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SOME COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS SATISFYING A GENERAL CONTRACTIVE CONDITION OF INTEGRAL TYPE

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Abstract: Inspired and motivated by the results using the concept of weak compatibility and commutativity, we prove some common fixed point theorems for six mapping in symmetric spaces which generalize several known corresponding results.

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1. Introduction

In 2002, Branciari [3] obtained a fixed point theorem for a single mapping satisfying an analogue of Banach's contraction principle for an integral type inequality. Aliouche [2] established a common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type and a property (E.A.) introduced by Aamri and. Moutawakil [1]. Boikanyo and Choudhary [4] have proved some common fixed point theorems for pointwise R-weakly commuting mappings in symmetric space with atleast one pair non compatible satisfying a contractive condition of integral type. They also have prove some results for weakly compatible mappings.[6]

Since then there have been many theorems dealing with mappings satisfying a general contractive condition of integral type. Some of these works are noted in Rhoades [8], Vijayaraju et al. [10], Gairola & Rawat [5].

The aim of the present paper is to obtain a common fixed point theorem by using the notion of weakly compatible mappings in symmetric space satisfying a contractive condition of integral type.

We recall that a symmetric on a set X is a non negative real valued function d on $X \times X$ such that

(i) $d(x, y)=0$ if and only if $x=y$,

(ii) $d(x, y)= d(y, x)$.

Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let $B(x, r)=\{y \in X: d(x, y) < r\}$. A topology $t(d)$ on X is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x, r) \subset U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x, r)$ is a neighbourhood of x in the topology $t(d)$. Note that $lim_{n\to\infty} d(x_n, x)=0$ if and only if $x_n \to x$ in the topology $t(d)$.

The following two axioms were given by Wilson [11]. Let (X, d) be a symmetric space.

(W.3) Given $\{x_n\}$, x and y in X, $lim_{n\to\infty} d(x_n, x)=0$ and $lim_{n\to\infty} d(x_n, y)=0$ implies $x=y$.

(W.4) Given $\{x_n\}$, $\{y_n\}$ and x in X, $lim_{n\to\infty} d(x_n, x)=0$ and $lim_{n\to\infty} d(x_n, y_n)=0$ implies that $lim_{n\to\infty} d(y_n, x)=0$.

It is easy to see that for a semi-metric d, if $t(d)$ is a Hausdorff, then (W.3) holds.

2. Preliminaries

In the sequel, we need a function $F^*=\{\varphi: R_+ \to R_+\}$ such that φ is a Lebesgue integrable mapping which is summable, non-negative and satisfy $\int_0^{\varepsilon} \varphi(t)dt > 0$ for all $\varepsilon > 0$ and ϕ will be a function defined by, $\phi: R_+ \to R_+$ such that $0 < \phi(t) < t$ for all $t > 0$.

Definition 1 Let S and T be two self mappings of a symmetric space (X, d). S and T are said to be compatible if $lim_{n\to\infty} d(STx_n, TSx_n)=0$ whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} d(Sx_n, t)= lim_{n\to\infty} d(Tx_n, t)=0$ for some $t \in X$.

Definition 2 Two self mappings S and T of a symmetric space (X, d) are said to be weakly compatible if they commute at their coincidence points.[2]

Definition 3 Let S and T be two self mappings of a symmetric space (X, d). We say that S and T satisfy the property (E.A) if there exist a sequence $\{x_n\}$ such that

$lim_{n\to\infty} d(Sx_n, t)= lim_{n\to\infty} d(Tx_n, t)=0$ for some $t \in X$.

Example 1. Let $X=[0, \infty)$. Let d be a symmetric on X defined by $d(x, y)=e^{|y-x|}-1$ for all x, y in X. Define $S, T: X \to X$ as follows:

$$Sx = 2x +1 \text{ and } Tx = x +2, \text{ for all } x \in X.$$

Note that the function d is not a metric. Consider the sequence $x_n=1+1/n$, $n=1, 2, \dots$

Clearly $lim_{n\to\infty} d(Sx_n, 3)= lim_{n\to\infty} d(Tx_n, 3)=0$.

