Journal of Rajasthan Academy of Physical Sciences
ISSN : 0972-6306; URL : http://raops.org.in
Vol.15, No.3, September, 2016, 139-148

A CHARACTERIZATION THEOREM IN ROTATORY HYDRODYNAMIC TRIPLY DIFFUSIVE CONVECTION WITH VISCOSITY VARIATIONS

Jyoti Prakash[1], Rajeev Kumar[2] and Pankaj Kumar[3]
Department of Mathematics and Statistics, Himachal Pradesh University, Shimla-171005, India.
Email: jpsmaths67@gmail.com, rajeevkumar2012math@gmail.com, pankajthakur28.85@gmail.com

Abstract: The paper mathematically establishes that rotatory hydrodynamic triply diffusive convection with variable viscosity and one of the components as heat with diffusivity κ, cannot manifest itself as oscillatory motions of growing amplitude in an initially bottom heavy configuration if the two concentration Rayleigh numbers R_1 and R_2, the Lewis numbers τ_1 and τ_2 for the two concentrations with diffusivities κ_1 and κ_2 respectively (with no loss of generality $\kappa > \kappa_1 > \kappa_2$), μ_{min} (the minimum value of viscosity μ in the closed interval $[0, 1]$) and the Prandtl number σ satisfy the inequality $R_1 + R_2 \leq \frac{27\pi^4}{4}\left\{\frac{\mu_{min} + \frac{(\tau_1+\tau_2)}{\sigma}}{1+\frac{\tau_1}{\tau_2^2}}\right\}$ provided $D^2\mu$ is positive everywhere and $\frac{(\tau_1+\tau_2)}{\sigma} \leq \mu_{min}$. It is further proved that this result is uniformly valid for the quite general nature of the bounding surfaces.

Keywords: Triply diffusive convection, variable viscosity, concentration Rayleigh number, oscillatory motion, initially bottom heavy configuration and Taylor number.
Mathematical Subject Classification Number:76E06, 76E20.

1. Introduction

The problem of thermosolutal instability, aside from its various applications in the fields of geophysics, astrophysics, oceanography, chemical engineering etc. has received considerable attention due to its complexities as double-diffusive phenomenon. For a broad view of the subject one may be referred to Turner [19], Brandt and Fernando [5], Radko[12], Sharma [14] and Bhatta [4].

These researchers have considered only the case of two component systems. However, it has been recognized later on [6, 9, 10, 17, 20] that there are many situations wherein more than two components are present. Examples of such multiple diffusive convection fluid systems include the solidification of molten alloys, Earth core, geothermally heated lakes, and magmas and their laboratory models and sea water etc. The presence of more than one salt in fluid mixtures is very often requested for describing natural phenomena such as underground water flow, contaminant transport, warming of the stratosphere, acid rain effects. Further a number of technologically important alloys such as nickel- based alloys [10] used in turbine blades and another high-strength applications, containing significant mass fraction of as many as seven metallic elements.

The recently established characterization theorem of Prakash et al. [11], which states that oscillatory motions (neutral or unstable) of growing amplitude cannot manifest in an initially bottom heavy triply diffusive convection whenever the sum of the concentration Rayleigh numbers is less than a critical value, has brought a fresh outlook to the subject matter of triply diffusive convection and paved the way for further theoretical and experimental investigations in this field of enquiry. The summary of Prakash et al.'s [11] characterization theorem is that it provides a classification of the neutral and unstable triply diffusive convection classes namely the bottom heavy class and the top heavy class and strikes a distinction between them by means of characterization theorems which disallow the existence of oscillatory motions in the former class. For the field of applications of the Prakash et al.'s [11]theorem in flows that are of interest in certain fields like geophysics, oceanography, astrophysics etc. it is necessary to extend the classical analysis where in the fluid viscosity is a function of temperature and /or depth because the effects of viscosity variation play an important role in several physical situations in these fields [2, 7, 8, 16, 18]. Since the variation of viscosity of liquids with temperature is extremely rapid, the inclusion of variation effects certainly extends the domain of validity of the existing results in the literature.

