(t)dt$$
$$\leq \lambda_m \int_0^{d(z_1,z_0)} \varphi(t)dt, \text{ where } \lambda_m \to 0 \text{ as } m \to \infty.$$

Using normality of cone, we get

$\left\|\int_0^{d(z_n,z_m)} \varphi(t)dt\right\| \leq \lambda_m.K \left\|\int_0^{d(z_1,z_0)} \varphi(t)dt\right\| \to 0$ as $m \to \infty$, where K is a normal constant.

[$\because 0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$ for $x, y \in E$]

which implies that

$\int_0^{d(z_n,z_m)} \varphi(t)dt \to 0$ as $m, n \to \infty$.

and which by def. of $'\varphi'$, implies that

$d(z_n, z_m) \to 0$ as $m, n \to \infty$.

i.e $\lim\limits_{m,n\to\infty} d(z_n, z_m) = 0$

Hence, $\{z_n\}$ is a Cauchy sequence in cone metric space (X, d).

Suppose that $J(X)$ is complete, then there exists a point $z \in J(X)$ such that

$z_{2n} = ABx_{2n} = Jx_{2n+1} \to z$ as $n \to \infty$.

Again, since $AB(X) \subseteq J(X), \ni a$ point $z' \in X$ such that $Jz' = z$ (If $AB(X)$ is complete, then there exists $z \in AB(X) \subseteq J(X)$, then the conclusion remains the same)

Now, we show that $STz' = z$.

By using (2), we have

$$\int_0^{d(STz',z)} \varphi(t)dt \le \int_0^{[d(ABx_{2n},STz')+d(ABx_{2n},z)]} \varphi(t)dt$$

$$\le \int_0^{d(ABx_{2n},STz')} \varphi(t)dt + \int_0^{d(ABx_{2n},z)} \varphi(t)dt$$

[By subadditivity of $'\varphi'$]

$$\le \int_0^{M_1(x_{2n},z')} \varphi(t)dt + \int_0^{M_2(x_{2n},z')} \varphi(t)dt + \int_0^{d(ABx_{2n},z)} \varphi(t)dt$$

where , $M_1(x_{2n}, z') = \beta_1 d(Ix_{2n}, Jz') + \beta_2 d(Ix_{2n}, ABx_{2n}) + \beta_3 d(Jz', STz')$

$+ \beta_4 d(Ix_{2n}, STz') + \beta_5 d(Jz', ABx_{2n})$

$= \beta_1 d(z_{2n-1}, z) + \beta_2 d(z_{2n-1}, z_{2n}) + \beta_3 d(z, STz')$

$+ \beta_4 d(z_{2n-1}, STz') + \beta_5 d(z, z_{2n})$

and $M_2(x_{2n}, z') = min\,\{d(Ix_{2n}, Jz'), d(Ix_{2n}, ABx_{2n}), d(Jz', STz'),$

$d(Ix_{2n}, STz'), d(Jz', ABx_{2n})\}$

$= min\,\{d(z_{2n-1}, z), d(z_{2n-1}, z_{2n}), d(z, STz'),$

$d(z_{2n-1}, STz'), d(z, z_n)\}$

$$\therefore \int_0^{d(z,STz')} \varphi(t)dt \le \int_0^{\{\beta_1[d(z_{2n-1},z_{2n})+d(z_{2n},z)]+\beta_2 d(z_{2n-1},z_{2n})+\beta_3 d(z,STz') +\beta_4[d(z_{2n-1},z_{2n})+d(z_{2n},z)+d(z,STz')]+\beta_5 d(z,z_{2n})\}} \varphi(t)dt$$

$$+ \int_0^{min\{d(z_{2n-1},z),d(z_{2n-1},z_{2n}),d(z,STz'), d(z_{2n-1},STz'),d(z,z_n)\}} \varphi(t)dt$$

