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# ON FINSLER SPACES SATISFYING THE CONDITION $L^5C = \delta^4$

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**Abstract:**In the year 1979, Matsumoto and Shiba [7] have discussed non-Riemannian Finsler spaces with vanishing T-tensor. In the paper, Matsumoto has shown that if a Finsler space ($M^n$,L) satisfy the T-condition i.e. $T_{hijk} = 0$, then for such a Finsler space $L^2C^2$ of $M^n$ is a function of position only (i.e.), $L^2C^2 = f(x)$ where L is the fundamental function and $C^2$ is square of length of torsion tensor $C_i$. In the light of above observation F.Ikeda in the year 1984, studied Finsler spaces, with $L^2C^2$ as a function of x in detail. In the continuity of above studied Ikeda in the year 1991, considered Finsler spaces with $L^2C^2$ as a nonzero constant, which is a stronger condition as compared to above (i.e., $L^2C^2 = f(x)$. Pandey et a. [8] studied Finsler spaces taking $L^2C^2$ as a sum of function of x and y i.e. $L^2C^2 = f(x) + g(y)$.

In the present paper, we shall consider combination of L and C differently and taking $L^5C = \delta^4$, where $\delta =$ $(a_{ijkl}(x)y^i y^j y^k y^l)^{(1/4)}$is a well known quartic metric.
**Keywords**: Finsler spaces, satisfying $L^5C = \delta^4$

## 1. Introduction

Matsumoto in the paper, [7] studied non-Riemannian Finsler spaces with the vanishing T-tensor, which are said to satisfy the T-condition(by T-condition we mean a Finsler space whose T-tensor vanishes),then the function $L^2C^2$ over $M^n$ is reduced to a function of the position only (i.e. $L^2C^2 = f(x)$), where L is the fundamental function and C is the length of torsion vector $C_i$. He has also quoted that if the metric tensor $g_{ij}$ has a special form as $g_{ij} = q_{ij}^{ls} l_l l_s$ then the function $L^2C^2$ becomes zero (i.e. $L^2C^2 = 0$). Because in this case the T-condition satisfies automatically and $C_i = 0$

In the light of above observation Ikeda[3] in the year 1984, studied Finsler spaces, $L^2C^2$ as a function of x in detail and come out with some interesting result specially for a two and three dimensional Finsler spaces. Actually Ikeda was examining the equivalence of T condition with $L^2C^2 = f(x)$. In continuity of above in the year 1991 Ikeda[4] considered Finsler spaces satisfying the condition $L^2C^2$ equals to nonzero constant, which is a stronger condition t $L^2C^2 = f(x)$. An example of such a Finsler space with the constant function $L^2C^2$ is a two-dimensional Berwald space. Pandey et al.[8] studied Finsler spaces taking LC equal to some known function of x and y i.e. LC = f(y) + g(x).

In the present paper, we shall consider combination of L and C differently. Actually, we are taking, $L^5C = \delta^4$, where $\delta$ is a well known quartic metric. For such a Finsler space, if it is C-reducible then it has been worked out under what condition T-tensor vanishes.[2,10] In the last section it has been worked out under what condition such a Finsler space ($L^5C = \delta^4$) will be Landesberg or Berwald space.[1]

Throughout the paper we shall confine ourselves to Cartan's connection, and the notations and terminology of the monograph [5] will be used without comment. In the paper monograph of Matsumoto[5] will be quoted by (#).

## 2. T - tensor of a Finsler space with $L^5C = \delta^4$

Let $l_i$, $h_{ij}$ and $C_{ijk}$ denote the unit vector (i.e. $l_i = \frac{y_i}{L}$), be angular metric tensor and the (h)hv-torsion tensor respectively.