The considerations of a temperature dependent viscosity on the pattern of density in the triply convection problems has the limitations that viscosity is a linear function of vertical coordinate which need not necessarily be so in a real physical situation. Therefore, in the governing equations of the problem, we consider viscosity as an arbitrary function of the vertical coordinate which is in accordance with the formulation regarding the role of viscosity in Rayleigh-Taylor instability problem. From the mathematical point of view the resulting differential equations have variable coefficients contrary to the case wherein viscosity is constant and therefore these more general problems introduce extra analytical complexities. In the present paper we make an attempt to mathematically handle these more complex problems in the context of Prakash et al.'s [11]theorem and extend the domain of validity of the earlier results in the literature.

2. Formulation of the problem

An infinite horizontal layer filled with a Boussinesq viscous fluid is statically confined between two horizontal boundaries $z = 0$ and$z = d$ (rotating with uniform angular velocity$\vec{\Omega}$), maintained at constant temperatures T_0 and $T_1 (< T_0)$ and uniform solute concentrationsS_{10}, S_{20} and$S_{11} (< S_{10})$, $S_{21} (< S_{20})$ at the lower and upper boundaries respectively(as shown in Figure 1). Let the origin be taken on the lower boundary $z =$

0with z-axis perpendicular to it. It is further assumed that cross diffusion effects may be neglected.

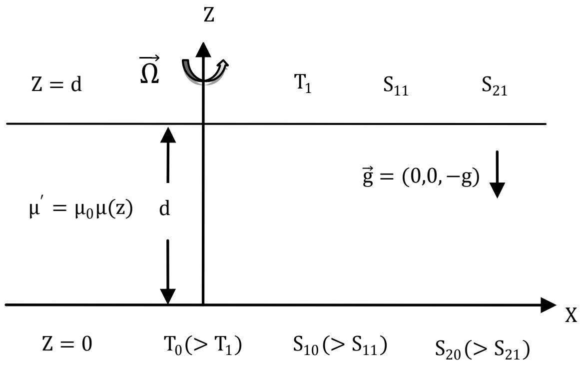

Figure 1: Physical Configuration

The basic hydrodynamic equations that governs rotatory hydrodynamic triply diffusive instability problem in non-dimensional form with variable viscosity are given by [11]

$$\mu(D^2-a^2)^2w + D^2\mu(D^2+a^2)w + 2D\mu D(D^2-a^2)w - \frac{p}{\sigma}(D^2-a^2)w = TD\zeta + Ra^2\theta - R_1a^2\phi_1 - R_2a^2\phi_2, \quad (1)$$

$$(D^2-a^2-p)\theta = -w, \quad (2)$$

$$\left(D^2-a^2-\frac{p}{\tau_1}\right)\phi_1 = -\frac{w}{\tau_1}, \quad (3)$$

$$\left(D^2-a^2-\frac{p}{\tau_2}\right)\phi_2 = -\frac{w}{\tau_2}, \quad (4)$$

and $D(\mu D\zeta) - \left(\mu a^2 + \frac{p}{\sigma}\right)\zeta = -Dw,$ (5)

with $w = 0 = \theta = \phi_1 = \phi_2 = D^2w = D\zeta$ at $z = 0$ and $z = 1$ (both the boundaries are free) (6)

or $w = 0 = \theta = \phi_1 = \phi_2 = Dw = \zeta$ at $z = 0$ and $z = 1$ (both the boundaries are rigid) (7)

or $w = 0 = \theta = \phi_1 = \phi_2 = Dw = \zeta$ at $z = 0$ (lower boundary is rigid)

and $w = 0 = \theta = \phi_1 = \phi_2 = D^2w = D\zeta$ at $z = 1$ (upper boundary is free) (8)

or $w = 0 = \theta = \phi_1 = \phi_2 = D^2w = D\zeta$ at $z = 0$ (lower boundary is free)

and $w = 0 = \theta = \phi_1 = \phi_2 = Dw = \zeta$ at $z = 1$ (upper boundary is rigid), (9)

where z is the vertical coordinate, $D = \frac{d}{dz}$ is the differentiation along the vertical direction, $a^2 > 0$ is the square of the wave number, $\sigma = \frac{\nu}{\kappa} > 0$ is the Prandtlnumber, $\tau_1 = \frac{\kappa_1}{\kappa} > 0$ and