$$+\int_0^{d(z,z_{2n})}\varphi(t)dt$$
$$\leq \beta_1\int_0^{d(z_{2n-1},z_{2n})}\varphi(t)dt+\beta_1\int_0^{d(z_{2n},z)}\varphi(t)dt$$
$$+\beta_2\int_0^{d(z_{2n-1},z_{2n})}\varphi(t)dt+\beta_3\int_0^{d(z,STz')}\varphi(t)dt$$
$$+\beta_4\int_0^{d(z_{2n-1},z_{2n})}\varphi(t)dt+\beta_4\int_0^{d(z_{2n},z)}\varphi(t)dt$$
$$+\beta_4\int_0^{d(z,STz')}\varphi(t)dt+(\beta_5+1)\int_0^{d(z,z_{2n})}\varphi(t)dt$$
$$+\int_0^{\substack{min\{d(z_{2n-1},z),d(z_{2n-1},z_{2n}),d(z,STz'),\\ d(z_{2n-1},STz'),d(z,z_n)\}}}\varphi(t)dt$$

[by subadditivity of $'\varphi'$]

or, $(1-\beta_3-\beta_4)\int_0^{d(z,STz')}\varphi(t)dt\leq(\beta_1+\beta_2+\beta_4)\int_0^{d(z_{2n-1},z_{2n})}\varphi(t)dt$
$$+(\beta_1+\beta_4+\beta_5+1)\int_0^{d(z_{2n},z)}\varphi(t)dt$$
$$+\int_0^{\substack{min\{d(z_{2n-1},z),d(z_{2n-1},z_{2n}),d(z,STz'),\\ d(z_{2n-1},STz'),d(z,z_n)\}}}\varphi(t)dt$$

So, using the condition of normality of cone, we have

$$(1-\beta_3-\beta_4)\left\|\int_0^{d(z,STz')}\varphi(t)dt\right\|\leq(\beta_1+\beta_2+\beta_4)\left\|\int_0^{d(z_{2n-1},z_{2n})}\varphi(t)dt\right\|$$
$$+(\beta_1+\beta_4+\beta_5+1)\left\|\int_0^{d(z_{2n},z)}\varphi(t)dt\right\|$$
$$+\int_0^{\substack{min\{\|d(z_{2n-1},z)\|,\|d(z_{2n-1},z_{2n})\|,\|d(z,STz')\|,\\ \|d(z_{2n-1},STz')\|,\|d(z,z_{2n})\|\}}}\varphi(t)dt$$
$$\to 0 \text{ as } n\to\infty.$$

We have $\left\|\int_0^{d(z,STz')}\varphi(t)dt\right\|\leq 0$, which is a contradiction.

Therefore, by def. of $'\varphi'$, we have

$$\int_0^{d(z,STz')}\varphi(t)dt=0$$

or, $\quad d(z,STz')=0$

or, $\quad z=STz'=Jz'$

Again, as $z=STz'\in ST(X)\subseteq I(X)$, there exist a point z'' in X such that $Iz''=z$.

By using (2), we have

$$\int_0^{d(ABz",z)} \varphi(t)dt = \int_0^{d(ABz",STz')} \varphi(t)dt$$

$$\leq \int_0^{M_1(z",z')} \varphi(t)dt + \int_0^{M_2(z",z')} \varphi(t)dt$$

where , $M_1(z",z') = \beta_1 d(Iz",Jz') + \beta_2 d(Iz",ABz") + \beta_3 d(Jz',STz')$

$+ \beta_4 d(Iz",STz') + \beta_5 d(Jz',ABz")$

$= \beta_1 d(z,z) + \beta_2 d(z,ABz") + \beta_3 d(z,z) + \beta_4 d(z,z)$

$+ \beta_5 d(z,ABz")$

$= (\beta_2 + \beta_5) d(z,ABz")$

and $M_2(z",z') = min\{d(Iz",Jz'), d(Iz",ABz"), d(Jz',STz'), d(Iz",STz'),$

$d(Jz',ABz")\}$

$= min\{d(z,z), d(z,ABz"), d(z,z), d(z,z), d(z,ABz")\}$

$= 0$

Therefore, $\int_0^{d(ABz",z)} \varphi(t)dt \leq (\beta_2 + \beta_5) \int_0^{d(z,ABz")} \varphi(t)dt$, which is a contradiction.