Then S and T satisfy property (E.A), but S and T are not weakly compatible.

Definition 4 Let (X, d) be a symmetric space. We say that (X, d) satisfy property (H.E) if given $\{x_n\}$, $\{y_n\}$ and x in X, $lim_{n\to\infty} d(x_n, x)=0$ and $lim_{n\to\infty} d(y_n, x)=0$ implies $lim_{n\to\infty} d(x_n, y_n)=0$.

Example 2.

(i) Every metric space (X, d) satisfies property (H.E).

(ii) Let $X=[0, \infty)$ with the symmetric function d defined in Example 1. It is easy to see that the symmetric space (X, d) satisfies property (H.E).

3. Main Result

Theorem 3.1 Let d be a symmetric for X that satisfy (W.3), (W.4) and H.E. Let A, B, S, T, I and J be self mappings on (X, d) satisfying the following conditions:

(i) $AB(X) \subset J(X)$, $ST(X) \subset I(X)$,

(ii) $$\int_0^{d(ABx,STy)} \varphi(t)dt \le \phi\left(\int_0^{M(x,y)} \varphi(t)dt\right) \quad \ldots (1)$$

for all $x, y \in X$, $\varphi \in F^*$ and

$$M(x, y) = \max\{d(Ix, Jy), d(ABx, Ix), d(STy, Jy), \frac{1}{2}[d(ABx, Jy)+d(STy, Ix)]\}$$

(iii) $I(X)$ or $J(X)$ is sequentially complete subspace of X.

(iv) (AB, I) and (ST, J) are weakly compatible and (AB, I) or (ST, J) satisfied the property (E.A).

Then AB, ST, I and J have a unique common fixed point [7].

Furthermore, if the pair (A, B), (A, I), (B, I), (S, T), (S, J) and (T, J) are commuting mappings. Then A, B, S, T, I and J have a unique common fixed point in X.

Proof: Suppose that, ST and J satisfy property (E.A.). Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(STx_n, z)= \lim_{n\to\infty} d(Jx_n, z)=0$ for some $z\in X$. Therefore, by (H.E.) we have $\lim_{n\to\infty} d(STx_n, Jx_n)=0$. Since $ST(X) \subset I(X)$, there exists in X a sequence $\{y_n\}$ such that $STx_n=Iy_n$. Hence, $\lim_{n\to\infty} d(Iy_n, z)=0$. Let us show that $\lim_{n\to\infty} d(ABy_n, z)=0$.

Suppose that $\lim_{n\to\infty} \text{Sup}\, d(ABy_n, STx_n) > 0$. Then, using (1), we have

$$\lim_{n\to\infty} Sup \int_0^{d(ABy_n,STx_n)} \varphi(t)dt \le \lim_{n\to\infty} Sup\,\phi\left(\int_0^{M(y_n,x_n)} \varphi(t)dt\right)$$

where $M(y_n, x_n)=\max\{d(Iy_n, Jx_n), d(ABy_n, Iy_n), d(STx_n, Jx_n), \frac{1}{2}[d(ABy_n, Jx_n)+d(STx_n, Iy_n)]\}$

$$=\max\{0, d(ABy_n, STx_n), 0, \frac{1}{2}[d(ABy_n, STx_n) + 0]\}$$

$$=d(ABy_n, STx_n).$$

$$\lim_{n\to\infty} Sup \int_0^{d(ABy_n, STx_n)} \varphi(t)dt \le \lim_{n\to\infty} Sup\phi\left(\int_0^{d(ABy_n, STx_n)} \varphi(t)dt\right) < \lim_{n\to\infty} Sup \int_0^{d(ABy_n, STx_n)} \varphi(t)dt$$

which is a contradiction. Hence $\int_0^{d(ABy_n, STx_n)} \varphi(t)dt$ =0 and $lim_{n\to\infty}\, d\,(ABy_n, STx_n)$=0. By (W.4), we have $lim_{n\to\infty}\, d\,(ABy_n, z)$=0.