$\tau_2 = \frac{\kappa_2}{\kappa} > 0$are the Lewis numbers for the two concentration components respectively, $R = \frac{g\alpha\beta d^4}{\kappa\nu} > 0$is the Rayleigh number, $R_1 = \frac{g\alpha_1\beta_1 d^4}{\kappa\nu} > 0$and $R_2 = \frac{g\alpha_2\beta_2 d^4}{\kappa\nu} > 0$are the concentration Rayleigh numbers for the concentration componentsS_1 and S_2respectively, $T = \frac{4\Omega^2 d^4}{\nu^2} > 0$ is a Taylor number, $p = p_r + ip_i$is the complex growth rate, w is the vertical velocity,θ is the temperature, ϕ_1 and $\phi_2$are the concentration of two components S_1 and S_2 respectively, w is the vertical velocity,θ is the temperature, ϕ_1 and ϕ_2 are the concentration of two components S_1 and S_2 respectively and $\mu' = \mu_0\mu(z)$ is the viscosity, where μ_0 is constant having the dimensions of viscosity and $\mu(z)$ is twice continuously differentiable function of z and is such that the ratio of the viscosities at the top and bottom boundaries is small [15] andζ is the vorticity. It may further be noted that equations (1)-(9) describe an eigen value problem for p and govern rotatory hydrodynamictriply diffusive convection with variable viscosity for any combination of rigid and dynamically free boundaries.

3.Mathematical Analysis

Now we prove the following theorems:

3.1 Theorem: If $R > 0, R_1 > 0, R_2 > 0, T > 0, \frac{(\tau_1+\tau_2)}{\sigma} \le \mu_{min}, p = p_r + ip_i, p_r \ge 0, p_i \ne 0$, $D^2\mu > 0$ and $R_s = R_1 + R_2 \le \frac{27\pi^4}{4}\left\{\frac{\mu_{min} + \frac{(\tau_1+\tau_2)}{\sigma}}{1+\frac{\tau_1}{\tau_2^2}}\right\}$, then a necessary condition for the existence of nontrivial solution $(w, \theta, \phi_1, \phi_2, \zeta, p)$ of equations (1)-(5) together with boundary conditions (6)-(9) is that $R_s = R_1 + R_2 < R$.

Proof: Equation (1) can further be simplified as

$$D(\mu D^3 w + D\mu D^2 w - 2a^2\mu Dw) + a^4\mu w - \frac{p}{\sigma}(D^2 - a^2)w + a^2(D^2\mu)w = Ra^2\theta - R_1 a^2\phi_1 - R_2 a^2\phi_2 + TD\zeta. \quad (10)$$

Multiplying both sides of equation (10) by w^*(the superscript $*$ henceforth denotes complex conjugation), integrating the resulting equation over the vertical range of z, we get

$$\int_0^1 w^* D(\mu D^3 w + D\mu D^2 w - 2a^2\mu Dw)dz + a^4\int_0^1 \mu|w|^2 dz - \frac{p}{\sigma}\int_0^1 w^*(D^2 - a^2)w dz + a^2\int_0^1 w^*(D^2\mu)w dz = Ra^2\int_0^1 w^*\theta dz - R_1 a^2\int_0^1 w^*\phi_1 dz - R_2 a^2\int_0^1 w^*\phi_2 dz + T\int_0^1 w^* D\zeta. \quad (11)$$

Making use of equations (2)-(5), we can write

$$\int_0^1 w^* D(\mu D^3 w + D\mu D^2 w - 2a^2\mu Dw)dz + a^4\int_0^1 \mu|w|^2 dz - \frac{p}{\sigma}\int_0^1 w^*(D^2 - a^2)w dz + a^2\int_0^1 w^*(D^2\mu)w dz = -Ra^2\int_0^1 \theta(D^2 - a^2 - p^*)\theta^* dz + R_1 a^2\tau_1\int_0^1 \phi_1\left(D^2 - a^2 - \frac{p^*}{\tau_1}\right)\phi_1^* dz + R_2 a^2\tau_2\int_0^1 \phi_2\left(D^2 - a^2 - \frac{p^*}{\tau_2}\right)\phi_2^* dz + T\int_0^1 \zeta\left[D(\mu D\zeta^*) - \left(\mu a^2 + \frac{p^*}{\sigma}\right)\zeta^*\right]dz. \quad (12)$$