The well known, T-tensor $T_{ijkh}$ ([5],#$eq^n$(28.20)) has been defined by :

$$T_{ijkh} = LC_{ijk}|_h + C_{ijk}l_h + C_{jkh}l_i + C_{khi}l_j + C_{hij}l_k \tag{1}$$

and the torsion tensor $C_i$ is given by $C_i = g^{jk}C_{ijk}$, where the symbol $|_h$ denote v-covariant differentiation and $g^{jk}$ is the reciprocal tensor of $g_{jk}$.

We are considering a Finsler space $F^n$, whose torsion tensor is such that

$$L^5C = \delta^4 \tag{2}$$

where $\delta^4 = a_{ijkl}(x)y^i y^j y^k y^l$ is a quartic metric.

Differentiation of equation (2) by $y^h$ yields

$$L^2C_{;h} + 5LCl_h = 4a_h \tag{3}$$

where the notation ';' denote the differentiation by $y^h$ of C and

$$\delta^3\delta_h = \delta^3 \frac{\partial\delta}{\partial y^h} = a_{hijk} y^i y^j y^k = L^3 a_{hijk} l^i l^j l^k = L^3 a_h$$

Also $a_h$ is defined as $L^3 a_h = a_{hijk} y^i y^j y^k$ .

Contracting equation (1) by $g^{jk}$ and multiplying with $C^i$ , we obtain

$$C^i T_{ih} = LC^i C_i|_h + l_h C^2 \tag{4}$$

Now, differentiate the equation $C^2 = g^{ij} C_i C_j$ with respect to $y^h$, which yields

$$C_{;h} = \frac{C^i C_i|_h}{C}$$

Substituting above value in equation (3), we get

$$LC^i C_i|_h = C\left(\frac{4a_h}{L} - 5Cl_h\right) \tag{5}$$

In the virtue of equation (4) and equation (2.5), we obtain

$$C^i T_{ih} = 4C\left(\frac{a_h}{L} - Cl_h\right) \tag{6}$$

Conversely, let $C^i T_{ih} = 4C\left(\frac{a_h}{L} - Cl_h\right)$

$$\Rightarrow \quad LC^i C_i|_h + C^2 l_h = 4C\left(\frac{a_h}{L} - Cl_h\right)$$

$$\Rightarrow \quad LC_{;h} + 5Cl_h = \frac{4a_h}{L} \quad (\because C^i C_i|_h = CC_{;h})$$

$$\Rightarrow \quad (L^2 C_{;h} + 5LCl_h = 4a_h)L^3$$

$$\Rightarrow \quad (L^5 C)_{;h} = 4L^3 a_h \quad [\because (L^5 C)_{;h} = \frac{\partial}{\partial y^h}(L^5 C) \;]$$

Multiplying both side by $y^h$ and using Euler's theorem, we get

$$L^5 C = \delta^4$$

Thus, we have

**Theorem 2.1** *For a Finsler space* $(M^n, L)$ *of dimension n, if torsion scalar C is such as* $L^5C = \delta^4$ *, then following relation*

$$T_{ih}C^i = 4C(\frac{a_h}{L} - Cl_h)$$

holds good.

Next, for a two dimensional Finsler space the T-tensor [5,11] can be written as:

$$T_{hijk} = I_{;2} m_h m_i m_j m_k \tag{7}$$

where, $I_{;2} = L\frac{\partial I}{\partial y^i} m_i$ , $LC_{ijk} = Im_i m_j m_k$ $([5], \#eq^n(28.3))$ and $LC = I$

Writing, $LC = I$ in equation (2) and differentiating with respect to $y^i$, we have

$$(L^4 I);_i = (\delta^4)_{;i} = 4L^3 a_i$$

$$\Rightarrow \quad 4l_i I + LI;_i = 4a_i \quad (L\frac{\partial I}{\partial y^i} = I_{;2}m_i) \Rightarrow \quad 4Il_i + I_{;2}m_i = 4a_i$$

Contracting both side by $m^i$, we get

$$I_{;2} = 3a_i m^i$$

Substituting the value of $I_{;2}$ in equation (5), we get

$$T_{hijk} = 4a_r m^r m_h m_i m_j m_k \tag{8}$$

**Corollary 2.1** For a two dimensional Finsler space if $L^5C = \delta^4$ and $a_i$ is parallel to $l_i$, then $T_{hijk} = 0$ .

