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A GENERAL VOLTERRA-TYPE INTEGRAL EQUATION ASSOCIATED WITH AN INTEGRAL OPERATOR INVOLVING THE PRODUCT OF GENERAL CLASS OF POLYNOMIALS AND MULTI-VARIABLE H-FUNCTION IN THE KERNEL

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Abstract: In this paper, we solve a general Volterra-type fractional equation associated with an integral operator involving a product of general class of polynomials and the multivariable H-function in its Kernel. We make use of convolution technique to solve the main equation. On account of the general nature of multivariable H-function and general class of polynomials, we can obtain a large number of integral equations involving products of several useful polynomials and special functions as its special cases. For the lack of space, we record here only two such special cases which involve the product of general class of polynomials S_N^M & Appell's function F_3 and a general class of polynomials. The main result derived in this paper also generalizes the results obtained by Gupta et al. [2] and Jain [3].

Keywords and Phrases: Convolution integral equation; multivariable H-function; General Class of Polynomials; Laplace Transform.
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1. Introduction

A GENERAL CLASS OF POLYNOMIALS

Srivastava has introduced the general class of polynomials [4, p.1, Eq. (1)]

$$S_N^M[x] = \sum_{R=0}^{[N/M]} \frac{(-N)_{MR}\, A_{N,R}}{R\,!}\, x^R \quad (N = 0, 1, 2 \ldots) \tag{1}$$

where M is an arbitrary positive integer, and the coefficients $A_{N,R}$ $(N, R \geq 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{N,R}$, $S_N^M[x]$ yields a number of known polynomials as its special cases. These include, among others, Jacobi polynomial, Laguerre polynomial and several others [8, p.158-161].

MULTIVARIABLE H-FUNCTION

A special cases of the H-function of r variables is defined as follows: [7, p.271, Eq.(4.1)]

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = H^{0,0:1,n_1;\cdots;1,n_r}_{p,q:p_1,q_1;\cdots;p_r,q_{r+1}}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,p} : \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q} : \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1\cdots\xi_r)\prod_{i=1}^{r}\left\{\phi_i(\xi_i)z_i^{\xi_i}\right\}\Gamma(-\xi_1)\ldots\Gamma(-\xi_r)d\xi_1\ldots d\xi_r \tag{2}$$

where(i = 1,…,r) $\omega = \sqrt{-1}$

Or equivalently [6, p.64, Eq.(1.3)]

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = \sum_{k_1,\ldots,k_r=0}^{\infty}\prod_{i=1}^{r}\frac{\phi_i(k_i)(-z_i)^{k_i}}{k_i\,!}\psi(k_1\ldots k_r) \tag{3}$$

$$\phi(k_1,\ldots,k_r) = \frac{1}{\prod_{j=1}^{q}\Gamma(1-b_j+\sum_{j=1}^{r}\beta_j^{(i)}k_i)\ \prod_{j=1}^{p}\Gamma(a_j-\sum_{j=1}^{r}\alpha_j^{(i)}k_i)} \tag{4}$$

$$\phi_i(k_i) = \frac{\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}k_i)}{\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}k_i)\prod_{j=1}^{q_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}k_i)},\ (i=1,2,\ldots.r) \tag{5}$$

For the convergence, existence conditions and other details of other details of the multivariable H-function refer the [5.p.252-253, Eq.(C.4-C.6)].

2. Main Result

A general Volterra-type integral equation associated with an integral operator involving a product of generalclass of polynomials and multivariable H-function in its kernel is given by:

$$\int_0^x (x-t)^{\beta-1} S_N^M H \begin{bmatrix} z_1(x-t) \\ \vdots \\ z_r(x-t) \end{bmatrix} y(t)dt + \frac{a}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} y(t)dt = g(x) \tag{6}$$

$\mathrm{Re}(\beta, \nu) > 0$

has the solution

$$y(x) = \int_0^x (x-t)^{l-\beta-\mu-1} \sum_{j=0}^{\infty} \frac{E_j (x-t)^j}{\Gamma(j+l-\beta-\mu)} g^{(l)}(t)dt \tag{7}$$