Integrating the various terms, by parts, for an appropriate number of times and making use of the boundary conditions (6)-(9), we get

$$\int_0^1 \mu\left(|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2\right)dz + \frac{p}{\sigma}\int_0^1\left(|Dw|^2 + a^2|w|^2\right)dz + a^2\int_0^1 D^2\mu\,|w|^2dz = Ra^2\int_0^1\left(|D\theta|^2 + a^2|\theta|^2 + p^*|\theta|^2\right)dz - R_1a^2\tau_1\int_0^1\left(|D\phi_1|^2 + a^2|\phi_1|^2 + \frac{p^*}{\tau_1}|\phi_1|^2\right)dz - R_2a^2\tau_2\int_0^1\left(|D\phi_2|^2 + a^2|\phi_2|^2 + \frac{p^*}{\tau_2}|\phi_2|^2\right)dz - T\int_0^1\left[\mu\left(|D\zeta|^2 + a^2|\zeta|^2\right) + \frac{p^*}{\sigma}|\zeta|^2\right]dz. \quad (13)$$

Equating the real and imaginary parts of both sides of equation (13) and cancelling $p_i(\neq 0)$throughout from the imaginary part, we have

$$\int_0^1 \mu\left(|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2\right)dz + \frac{p_r}{\sigma}\int_0^1\left(|Dw|^2 + a^2|w|^2\right)dz + a^2\int_0^1 D^2\mu\,|w|^2dz = Ra^2\int_0^1\left(|D\theta|^2 + a^2|\theta|^2 + p_r|\theta|^2\right)dz - R_1a^2\tau_1\int_0^1\left(|D\phi_1|^2 + a^2|\phi_1|^2 + \frac{p_r}{\tau_1}|\phi_1|^2\right)dz - R_2a^2\tau_2\int_0^1\left(|D\phi_2|^2 + a^2|\phi_2|^2 + \frac{p_r}{\tau_2}|\phi_2|^2\right)dz - T\int_0^1\left[\mu\left(|D\zeta|^2 + a^2|\zeta|^2\right) + \frac{p_r}{\sigma}|\zeta|^2\right]dz. \quad (14)$$

and

$$\frac{1}{\sigma}\int_0^1\left(|Dw|^2 + a^2|w|^2\right)dz = Ra^2\int_0^1|\theta|^2dz + R_1a^2\int_0^1|\phi_1|^2dz + R_2a^2\int_0^1|\phi_2|^2dz + \frac{T}{\sigma}\int_0^1|\zeta|^2dz. \quad (15)$$

Rearranging equation (14), we get

$$\int_0^1 \mu\left(|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2\right)dz + \frac{p_r}{\sigma}\int_0^1\left(|Dw|^2 + a^2|w|^2\right)dz + a^2\int_0^1 D^2\mu\,|w|^2dz = Ra^2\int_0^1\left(|D\theta|^2 + a^2|\theta|^2\right)dz - R_1a^2\tau_1\int_0^1\left(|D\phi_1|^2 + a^2|\phi_1|^2\right)dz - R_2a^2\tau_2\int_0^1\left(|D\phi_2|^2 + a^2|\phi_2|^2\right)dz - T\int_0^1\left[\mu\left(|D\zeta|^2 + a^2|\zeta|^2\right)\right]dz + p_r\left(Ra^2\int_0^1|\theta|^2dz - R_1a^2\int_0^1|\phi_1|^2dz - R_2a^2\int_0^1|\phi_2|^2dz - \frac{T}{\sigma}\int_0^1|\zeta|^2dz\right). \quad (16)$$

We derive the validity of the theorem from the inequality obtained by replacing each one of the terms of (16)by its appropriate estimate.

Utilizing Rayleigh-Ritz's inequality [13], we have

$$\int_0^1|Dw|^2dz \geq \pi^2\int_0^1|w|^2dz, \quad (17)$$

$$\int_0^1|D\theta|^2dz \geq \pi^2\int_0^1|\theta|^2dz, \quad (18)$$

$$\int_0^1|D\phi_1|^2dz \geq \pi^2\int_0^1|\phi_1|^2dz, \quad (19)$$

and $\int_0^1|D\phi_2|^2dz \geq \pi^2\int_0^1|\phi_2|^2dz$. (20)

Further, we also have [1]

$$\int_0^1 |D^2 w|^2\, dz \geq \pi^4 \int_0^1 |w|^2\, dz, \tag{21}$$

and thus making use of inequalities (17) and (21), we get

$$\int_0^1 \mu\,(|D^2 w|^2 + 2a^2 |Dw|^2 + a^4 |w|^2) dz \geq \mu_{min}\,(\pi^2 + a^2)^2 \int_0^1 |w|^2\, dz, \tag{22}$$

whereμ_{min} is the minimum value of μ in the closed interval $[0, 1]$.