Hence, $\int_0^{d(z,ABz")} \varphi(t)dt = 0$

or, $d(z,ABz") = 0$ [by def. of $'\varphi'$]

or, $ABz" = Iz" = z$

Now, assume that if $I(X)$ is complete, then there exists a point $z \in I(X)$ such that $z_{2n+1} = Ix_{2n+2} = STx_{2n+1} \to z$ as $n \to \infty$. So, we can find a point $z" \in X$ such that $Iz" = z$ (If $S(X)$ is complete, then there exists $z \in ST(X) \subseteq I(X)$, then the conclusions remains the same)

Now, we show that $ABz" = z$.

By using (2), we have

$$\int_0^{d(ABz",z)} \varphi(t)dt \leq \int_0^{[d(ABz",STx_{2n+1})+d(STx_{2n+1},z)]} \varphi(t)dt$$

$$\leq \int_0^{d(ABz",STx_{2n+1})} \varphi(t)dt + \int_0^{d(STx_{2n+1},z)} \varphi(t)dt$$

$$\leq \int_0^{M_1(z",x_{2n+1})} \varphi(t)dt + \int_0^{M_2(z",x_{2n+1})} \varphi(t)dt + \int_0^{d(z_{2n+1},z)} \varphi(t)dt$$

where , $M_1(z",x_{2n+1}) = \beta_1 d(Iz",Jx_{2n+1}) + \beta_2 d(Iz",ABz")$

$+ \beta_3 d(Jx_{2n+1},STx_{2n+1}) + \beta_4 d(Iz",STx_{2n+1})$

$+ \beta_5 d(Jx_{2n+1},ABz")$

$= \beta_1 d(z,z_{2n}) + \beta_2 d(z,ABz") + \beta_3 d(z_{2n},z_{2n+1})$

$+ \beta_4 d(z, z_{2n+1}) + \beta_5 d(z_{2n}, ABz")$

$= \beta_1[d(z_{2n+1}, z) + d(z_{2n+1}, z_{2n})] + \beta_2 d(z, ABz")$

$+ \beta_3 d(z_{2n}, z_{2n+1}) + \beta_4 d(z, z_{2n+1}) + \beta_5[d(z_{2n}, z_{2n+1})$

$+d(z_{2n+1}, z_{2n}) + d(z, ABz")]$

and $M_2(z", x_{2n+1}) = min\{d(Iz", Jx_{2n+1}), d(Iz", ABz"), d(Jx_{2n+1}, STx_{2n+1}),$

$d(Iz", STx_{2n+1}), d(Jx_{2n+1}, ABz")\}$

$= min\{d(z, z_n), d(z, ABz"), d(z_{2n}, z_{2n+1}),$

$d(z, z_{2n+1}), d(z_{2n}, ABz")\}$

Therefore,

$$\int_0^{d(ABz",z)} \varphi(t)dt \leq \beta_1 \int_0^{[d(z_{2n+1},z)+d(z_{2n+1},z_{2n})]} \varphi(t)dt$$

$$+ \beta_2 \int_0^{d(z,ABz")} \varphi(t)dt + \beta_3 \int_0^{d(z_{2n},z_{2n+1})} \varphi(t)dt$$

$$+ (\beta_4 + 1) \int_0^{d(z,z_{2n+1})} \varphi(t)dt$$

$$+ \beta_5 \int_0^{[d(z_{2n},z_{2n+1})+d(z_{2n+1},z)+d(z,ABz")]} \varphi(t)dt$$

$$+ \int_0^{\min\{d(z,z_{2n}),d(z,ABz"),d(z_{2n},z_{2n+1}), d(z,z_{2n+1}),d(z_{2n},ABz")\}} \varphi(t)dt$$