$\mathrm{Re}(l-\beta-\mu) > 0$

provided that

$g^{(i)}(0) = 0$ for $0 \le i \le l-1$, l being a positive integer and $\nu - \beta$ is an integer. Also,

$$E_j = (-1)^j (\lambda)^{-j-1} \det \begin{bmatrix} \lambda_{\mu+1} & \lambda_{\mu} & \cdots & \cdots & 0 & 0 \\ \lambda_{\mu+2} & \lambda_{\mu+1} & & & & \\ \vdots & . & & & & \\ \lambda_{\mu+\nu-\beta} + a & . & & & & \\ \vdots & . & & & & \\ \lambda_{\mu+j} & \lambda_{\mu+j-1} & \cdots & \lambda_{\mu+\nu-\beta} + a & \cdots & \lambda_{\mu+1} \end{bmatrix} \tag{8}$$

and μ is the least B for which

$$\lambda_B = (-1)^B \sum_{k_1+k_2+\ldots+k_{r+1}=B} \square(k_1, \ldots, k_{r+1}) \frac{z_1^{k_1}}{k_1\,!} \cdots \frac{z_{r+1}^{k_{r+1}}}{k_{r+1}\,!} \tag{9}$$

where

$$\square(k_1, \ldots, k_{r+1}) = \Phi_1(k_1) \ldots \Phi_{r+1}(k_{r+1}) \varphi(k_1, \ldots, k_{r+1}) \tag{10}$$

$$\varphi(k_1, \ldots k_{r+1}) = \Gamma(\beta + k_1 + \ldots + k_{r+1}) \left\{ \prod_{j=1}^{p} \Gamma\left(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} k_i \right) \prod_{j=1}^{q} \Gamma\left(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} k_i \right) \right\}^{-1} \tag{11}$$

$$\Phi_i(k_i) = \prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} k_i) \left\{ \prod_{j=n_i+1}^{p_i} \Gamma\left(c_j - \gamma_j^{(i)} k_i \right) \prod_{j=1}^{q} \Gamma\left(1 - d_j + \delta_j^{(i)} k_i \right) \right\}^{-1} \quad (i=1,\ldots,r) \tag{12}$$

$$\Phi_{r+1}(k_{r+1}) = \begin{cases} (-N)_M k_{r+1} A_{N,k_{r+1}}, 0 \le k_{r+1} \le \left[\frac{N}{M}\right] \\ 0, k_{r+1} > \left[\frac{N}{M}\right] \end{cases} \tag{13}$$

Proof. To solve (6) we first take Laplace transform [1] of its both sides as a result of which the following is obtained:

$$\int_0^\infty e^{-sx} x^{\beta-1} S_N^M[-z_{r+1}x] H\begin{bmatrix} z_1 x \\ \\ z_r x \end{bmatrix} dx Y(s) + as^{-\nu} Y(s) = G(s) \tag{14}$$

Now expressing the $S_N^M[-z_{r+1}x]$ and by $H\begin{bmatrix} z_1 x \\ \vdots \\ \vdots \\ z_r x \end{bmatrix}$ involved in (14) in series using (1) and (3), changing the order of series and integration and evaluating the x-integral, we obtain

$$\left[\sum_{k_1,\dots,k_r=0}^{\infty} \Delta(k_1 \dots k_{r+1}) \frac{(-z_1)^{k_1}}{k_1!} \dots \frac{(-z_{r+1})^{k_{r+1}}}{k_{r+1}!} s - \beta - (k_1 \dots k_{r+1}) + as^{-\nu}\right] Y(s) = G(s) \tag{15}$$

where $\Delta(k_1, \dots k_{r+1})$ is defined by (10). Re-writing (15), we get

$$s^{-\beta}\left[\sum_{B=0}^{\infty} \lambda_B s^{-B} + as^{-\nu+\beta}\right] Y(s) = G(s) \tag{16}$$

where λ_B is given by (9). Again, (16) is equivalent to

$$Y(s) = s^{\beta}\left[\sum_{B=0}^{\infty} \lambda_B s^{-B} + as^{-\nu+\beta}\right]^{-1} G(s) \tag{17}$$

If μ denotes the least B for which $\lambda_B \neq 0$, the series given by (17) can be reciprocated. Writing

$$\left[\sum_{B=0}^{\infty} \lambda_{B+\mu} s^{-B} + as^{-\nu+\beta}\right]^{-1} = \sum_{j=0}^{\infty} E_j s^{-j} \tag{18}$$

(17) takes the following form,

$$Y(s)=s^{-\beta-\ell+\mu}\sum_{j=0}^{\infty}E_j s^{-j}\left[s^{\ell}G(s)\right] \tag{19}$$

(19)can be written as

$$L\{y(x);s\}=L\left\{\sum_{j=0}^{\infty}E_j\frac{x^{j+\ell-\mu-\beta-1}}{\Gamma(j+\ell-\mu-\beta)};s\right\}L\left\{g^{(\ell)}(x);s\right\} \tag{20}$$

Now using the convolution theorem in the RHS of (20) we get

$$L\{y(x);s\}=L\left\{\int_0^x\sum_{j=0}^{\infty}E_j\frac{(x-t)^{j+\ell-\mu-\beta-1}}{\Gamma(j+\ell-\mu-\beta)}g^{(\ell)}(t)dt;s\right\} \tag{21}$$

Finally, on taking the inverse of the Laplace transform [1] of both sides of (21) we arrive at the desired result (7).