Since $p_r \geq 0$, we get

$$\frac{p_r}{\sigma} \int_0^1 (|Dw|^2 + a^2 |w|^2)\, dz \geq 0. \tag{23}$$

Let $D^2 \mu > 0$, we have

$$a^2 \int_0^1 (D^2 \mu)\, |w|^2 dz > 0. \tag{24}$$

Now multiplying equation (2) byθ^*, integrating the resulting equation over the vertical range of z for an appropriate number of times and utilizing the boundary conditions on θ, we have from the real part of final equation

$$\int_0^1 (|D\theta|^2 + a^2 |\theta|^2)\, dz + p_r \int_0^1 |\theta|^2\, dz = \text{Real part of} \left(\int_0^1 \theta^* w dz \right) \leq \left| \int_0^1 \theta^* w dz \right|$$

$$\leq \int_0^1 |\theta^* w| dz \leq \int_0^1 |\theta^*||w| dz \leq \int_0^1 |\theta||w| dz \leq \left(\int_0^1 |\theta|^2 dz \right)^{\frac{1}{2}} \left(\int_0^1 |w|^2 dz \right)^{\frac{1}{2}}. \tag{25}$$

(using Schwartz Inequality)

Combining inequality (25) with inequality (18), and the fact that $p_r \geq 0$, we get

$$(\pi^2 + a^2) \int_0^1 |\theta|^2\, dz \leq \left(\int_0^1 |\theta|^2 dz \right)^{\frac{1}{2}} \left(\int_0^1 |w|^2 dz \right)^{\frac{1}{2}},$$

which gives that

$$\left(\int_0^1 |\theta|^2 dz \right)^{\frac{1}{2}} \leq \frac{1}{(\pi^2 + a^2)} \left(\int_0^1 |w|^2 dz \right)^{\frac{1}{2}}$$

and hence

$$\int_0^1 (|D\theta|^2 + a^2 |\theta|^2)\, dz \leq \frac{1}{(\pi^2 + a^2)} \left(\int_0^1 |w|^2 dz \right). \tag{26}$$

Now for the case of rigid boundaries,$\zeta(0) = 0 = \zeta(1)$, again by Rayleigh - Ritz inequality [13] we obtain

$$\int_0^1 |D\zeta|^2 dz \geq \pi^2 \int_0^1 |\zeta|^2 dz. \tag{27}$$

Making use of equation (15), inequalities (19), (20), (27) and the result $\int_0^1 |D\zeta|^2 dz = \pi^2 \int_0^1 |\zeta|^2 dz$ ([3]), it follows that

$$R_1 a^2 \tau_1 \int_0^1 (|D\phi_1|^2 + a^2|\phi_1|^2)\,dz + R_2 a^2 \tau_2 \int_0^1 (|D\phi_2|^2 + a^2|\phi_2|^2)\,dz + T\int_0^1 [\mu(|D\zeta|^2 + a^2|\zeta|^2)]dz$$

$$\geq (\pi^2 + a^2)\left\{R_1 a^2 \tau_1 \int_0^1 |\phi_1|^2 dz + R_2 a^2 \tau_2 \int_0^1 |\phi_2|^2 dz + T\mu_{min}\int_0^1 |\zeta|^2 dz\right\}$$

$$\geq (\pi^2 + a^2)\left\{\frac{\tau_1}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)dz - R_2 a^2\tau_1 \int_0^1 |\phi_2|^2 dz - \frac{T\tau_1}{\sigma}\int_0^1 |\zeta|^2 dz + \frac{\tau_2}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)dz - R_1 a^2 \tau_2 \int_0^1 |\phi_1|^2 dz - \frac{T\tau_2}{\sigma}\int_0^1 |\zeta|^2 dz + T\mu_{min}\int_0^1 |\zeta|^2 dz\right\}$$

$$\geq (\pi^2 + a^2)^2 \frac{(\tau_1+\tau_2)}{\sigma}\int_0^1 |w|^2 dz - (\pi^2 + a^2)\left(R_2 a^2 \tau_1 \int_0^1 |\phi_2|^2 dz + R_1 a^2 \tau_2 \int_0^1 |\phi_1|^2 dz\right) - (\pi^2 + a^2)T\left\{\frac{(\tau_1+\tau_2)}{\sigma} - \mu_{min}\right\}\int_0^1 |\zeta|^2 dz. \quad (28)$$