or, $(1 - \beta_2 - \beta_5) \int_0^{d(z,ABz")} \varphi(t)dt \leq (\beta_1 + \beta_3 + \beta_5) \int_0^{d(z_{2n+1},z_{2n})} \varphi(t)dt$

$$+(\beta_1 + \beta_4 + \beta_5 + 1) \int_0^{d(z_{2n+1},z)} \varphi(t)dt$$

$$+ \int_0^{\min\{d(z,z_{2n}),d(z,ABz"),d(z_{2n},z_{2n+1}), d(z,z_{2n+1}),d(z_{2n},ABz")\}} \varphi(t)dt$$

Using normality of cone, we get

$$(1 - \beta_2 - \beta_5) \left\| \int_0^{d(z,ABz")} \varphi(t)dt \right\| \leq (\beta_1 + \beta_3 + \beta_5) \left\| \int_0^{d(z_{2n+1},z_{2n})} \varphi(t)dt \right\|$$

$$+(\beta_1 + \beta_4 + \beta_5 + 1) \left\| \int_0^{d(z_{2n+1},z)} \varphi(t)dt \right\|$$

$$+ \int_0^{min\{\|d(z,z_{2n})\|,\|d(z,ABz")\|,\|d(z_{2n},z_{2n+1})\|, \|d(z,z_{2n+1})\|,\|d(z_{2n},ABz")\|\}} \varphi(t)dt$$

$\rightarrow 0$ as $n \rightarrow \infty$.

$\therefore \left\| \int_0^{d(z,ABz")} \varphi(t)dt \right\| \leq 0$, which is a contradiction.

Therefore, by def. of $'\varphi'$, we get

$$\int_0^{d(z,ABz")} \varphi(t)dt = 0$$

or, $d(z, ABz") = 0$

or, $ABz" = z = Iz"$

Again, $ABz" = z \in AB(X) \subseteq J(X)$, then there exists a point $z' \in X$ such that $Jz' = z$.

By using condition (6), we get

$$\int_0^{d(STz',z)} \varphi(t)dt = \int_0^{d(ABz",STz')} \varphi(t)dt$$

$$\leq \int_0^{M_1(z",z')} \varphi(t)dt + \int_0^{M_2(z",z')} \varphi(t)dt$$

where , $M_1(z", z') = \beta_1 d(Iz", Jz') + \beta_2 d(Iz", ABz") + \beta_3 d(Jz', STz')$
$+ \beta_4 d(Iz", STz') + \beta_5 d(Jz', ABz")$
$= \beta_1 d(z, z) + \beta_2 d(z, z) + \beta_3 d(z, STz') + \beta_4 d(z, STz')$
$+ \beta_5 d(z, z)$
$= (\beta_3 + \beta_4) d(z, STz')$

and $M_2(z", z') = min \{d(Iz", Jz'), d(Iz", ABz"), d(Jz', STz'),$
$d(Iz", STz'), d(Jz', ABz")\}$
$= min \{d(z, z), d(z, z), d(z, STz'), d(z, STz'), d(z, z)\}$
$= min \{0,0, d(z, STz'), d(z, STz'), 0\}$
$= 0$

Therefore,

$$\int_0^{d(STz',z)} \varphi(t)dt \leq (\beta_3 + \beta_4) \int_0^{d(z,STz")} \varphi(t)dt$$

[by subadditivity of $'\varphi'$]

which is a contradiction.

Hence,

$$\int_0^{d(STz',z)} \varphi(t)dt = 0$$

or, $d(STz', z) = 0$

or, $STz' = z = Iz'$.