Suppose that, $I(X)$ is complete subspace of X, then there exists $u\in X$ such that $Iu=z$. We have,

$$lim_{n\to\infty}\, d\,(ABy_n, Iu)=lim_{n\to\infty}\, d\,(STx_n, Iu)=lim_{n\to\infty}\, d\,(Jx_n, Iu)=lim_{n\to\infty}\, d\,(Iy_n, Iu)=0.$$

Now, we claim that $ABu=Iu$. If not, then from (1), we have

$$\int_0^{d(ABu, STx_n)} \varphi(t)dt \le \phi\left(\int_0^{M(u, x_n)} \varphi(t)dt\right)$$

where $M(u, x_n)$ =max$\{d(Iu, Jx_n), d(ABu, Iu), d(STx_n, Jx_n), \frac{1}{2}[d(ABu, Jx_n)+d(STx_n, Iu)]\}$

$$=\max\{0, d(ABu, STx_n), 0, \frac{1}{2}[d(ABu, STx_n) + 0]\}$$

$$=d(ABu, STx_n).$$

$\int_0^{d(ABu, STx_n)} \varphi(t)dt \le \phi\left(\int_0^{d(ABu, STx_n)} \varphi(t)dt\right)$. Letting $n\to\infty$, we obtain $\lim_{n\to\infty} \int_0^{d(ABu, STx_n)} \varphi(t)dt = 0$, which implies $lim_{n\to\infty}\, d\,(ABu, STx_n)$=0. By (W.3), we have $ABu=z=Iu$.

Using the weak compatibility of AB and I implies that $IABu=ABIu$ i.e. $Iz=ABz$. On the other hand since $AB(X) \subset J(X)$, there exists $v\in X$ such that $ABu=Jv$.

We claim that $STv=Jv$. If not then from (1), we have

$$\int_0^{d(Jv, STv)} \varphi(t)dt = \int_0^{d(ABu, STv)} \varphi(t)dt \le \phi\left(\int_0^{M(u,v)} \varphi(t)dt\right)$$

where $M(u, v)$ = max$\{d(Iu, Jv), d(ABu, Iu), d(STv, Jv), \frac{1}{2}[d(ABu, Jv)+d(STv, Iu)]\}$

$$=\max\{0,\ 0,\ d(STv, Jv),\ \frac{1}{2}[\,0 + d(STv, Jv)]\}$$

$$=d(Jv, STv).$$

$\int_0^{d(Jv,STv)} \varphi(t)dt \le \phi \int_0^{d(Jv,STv)} \varphi(t)dt < \int_0^{d(Jv,STv)} \varphi(t)dt$ which is a contradiction. Hence $\int_0^{d(Jv,STv)} \varphi(t)dt = 0$ which implies that $d(Jv, STv)=0$. Then $z=ABu=Iu=Jv=STv$.

Now using the weak compatibility of *ST* and *J* implies that $STJv=JSTv$ i.e. $STz=Jz$. Let us show that *z* is a common fixed point of *AB*, *ST*, *I* and *J*.

If $z\neq ABz$, using (1), we get

$$\int_0^{d(z,ABz)} \varphi(t)dt = \int_0^{d(ABz,STv)} \varphi(t)dt \le \phi\left(\int_0^{M(z,v)} \varphi(t)dt\right)$$

where $M(z, v) = \max\{d(Iz, Jv), d(ABz, Iz), d(STv, Jv), \frac{1}{2}[d(ABz, Jv)+d(STv, Iz)]\}$

$$=\max\{d(ABz, z),\ 0,\ 0,\ \frac{1}{2}[d(ABz, z) + d(z, ABz)]\}$$

$$=d(z, ABz).$$

Therefore, $\int_0^{d(z,ABz)} \varphi(t)dt \le \phi\left(\int_0^{d(z,ABz)} \varphi(t)dt\right) < \int_0^{d(z,ABz)} \varphi(t)dt$, which is a contradiction. Thus, $z=ABz=Iz$.