Thus $-R_1 a^2 \tau_1 \int_0^1 (|D\phi_1|^2 + a^2|\phi_1|^2)\,dz - R_2 a^2 \tau_2 \int_0^1 (|D\phi_2|^2 + a^2|\phi_2|^2)\,dz - T\int_0^1 [\mu(|D\zeta|^2 + a^2|\zeta|^2)]dz \leq$

$$-(\pi^2 + a^2)^2 \frac{(\tau_1+\tau_2)}{\sigma}\int_0^1 |w|^2 dz + (\pi^2 + a^2)\left(R_2 a^2 \tau_1 \int_0^1 |\phi_2|^2 dz + R_1 a^2 \tau_2 \int_0^1 |\phi_1|^2 dz\right) - (\pi^2 + a^2)T\left\{\mu_{min} - \frac{(\tau_1+\tau_2)}{\sigma}\right\}\int_0^1 |\zeta|^2 dz. \quad (29)$$

Now multiplying equations (3) by its complex conjugate, integrating the resulting equation over the vertical range of z for an appropriate number of times and utilizing the boundary conditions on ϕ_1, we have

$$\int_0^1 (|D^2\phi_1|^2 + 2a^2|D\phi_1|^2 + a^4|\phi_1|^2)dz + \frac{2p_r}{\tau_1}\int_0^1 (|D\phi_1|^2 + a^2|\phi_1|^2)\,dz + \frac{|p|^2}{\tau_1^2}\int_0^1 |\phi_1|^2\,dz = \frac{1}{\tau_1^2}\int_0^1 |w|^2\,dz. \quad (30)$$

We can easily show that ([1])

$$\int_0^1 |D^2\phi_1|^2 dz \geq \pi^4 \int_0^1 |\phi_1|^2 dz\,. \quad (31)$$

Further, since $p_r \geq 0$,making use of inequalities (19) and (31) in equation (30), we have

$$\int_0^1 |\phi_1|^2 dz < \frac{1}{\tau_1^2(\pi^2+a^2)^2}\int_0^1 |Dw|^2\,dz. \quad (32)$$

In the same manner by using (4), we obtain

$$\int_0^1 |\phi_2|^2 dz < \frac{1}{\tau_2^2(\pi^2+a^2)^2} \int_0^1 |Dw|^2\, dz. \quad (33)$$

Now using inequalities (32) and (33), we have

$$R_2 a^2 \tau_1 \int_0^1 |\phi_2|^2 dz + R_1 a^2 \tau_2 \int_0^1 |\phi_1|^2 dz \le \frac{a^2}{(\pi^2+a^2)^2}\left(\frac{R_2\tau_1^3 + R_1\tau_2^3}{\tau_1^2\tau_2^2}\right) \int_0^1 |w|^2 dz\,. \quad (34)$$

Making use of inequality (34) in inequality (29), we have

$$-R_1 a^2 \tau_1 \int_0^1 (|D\phi_1|^2 + a^2|\phi_1|^2)\, dz - R_2 a^2 \tau_2 \int_0^1 (|D\phi_2|^2 + a^2|\phi_2|^2)\, dz - T\int_0^1 [\mu(|D\zeta|^2 + a^2|\zeta|^2)] dz \le -(\pi^2+a^2)^2 \frac{(\tau_1+\tau_2)}{\sigma} \int_0^1 |w|^2 dz + \frac{a^2}{(\pi^2+a^2)^2}\left(\frac{R_2\tau_1^3 + R_1\tau_2^3}{\tau_1^2\tau_2^2}\right) \int_0^1 |w|^2 dz - (\pi^2+a^2) T\left\{\mu_{min} - \frac{(\tau_1+\tau_2)}{\sigma}\right\} \int_0^1 |\zeta|^2 dz. \quad (35)$$

Let $R_s = R_1 + R_2$, thus $R_2 = R_s - R_1$.