Finally, we show that AB and I, ST and J have a unique point of coincidence in X. Assume that there exists an another point $p \in X$ such that $STx = Jx = p$, then

$$\int_0^{d(z,p)} \varphi(t)dt = \int_0^{d(ABz",STx)} \varphi(t)dt$$

$\leq \int_0^{M_1(z'',x)} \varphi(t)dt + \int_0^{M_2(z'',x)} \varphi(t)dt$

where , $M_1(z'',x) = \beta_1 d(Iz'',Jx) + \beta_2 d(Iz'',ABz'') + \beta_3 d(Jx,STx)$
$+ \beta_4 d(Iz'',STx) + \beta_5 d(Jx,ABz'')$
$= \beta_1 d(z,p) + \beta_2 d(z,z) + \beta_3 d(p,p) + \beta_4 d(z,p)$
$+ \beta_5 d(p,z)$
$= (\beta_1 + \beta_4 + \beta_5) d(z,p)$

and $M_2(z'',x) = min\{d(Iz'',Jx), d(Iz'',ABz''), d(Jx,STx),$
$d(Iz'',STx), d(Jx,ABz'')\}$
$= min\{d(z,p), d(z,z), d(p,p), d(z,p), d(p,z)\}$
$= min\{d(z,p), 0, 0, d(z,p), d(z,p)\}$
$= 0$

$\therefore \int_0^{d(z,p)} \varphi(t)dt = \int_0^{d(ABz'',STx)} \varphi(t)dt$
$\leq (\beta_1 + \beta_4 + \beta_5) \int_0^{d(z,p)} \varphi(t)dt$, which is a contradiction.

Hence,

$\int_0^{d(z,p)} \varphi(t)dt = 0$

or, $d(z,p) = 0$

or, $z = p$.

Therefore, $STx = STz' = Jx = Jz' = z = p$.

Hence, it shows that z is a unique point of coincidence of ST and J in X.

Again, suppose there exists an another point $q \in X$ such that $ABx = Ix = q$, then by using condition (6), we get

$\int_0^{d(z,q)} \varphi(t)dt = \int_0^{d(STz',ABx)} \varphi(t)dt$
$\leq \int_0^{M_1(x,z')} \varphi(t)dt + \int_0^{M_2(x,z')} \varphi(t)dt$

where, $M_1(x,z') = \beta_1 d(Ix,Jz') + \beta_2 d(Ix,ABx) + \beta_3 d(Jz',STz')$
$+ \beta_4 d(Ix,STz') + \beta_5 d(Jz',ABx)$
$= \beta_1 d(q,z) + \beta_2 d(q,q) + \beta_3 d(z,z) + \beta_4 d(q,z)$
$+ \beta_5 d(q,z)$
$= (\beta_1 + \beta_4 + \beta_5) d(q,z)$

and $M_2(x, z') = min\{d(Ix, Jz'), d(Ix, ABx), d(Jz', STz'),$
$d(Ix, STz'), d(Jz', ABx)\}$
$= min\{d(q, z), d(q, q), d(z, z), d(q, z), d(z, q)\}$
$= min\{d(q, z), 0, 0, d(q, z), d(q, z)\}$
$= 0$

Therefore,

$\int_0^{d(z,q)} \varphi(t)dt \leq (\beta_1 + \beta_4 + \beta_5) \int_0^{d(z,q)} \varphi(t)dt$, which is a contradiction.

Hence,

$$\int_0^{d(z,q)} \varphi(t)dt = 0$$

or, $d(z, q) = 0$

or, $z = q$.

So, $ABx = ABz$=Ix=Iz $= z = q$.

Hence, z is a unique point of coincidence of AB and I. So, according to Lemma (2.7), z is a unique common fixed point of $\{ST, J\}$ and $\{AB, I\}$. Therefore z is a unique common fixed point of mappings AB, ST, I and J.

Finally, we prove that z is also a common fixed point of A, B, S, T, I and J.

Let both the pairs (AB, I) and (ST, J) have a unique common fixed point z.

Then,

$$Az = A(ABz) = A(BAz) = AB(Az)$$
$$Az = A(Iz) = I(Az)$$
$$Bz = B(ABz) = B\big(A(Bz)\big) = BA(Bz) = AB(Bz)$$
$$Bz = B(Iz) = I(Bz)$$

which implies that (AB, I) has common fixed points which are Az and Bz. We get thereby $Az = z = Bz = Iz = ABz$.