If $z\neq STz$, using (1), we get

$$\int_0^{d(z,STz)} \varphi(t)dt = \int_0^{d(ABz,STz)} \varphi(t)dt \le \phi\left(\int_0^{M(z,z)} \varphi(t)dt\right)$$

where $M(z, z) = \max\{d(Iz, Jz), d(ABz, Iz), d(STz, Jz), \frac{1}{2}[d(ABz, Jz)+d(STz, Iz)]\}$

$$=\max\{d(z, STz),\ 0,\ 0,\ \frac{1}{2}[d(z, STz) + d(STz, z)]\}$$

$$=d(z, STz).$$

Therefore, $\int_0^{d(z,STz)} \varphi(t)dt \le \phi\left(\int_0^{d(z,STz)} \varphi(t)dt\right) < \int_0^{d(z,STz)} \varphi(t)dt$, which is a contradiction. Thus, $z=STz=Jz$.

Therefore, $z=STz=Jz=ABz=Iz$. i.e. z is the common fixed point of AB, ST, I and J. For the uniqueness of z, suppose that $z\neq \omega$ is another common fixed point of AB, ST, I and J. Using (1), we have

$$\int_0^{d(z,\omega)} \varphi(t)dt = \int_0^{d(ABz,ST\omega)} \varphi(t)dt \le \phi\left(\int_0^{M(z,\omega)} \varphi(t)dt\right)$$

where $M(z, \omega) = \max\{d(Iz, J\omega), d(ABz, Iz), d(ST\omega, J\omega), \frac{1}{2}[d(ABz, J\omega)+d(ST\omega, Iz)]\}$

$= d(z, \omega)$.

Therefore, $\int_0^{d(z,\omega)} \varphi(t)dt \le \phi\left(\int_0^{d(z,\omega)} \varphi(t)dt\right) < \int_0^{d(z,\omega)} \varphi(t)dt$, which is a contradiction. Therefore, $\int_0^{d(z,\omega)} \varphi(t)dt = 0$, which implies that $z=\omega$.

Now we prove that z is a common fixed point of A, B, S, T, I and J. For this let z is a unique common fixed point of both the pair (AB, I) and (ST, J). Using the commutativity[9] of (A, B), (A, I) and (B, I) then

$$Az = A(ABz) = A(BAz) = AB(Az), \qquad Az = A(Iz) = I(Az)$$

and $$Bz = B(ABz) = BA(Bz) = AB(Bz), \qquad Bz = B(Iz) = I(Bz)$$

which shows that Az and Bz are a common fixed point of (AB, I), yielding thereby $Az=z=Bz=Iz=ABz$. Similarly, using the commutativity of (S, T), (S, J) and (T, J) it can be shown that $Sz=z=Tz=Jz=ABz$.

Now, we need to show that $Az = Sz$ ($Bz = Tz$). For this let $Az \neq Sz$, using (1), we get

$$\int_0^{d(Az,Sz)} \varphi(t)dt = \int_0^{d(A(BAz),S(TSz))} \varphi(t)dt = \int_0^{d(AB(Az),ST(Sz))} \varphi(t)dt$$

$$\le \phi\left(\int_0^{M(Az,Sz)} \varphi(t)dt\right)$$

where $M(Az, Sz) = \max\{d(I(Az), J(Sz)), d(AB(Az), I(Az)), d(ST(Sz), J(Sz)),$

$$\frac{1}{2}[d(AB(Az), J(Sz))+d(ST(Sz), I(Az))]\}$$

$$=d(Az, Sz).$$

Therefore, $\int_0^{d(Az,Sz)} \varphi(t)dt \le \phi\left(\int_0^{d(Az,Sz)} \varphi(t)dt\right) < \int_0^{d(Az,Sz)} \varphi(t)dt$, which is a contradiction. Therefore, $\int_0^{d(Az,Sz)} \varphi(t)dt = 0$ which implies that *Az*=*Sz*. Similarly, it can be shown that *Bz*=*Tz*. Thus, *z* is the unique common fixed point of *A*, *B*, *S*, *T*, *I* and *J*. This completes the proof.