So that $\quad R_2\tau_1^3 + R_1\tau_2^3 = R_s\tau_1^3 + R_1(\tau_2^3 - \tau_1^3)$. (36)

Utilizing the equation (36) and the fact that $\kappa_1 > \kappa_2$, (i.e.$\tau_1 > \tau_2$), inequality (35) implies

$$-R_1 a^2 \tau_1 \int_0^1 (|D\phi_1|^2 + a^2|\phi_1|^2)\, dz - R_2 a^2 \tau_2 \int_0^1 (|D\phi_2|^2 + a^2|\phi_2|^2)\, dz - T\int_0^1 [\mu(|D\zeta|^2 + a^2|\zeta|^2)] dz \le \left\{-(\pi^2+a^2)^2 \frac{(\tau_1+\tau_2)}{\sigma} + \frac{a^2 R_s \tau_1}{(\pi^2+a^2)\tau_2^2}\right\} \int_0^1 |w|^2 dz - (\pi^2+a^2) T\left\{\mu_{min} - \frac{(\tau_1+\tau_2)}{\sigma}\right\} \int_0^1 |\zeta|^2 dz. \quad (37)$$

Also from equation (15) and the fact $p_r \ge 0$, we have

$$p_r\left(Ra^2 \int_0^1 |\theta|^2\, dz - R_1 a^2 \int_0^1 |\phi_1|^2\, dz - R_2 a^2 \int_0^1 |\phi_2|^2\, dz - \frac{T}{\sigma}\int_0^1 |\zeta|^2 dz\right) < 0. \quad (38)$$

Now, if permissible, let $R_s = R_1 + R_2 \ge R$. Then, in that case, we derive from equation (16) and the inequalities (22)-(24), (37) and (38) that

$$\left[(\pi^2+a^2)^2\left(\mu_{min} + \frac{(\tau_1+\tau_2)}{\sigma}\right) - \frac{R_s a^2}{(\pi^2+a^2)}\left(1 + \frac{\tau_1}{\tau_2^2}\right)\right] \int_0^1 |w|^2 dz + (\pi^2+a^2) T\left\{\mu_{min} - \frac{(\tau_1+\tau_2)}{\sigma}\right\} \int_0^1 |\zeta|^2 dz < 0. \quad (39)$$

Since $\frac{(\tau_1+\tau_2)}{\sigma} \le \mu_{min}$, inequality (39) implies that

$$R_s > \frac{(\pi^2+a^2)^3}{a^2}\left(\frac{\mu_{min} + \frac{(\tau_1+\tau_2)}{\sigma}}{1+\frac{\tau_1}{\tau_2^2}}\right),$$

so that we necessarily have $R_s > \frac{27\pi^4}{4}\left\{\frac{\mu_{min} + \frac{(\tau_1+\tau_2)}{\sigma}}{1+\frac{\tau_1}{\tau_2^2}}\right\}$, (40)

since the minimum value of $\frac{(\pi^2+a^2)^3}{a^2}$ is $\frac{27\pi^4}{4}$ (for $a^2 = \frac{\pi^2}{2}$).

Hence if $R_s = R_1 + R_2 \leq \frac{27\pi^4}{4}\left\{\frac{\mu_{min} + \frac{(\tau_1+\tau_2)}{\sigma}}{1+\frac{\tau_1}{\tau_2^2}}\right\}$, then we must have $R_s = R_1 + R_2 < R$.

This establishes the theorem.

The essential content of the theorem from the physical point of view are that rotatory hydrodynamic triply diffusive convection with one of the components as heat with diffusivity κ cannot manifest as oscillatory motions of growing amplitude in an initially bottom heavy configuration if the two concentration Rayleigh numbers R_1 and R_2, the Lewis numbers τ_1 and τ_2 for the two concentrations with diffusivities κ_1 and κ_2 respectively (with no loss of generality $\kappa > \kappa_1 > \kappa_2$) and the Prandtl number σ satisfy the inequality $R_1 + R_2 \leq \frac{27\pi^4}{4}\left\{\frac{\mu_{min} + \frac{(\tau_1+\tau_2)}{\sigma}}{1+\frac{\tau_1}{\tau_2^2}}\right\}$ provided $D^2\mu$ is positive everywhere and $\frac{(\tau_1+\tau_2)}{\sigma} \leq \mu_{min}$. It is further established that this result is uniformly valid for the quite general nature of the bounding surfaces.

3.2 Note: For the constant viscosity case it is clear that $\mu' = \mu_0\mu(z)$ where $\mu(z) = 1$ for all z; therefore $\mu_{min}(z) = 1$ and we recover the result due to Prakash et al. [11] from the above theorem.