Similarly, using the commutativity of $(S, T), (S, J)$ and (T, J), $Sz = z = Tz = Jz = STz$ can be shown.

Now, we need to show that $Az = Sz$ $(Bz = Tz)$.

By using condition (6), we have

$$\int_0^{d(Az,Sz)} \varphi(t)dt = \int_0^{d(A(BAz),S(TSz))} \varphi(t)dt$$

$$= \int_0^{d(AB(Az),ST(Sz))} \varphi(t)dt$$
$$\leq \int_0^{M_1(Az,Sz)} \varphi(t)dt + \int_0^{M_2(Az,Sz)} \varphi(t)dt$$

where , $M_1(Az, Sz) = \beta_1 d(I(Az), J(Sz)) + \beta_2 d(I(Az), AB(Az))$
$+\beta_3 d(I(Sz), ST(Sz)) + \beta_4 d(I(Az), ST(Sz))$
$+\beta_5 d(J(Sz), AB(Az))$
$= \beta_1 d(Az, Sz) + \beta_2 d(Az, Az) + \beta_3 d(Sz, Sz)$
$+ \beta_4 d(Az, Sz) + \beta_5 d(Sz, Az)$
$= (\beta_1 + \beta_4 + \beta_5) d(Az, Sz)$

and $M_2(Az, Sz) = min\,\{d(Az, Sz), d(Az, Az), d(Sz, Sz), d(Az, Sz), d(Sz, Az)\}$
$= min\,\{d(Az, Sz), 0{,}0, d(Az, Sz), d(Az, Sz)\}$
$= 0$

$$\therefore \int_0^{d(Az,Sz)} \varphi(t)dt \leq (\beta_1 + \beta_4 + \beta_5) \int_0^{d(Az,Sz)} \varphi(t)dt$$

which is a contradiction.

Hence,

$$\int_0^{d(Az,Sz)} \varphi(t)dt = 0$$

or, $d(Az, Sz) = 0$

or, $Az = Sz$.

Similarly, $Bz = Tz$ can be shown.

Consequently, z is a unique common fixed point of A, B, S, T, I and J.

Our first corollary is obtained by putting $AB = S, ST = T, M_1(x, y) = M(x, y)$ and $M_2(x, y) = 0\ \ \forall\, x, y \in X$ in theorem (3.2), which is a generalization of result of Han and Xu [3] to integral type contraction in a normal cone metric space.

Corollary 3.3: Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Suppose that $\varphi: P \to P$ is a non-vanishing map and a subadditive cone integrable on each $[a, b] \subset P$ such that for each $\in > 0$, $\int_0^{\in} \varphi(t)dt > 0$.

Let S, T, I and J be self mappings of (X, d) satisfying

(i) $S(X) \subset J(X),\ T(X) \subset I(X)$

(ii) $\int_0^{d(Sx,Ty)} \varphi(t)dt \leq \int_0^{M(x,y)} \varphi(t)dt$

where $M(x, y) = \beta_1 d(Ix, Jy) + \beta_2 d(Ix, Sx) + \beta_3 d(Jy, Ty)$

$+ \beta_4 d(Ix, Ty) + \beta_5 d(Jy, Sx)$

for all $x, y \in X$ where $\beta_i \geq 0$ $(i = 1,2,3,4,5)$ satisfying

$\beta_1 + \beta_2 + \beta_3 + 2max\{\beta_4, \beta_5\} < 1$

or, $\beta_1 + 2max\{\beta_2, \beta_3\} + \beta_4 + \beta_5 < 1$

If one of $S(X)$, $I(X)$, $T(X)$, $J(X)$ is a complete subspace of X then the mappings S, T, I and J have a unique point of coincidence in X.

Moreover, if the pairs $\{S, I\}$ and $\{T, J\}$ are weakly compatible respectively, then all of the mappings S, T, I and J have unique common fixed point.