For a C-reducible Finsler space the T-tensor [7] can be written as,

$$T_{hijk} = \frac{LC^*}{n^2 - 1} \mathrm{e}_{(hijk)} h_{hi} h_{jk} \quad (\#equation(30.28)) \tag{9}$$

where, $C^* = g^{ij} C_i|_j$ and $\mathrm{e}_{(ijk)}$ represents sum of cyclic permutation in the indices h,i,j.

Contracting equation (9) by $g^{jk}$, we get

$$T_{hi} = \frac{LC^*}{n-1} h_{hi}$$

Using equation (4),

$$C^i T_{ih} = \frac{LC^*}{n-1} C_h = \frac{4C}{L} a_h - 4C^2 l_h \tag{10}$$

$$\Rightarrow \quad a_h = \frac{L}{4}\left(\frac{LC^* C_h}{C(n-1)} + Cl_h\right)$$

$$\Rightarrow \quad a_h C^h = \frac{L}{4}\left(\frac{LC^* C}{n-1}\right)$$

**Corollary 2.2** *For a C-reducible Finsler space* $(M^n, L)$ [6,9] *with* $L^5 C = \delta^4$ *and* $a_i$ *is parallel to* $l_i$*, then* $T_{hijk} = 0.$

**Remark 2.1** *It is an open problem to examine* $L^2 C^2 = f(x)$ *and* $L^5 C = \delta^4$ *for two dimensional and C-reducible Finsler spaces with* $a_i$ *is along* $y_i$ *which one is the stronger condition.*

**3. Landsberg and Berwald spaces satisfying the condition** $L^5 C = \delta^4$

Let us consider a Finsler space $M^n$, where C is such that :

$$L^5 C = \delta^4$$

Differentiation of above equation with respect to $y^i$ gives

$$L^5 C_{;i} + 5L^4 C l_i = 4\delta^3 \delta_i$$

where, we put $C_{;i} = \frac{\partial C}{\partial y^i}$ and $\delta_i = \frac{\partial \delta}{\partial y^i}$

Again, differentiating above equation with respect to $y^j$ and using well-known relation, $h_{ij} = g_{ij} - l_i l_j$, we have

$$g_{ij} = \frac{1}{5L^3 C}\left[4\delta^2\left(\delta\delta_{ij} + 3\delta_i\delta_j\right) - \left\{L^5 C_{;i;j} + 5L^4\left(l_i C_{;j} + l_j C_{;i}\right) + 15L^3 C l_i l_j\right\}\right] \tag{11}$$

$$C_{ijk} = \frac{1}{10L^3 C}\left\{4\delta\left(\delta^2\delta_{ijk} + 3\delta \mathrm{e}_{ijk}\delta_i\delta_{jk} + 6\delta_i\delta_j\delta_k\right) - L^5 C_{;i;j;k}\right\} \tag{12}$$

$$-\frac{1}{2L^2 C}(L^2 \mathrm{e}_{ijk} C_{;i;j} l_k + L\mathrm{e}_{ijk} h_{ij} C_{|k} + 4L\mathrm{e}_{ijk} C_{|i} l_j l_k$$

$$+3C\epsilon_{ijk}h_{ij}l_k+12Cl_il_jl_k)$$

Rearranging the terms , above equation can also be rewritten as :

$$C_{ijk}=\frac{1}{10L^3C}\left\{4L\left(L^2a_{ijk}+3LC_{ijk}a_ia_{jk}+6a_ia_ja_k\right)-L^5C_{;i;j;k}\right\}$$
$$-\frac{1}{2L^2C}(L^2\epsilon_{ijk}C_{;i;j}l_k+L\epsilon_{ijk}h_{ij}C_{|k}+4L\epsilon_{ijk}C_{|i}l_jl_k+3C\epsilon_{ijk}h_{ij}l_k+12Cl_il_jl_k) \qquad (13)$$