4. Conclusion: Linear stability analysis of rotatory hydrodynamic triply diffusive convection with variable viscosity has been performed. A classification of the neutral or unstable rotatory hydrodynamic triply diffusive convection configuration with variable viscosity has been made into two classes namely, the bottom heavy class and top heavy class and then a distinction between these classes is made by means of a characterization theorem which disallow the existence of oscillatory motions in the former class. The work done in the present paper will certainly pave the way for further theoretical and experimental investigation in this field of enquiry.

Acknowledgement: The authors are thankful to the Referee for valuable suggestions.

References

[1] Banerjee, M. B., Gupta, J. R. and Prakash, J. (1992). Upper limits to the complex growth rate in thermohaline instability, *J. Math. Anal. Appl.* **167 (1)**, 66-73.

[2] Banerjee, M. B., Gupta, J. R. and Shandil, R. G. (1977). Generalized thermal convection with viscosity variations, *J. Math. Phys.Soc.* **11**, 421-442.

[3] Banerjee, M.B., Shandil, R.G., Lal, P. and Kumar, V. (1995). A mathematical theorem in rotatory thermohaline convection, *J. Math. Anal. Appln*. **189**, 351-361.

[4] Bhatta, D. (2013). Hydro-thermal convective solutions for an aquifer system heated from below, *Appl. Appl. Math*. **8(1)**, 1-15.

[5] Brandt, A. and Fernando, H. J. S. (1996). *Double Diffusive Convection*, Am. Geophys. Union, Washington, DC.

[6] Griffiths, R. W. (1979). The influence of a third diffusing component upon the onset of Convection, *J. Fluid Mech.* **92**, 659-670.

[7] Hooman, K. and Gurgenci, H. (2008). Effects of temperature dependent viscosity on benard convection in a porous medium using a non-Darcy model, *Int. J. Heat Mass Transfer* **51(5-6)**, pp. 1139-1149.

[8] Korenga, J. and Jordhan, T. H. (2002). Onset of convection with temperature and depth dependent Viscosity, *Geophysical Research Letters* **29(19)**, 1923 doi:10.1029 /2002 GL 015672.

[9] Lopez, A. R., Romero, L. A. and Pearlstein, A. J. (1990). Effect of rigid boundaries on the onset of convective instability in a triply diffusive fluid layer, *Physics of fluids A*. **2(6)**, 897-902.

[10] Pearlstein, A. J., Harris, R. M. and Terrones, G. (1989).The onset of convective instability in a triply diffusive fluid layer, *J. Fluid Mech.* **202,** 443-465.

[11] Prakash J., Vaid K., Bala R. and Kumar V. (2015). Characterization of rotatory hydrodynamic triply diffusive convection, *Z. Angew. Math. Phys.* **66**, 2665-2675.

[12] Radko, T. (2013). *Double Diffusive Convection*, Cambridge University Press.

[13] Schultz, M. H. (1973). *Spline Analysis*, Prentice-Hall Inc., Englewood Cliffs., NJ.

[14] Sharma, H. M. (2010). Bound for the Complex Growth Rate in Thermosolutal Convection Coupled with Cross-diffusions, *Appl. Appl. Math.* **05 (10)**, 1428 – 1441.

[15] Stengel, K. C., Oliver, D. S. and Booker, J. R. (1982). Onset of convection in variable viscosity Fluid, *J. Fluid Mech*. **120**, 411-431.

[16] Sunil and Choudhary, S. (2013). Global stability for thermal convection in a couple stress fluid with temperature and pressure dependent viscosity, *Studia Geotechanica et Mechanica***35(3)**, doi:10.2478/sgem-2013-0032.

[17] Terrones, G. (1993). Cross-diffusion effects on the stability criteria in a triply diffusive system, *Phys. Fluids A.* **5(9)**, 2172-2182.

[18] Torrance, K. E. and Turcotte, D. L. (1971).Thermal convection with large viscosityvariations, *J. Fluid. Mech.* **47**, 113-125.

[19] Turner, J. S. (1974). Double-diffusive phenomena, *Ann. Rev. Fluid Mech.* **6**, 37-56.

[20] Turner, J. S. (1985). Multicomponent convection, *Ann. Rev. Fluid Mech*. **17**, 11-44.