If we take *B*=*T*=identity map in our Theorem 3.1, we obtain the following result of Boikanya and Chaudhary [4]:

Corollary 3.2 Let *d* be a symmetric for *X* that satisfy (W.3), (W.4) and H.E. Let *A*, *S*, *I*, and *J* be self mappings on (*X*, *d*) satisfying the following conditions:

(i) $A(X) \subset J(X)$, $S(X) \subset I(X)$,

(ii) $\int_0^{d(Ax,Sy)} \varphi(t)dt \le \phi\left(\int_0^{M(x,y)} \varphi(t)dt\right)$ for all $x, y \in X$, $\varphi \in F^*$ and

$M(x, y)=\max\{d(Ix, Jy), d(Ax, Ix), d(Sy, Jy), \frac{1}{2}[d(Ax, Jy)+d(Sy, Ix)]\}$

(iii) *I*(*X*) or *J*(*X*) is sequentially complete subspace of *X*.

(iv) (*A*, *I*) and (*S*, *J*) are weakly compatible and (*A*, *I*) or (*S*, *J*) satisfied the property (E.A).

Then *A*, *S*, *I* and *J* have a unique common fixed point.

Now let us take *B*=*T*=*I*=*J*=identity map in our Theorem 3.1, we obtain the following corollary:

Corollary 3.3 Let *d* be a symmetric for *X* that satisfy (W.3), (W.4) and H.E. Let *A* and *S* be self mappings on (*X*, *d*) satisfying the following conditions:

$\int_0^{d(Ax,Sy)} \varphi(t)dt \le \phi\left(\int_0^{M(x,y)} \varphi(t)dt\right)$ for all $x, y \in X$, $\varphi \in F^*$ and

$M(x, y)=\max\{d(x, y), d(Ax, x), d(Sy, y), \frac{1}{2}[d(Ax, y)+d(Sy, x)]\}$

If the range of *A* or *S* is a complete subspace of *X*, then *A* and *S* have a unique common fixed point in *X*.

We note that Corollary 3.3 is more general then Theorem 2 of Vijayraju *et al*. [10].

Again taking B=T=I=J=S=identity map in our Theorem 3.1, we obtain the following corollary:

Corollary 3.4 Let d be a symmetric for X that satisfy (W.3), (W.4) and H.E. Let A be self mappings on (X, d) satisfying the following conditions:

$$\int_0^{d(Ax,Ay)} \varphi(t)dt \le \phi\left(\int_0^{M(x,y)} \varphi(t)dt\right) \text{ for all } x, y \in X,\ \varphi \in F^* \text{ and}$$

$$M(x, y)=\max\{d(x, y), d(Ax, x), d(Ay, y), \frac{1}{2}[d(Ax, y)+d(Ay, x)]\}$$

If the range of A is a complete subspace of X, then A has a unique fixed point in X.

Clearly, Corollary 3.4 is more general than Theorem 2 of Rhoades [8].

Theorem 3.5 Let d be a symmetric for X that satisfy (W.3), (W.4) and H.E. Let A, B, S, T, I and J be self mappings on (X, d) satisfying the following conditions:

(i) $AB(X) \subset J(X)$, $ST(X) \subset I(X)$,

(ii) $$\int_0^{d(ABx,STy)} \varphi(t)dt \le \phi\left(\int_0^{M(x,y)} \varphi(t)dt\right) \qquad \dots (2)$$

for all $x, y \in X$, $\varphi \in F^*$ and

$$M(x, y)=\max\{d(Ix, Jy), d(ABx, Ix), d(STy, Jy), \frac{1}{2}d(ABx, Jy), \frac{1}{2}d(STy, Ix)\}$$

(iii) $I(X)$ or $J(X)$ is sequentially complete subspace of X.

(iv) (AB, I) and (ST, J) are weakly compatible and (AB, I) or (ST, J) satisfied the property (E.A).

Then AB, ST, I and J have a unique common fixed point.

Furthermore, if the pair (A, B), (A, I), (B, I), (S, T), (S, J) and (T, J) are commuting mappings. Then A, B, S, T, I and J have a unique common fixed point in X.

Proof: By a method similar to that in the proof of Theorem 3.1, the proof follows.

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