Our next corollary is obtained by putting $AB = S^m, ST = T^n, I = I^m, J = J^n$, $M_1(x, y) = M(x, y)$ and $M_2(x, y) = 0\ \forall\, x, y \in X$ in theorem (3.2).

Corollary 3.4: Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Suppose that $\varphi: P \to P$ is a non-vanishing map and a subadditive cone integrable on each $[a, b] \subset P$ such that for each $\in > 0$, $\int_0^{\in} \varphi(t)dt > 0$.

Let S, T, I, J: $X \to X$ satisfying

(i) $S(X) \subset J(X)$, $T(X) \subset I(X)$

(ii) $\int_0^{d(S^m x, T^n y)} \varphi(t)dt \leq \int_0^{M(x,y)} \varphi(t)dt$

where $M(x, y) = \beta_1 d(I^m x, J^n y) + \beta_2 d(I^m x, S^m x) + \beta_3 d(J^n y, T^n y)$

$+ \beta_4 d(I^m x, T^n y) + \beta_5 d(J^n y, S^m x)$

for all $x, y \in X$ where $\beta_i \geq 0$ $(i = 1,2,3,4,5)$ satisfying

$\beta_1 + \beta_2 + \beta_3 + 2max\{\beta_4, \beta_5\} < 1$

or, $\beta_1 + 2max\{\beta_2, \beta_3\} + \beta_4 + \beta_5 < 1$

(iii) One of $S(X)$, $I(X)$, $J(X)$, $T(X)$ is a complete subspace of X then the four mappings S, T, I and J have a unique point of coincidence in X.

Moreover, if the pairs $\{T, J\}$ and $\{S, I\}$ are weakly compatible respectively, then all of the mappings S, T, I and J have unique common fixed point.

Proof: From above corollary, it is clear that the four mappings S^m, I^m, T^n and J^n have a unique common fixed point z. Now,

$Sz = S(S^m z) = S^{m+1}z = S^m(Sz)$

and $Iz = I(I^m z) = I^{m+1}z = I^m(Iz)$, which shows that Sz and Iz are also fixed point of S^m and I^m.

Hence, $Sz = Iz = z$. By using same process, we get $Tz = Jz = z$.

Now we give an example to illustrate our corollary (3.3).

Example 3.5: Consider $X = [0,6]$ with the usual metric defined by

$d(x, y) = \|x - y\| = |x - y|$ where $x, y \in X$ and $F = R$ = Real Banach space.

Define self mappings S, T, I and J on X as

$S0 = 0, \quad Sx = 1, \quad 0 < x \leq 6$

$T0 = 0, \quad Tx = 3, \quad 0 < x < 6, \quad T6 = 0$

$I0 = 0, \quad Ix = 5, \quad 0 < x < 6, \quad I6 = 3$ and

$J0 = 0, \quad Jx = 6, \quad 0 < x < 6, \quad J6 = 1$

Take $\varphi(t) = 1$.

Here, $S(X) = \{0,1\} \subseteq J(X) = \{0,1,6\}$

And $T(X) = \{0,3\} \subseteq I(X) = \{0,3,5\}$

Also, the pairs (S, I) and (T, J) are weakly compatible (coincidentally commuting) at $x = 0$ which is their common coincidence point.

Now, we verify that contractive condition are satisfied for

$\beta_1 = \frac{1}{4}, \ \beta_2 = \frac{1}{6}, \beta_3 = \frac{1}{8}, \beta_4 = \frac{1}{10}$ and. $\beta_5 = \frac{1}{9}$

where $\beta_1 + \beta_2 + \beta_3 + 2max\{\beta_4, \beta_5\} < 1$

and $\beta_1 + 2max\{\beta_2, \beta_3\} + \beta_4 + \beta_5 < 1$.

Thus, all the conditions of the corollary (3.3) are satisfied and 0 is the unique common fixed point of S, T, I and J.

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