Remark 1: Note here that for a = 0 in (6) we get the result obtained by Gupta et al. [2].

2.1 SPECIAL CASES

1. If we put r = 2 in (6) and reduce the H-function of two variables thus obtained to Appell's functionF_3 [5, p.89, eq.(6.4.6)] we find after a little simplification that the Volterra-type integral equation given by

$$\begin{aligned}&\int_0^x(x-t)^{\beta-1}S_N^M\left[-z_{r+1}(x-t)\right]F_3\left[c_1^{(1)},c_1^{(2)},c_2^{(1)},c_2^{(2)};b;-z_1(x-t),-z_2(x-t)\right]y(t)dt\\&+\frac{a}{\Gamma(\nu)}\int_0^x(x-t)^{\nu-1}y(t)\,dt=g(x)\end{aligned} \tag{22}$$

has the solution

$$y(x)=\frac{\Gamma(c_1^{(1)})\Gamma(c_1^{(2)})\Gamma(c_2^{(1)})\Gamma(c_2^{(2)})}{\Gamma(b)}\int_0^x(x-t)^{\ell-\beta-\mu-1}\sum_{j=0}^{\infty}\frac{E_j(x-t)^j}{\Gamma(j+\ell-\beta-\mu)}g^{(\ell)}(t)dt \tag{23}$$

where $\mathrm{Re}(\ell-\beta-\mu)>0,|z_1(x-t)|<1,|z_2(x-t)|<1$

provided that $g^{(i)}(0)=0$ for $0\le i\le\ell-1$, ℓ being a positive integer and $\nu-\beta$ is an integer and E_j are given by the relation (18) and μ is least B for which $\lambda_B\neq 0$

$$\lambda_B=(-1)^B\sum_{k_1+k_2+k_3=B}\Delta(k_1,k_2,k_3)\frac{z_1^{k_1}}{k_1!}\frac{z_2^{k_2}}{k_2!}\frac{z_3^{k_3}}{k_3!} \tag{24}$$

Where in (24)

$$\Delta\left(k_1,k_2,k_3\right)=\frac{\Gamma(c_1^{(1)}+k_1)\Gamma(c_1^{(2)}+k_2)\Gamma(c_2^{(1)}+k_1)\Gamma(c_2^{(2)}+k_2)\Gamma(\beta+k_1+k_2+k_3)}{\Gamma(b+k_1+k_2)}\varphi_3(k_3) \quad (25)$$

and

$$\varphi_3(k_3)=\begin{cases}(-N)_M\, k_3 A_{N,k_3},\, 0\le k_3\le\left[\dfrac{N}{M}\right]\\ \qquad 0,\, k_3>\left[\dfrac{N}{M}\right]\end{cases} \quad (26)$$

2. If we put r = 1; p = q = 0; z_2 = -1 in the LHS of (6), and further reduce the Fox's H-function thus obtained to e^{-z1} [5, p.18, eq. (2.6.2)] and let $z_1\to 0$, the Fox's H-function reduces to unity then we arrive at the following special case of (6):

$$\int_0^x (x-t)^{\beta-1}S_N^M\left[(x-t)\right]y(t)dt+\frac{a}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}y\left(t\right)dt=g\left(x\right) \quad (27)$$

has the solution $y(x)=\int_0^x\left(x-t\right)^{\ell-\beta-\mu-1}\sum_{j=0}^{\infty}\frac{E_j(x-t)^j}{\Gamma\left(j+\ell-\beta-\mu\right)}g^{(\ell)}(t)dt$ (28)

where $\operatorname{Re}(\ell-\beta-\mu)>0, \operatorname{Re}(\beta)>0$

provided that $g^{(i)}(0)=0$ for $0\le i\le\ell-1$, ℓ being a positive integer and ν – β is an integer and E_j are given by the relation (8) and μ is least k for which $\lambda_k \neq 0$

$$\lambda_k=\frac{(-N)_M\, kA_{N,k}\Gamma\left(\beta+k\right)}{k!} \quad (29)$$

$$k=0,1,...,\left[\frac{N}{M}\right], N=0,1,2,..$$

Remark 2: If we put a = 0 in (27) we get the result obtained by Jain[3, p. 102-103]

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