Taking h-derivative with respect to $y^h$ of above equation, we get

$$C_{ijk|h}=\frac{2}{5L^2C}[L^2a_{ijk|h}+3L\epsilon_{ijk}\left(a_{i|h}a_{jk}+a_ia_{jk|h}\right)+6\epsilon_{ijk}a_{i|h}a_ja_k]-\frac{L^2}{10C}C_{i;j;k|h}$$
$$-\frac{1}{2L^2C}(L^2\epsilon_{ijk}l_kC_{i;j|h}+L\epsilon_{ijk}h_{ij}C_{k|h}+4L\epsilon_{ijk}C_{;i|h}l_jl_k \qquad (14)$$
$$+2C_{|h}l_il_jl_k+3C_{|h}\epsilon_{ijk}h_{ij}l_k+CC_{|h}C_{ijk})$$

Contracting above equation by $y^h$, we get

$$P_{ijk}=C_{ijk|0}=\frac{2}{5L^2C}[L^2a_{ijk|0}+3L\epsilon_{ijk}\left(a_{i|0}a_{jk}+a_ia_{jk|0}\right)$$
$$+6\epsilon_{ijk}a_{i|0}a_ja_k]-\frac{L^2}{10C}C_{i;j;k|0}$$
$$-\frac{1}{2L^2C}(L^2\epsilon_{ijk}l_kC_{i;j|0}+L\epsilon_{ijk}0_{ij}C_{k|0}+4L\epsilon_{ijk}C_{;i|0}l_jl_k \qquad (15)$$
$$+2C_{|0}l_il_jl_k+3C_{|0}\epsilon_{ijk}0_{ij}l_k+CC_{|0}C_{ijk})$$

where, $P_{ijk}$ is the (v)hv-torsion tensorand the index '0' means the contraction by $y^i$i.e. $C_{ijk|h}y^h=C_{ijk|h}$.

If, we put $C_{ijk|h}=0$ and $P_{ijk}=0$ respectively, then we obtain

$$\frac{2}{5L^2C}\{L^2a_{ijk|h}+3L\epsilon_{ijk}\left(a_{i|h}a_{jk}+a_ia_{jk|h}\right)+6\epsilon_{ijk}a_{i|h}a_ja_k\} \qquad (16)$$
$$-\frac{L^2}{10C}C_{i;j;k|h}-\frac{1}{2L^2C}(L^2\epsilon_{ijk}l_kC_{i;j|h}+L\epsilon_{ijk}h_{ij}C_{k|h}+4L\epsilon_{ijk}C_{;i|h}l_jl_k$$

$$+2C_{|h}l_i l_j l_k + 3C_{|h}\mathrm{e}_{ijk}h_{ij}l_k + CC_{|h}C_{ijk}) = 0$$

$$\frac{2}{5L^2C}\{L^2 a_{ijk|0} + 3L\mathrm{e}_{ijk}\left(a_{i|0}a_{jk} + a_i a_{jk|0}\right) + 6\mathrm{e}_{ijk}a_{i|0}a_j a_k\} \tag{17}$$

$$-\frac{L^2}{10C}C_{i;j;k|0} - \frac{1}{2L^2C}(L^2\mathrm{e}_{ijk}l_k C_{i;j|0} + L\mathrm{e}_{ijk}0_{ij}C_{k|0} = 0$$

So, we have

**Theorem 3.1** The necessary and sufficient condition for a Finsler space with torsion scalar C such thatL$^5$C = δ$^4$(δ$^4$ = $a_{ijkl}y^iy^jy^ky^l$)Berwald space, equation (16) holds good.

**Theorem 3.2** The necessary and sufficient condition for a Finsler space with torsion scalar C such that L$^5$C = δ$^4$(δ$^4$ = $a_{ijkl}y^iy^jy^ky^l$) Landsberg space, equation (16) holds good.

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