Journal of Rajasthan Academy of Physical Sciences
ISSN : 0972-6306; URL : http://raops.org.in
Vol.15, No.3, September, 2016, 117-126

CERTAIN RELATION OF GENERALIZED FRACTIONAL CALCULUS ASSOCIATED WITH THE GENERALIZED MITTAG-LEFFLER FUNCTION

Rajeev Kumar Gupta, Bhupender Singh Shaktawat and Dinesh Kumar
Department of Mathematics & Statistics, Jai Narain Vyas University,
Jodhpur - 342005, India
E-mail: drrkbgupta@yahoo.co.in, b.s.shaktawatmath@gmail.com,
dinesh_dino03@yahoo.com

Abstract: In this paper, we have applied the generalized fractional calculus operators (Saigo-Maeda [13]) involving Appell's function $F_3(.)$, on the generalized Mittag-Leffler function introduced by Prabhakar [11]. Our main results are obtained in the terms of generalized Wright function. Recent results of Ahmed [1], Chaurasia and Pandey [2], Saxena et al. [15-16] are the special cases of our main findings.

Keywords and Phrase: Generalized fractional calculus operators, Generalized Mittag-Leffler function, Generalized Wright hypergeometric function.
Mathematics Subject Classification:Primary: 26A33, 33C45; Secondary: 33E12.

1. Introduction and Preliminaries

The Swedish mathematician Gosta Mittag-Leffler[9] introduced the function $E_\upsilon(z)$ as:

$$E_\upsilon(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\upsilon n+1)},\left(\upsilon\in\mathrm{C},\Re(\upsilon)>0,z\in\mathrm{C}\right), \tag{1}$$

the Mittag-Leffler function (1) is a direct generalization of $\exp(z)$ in which $\upsilon=1.$

The generalization of $E_\upsilon(z)$ was studied by Wiman [17], given by

$$E_{\upsilon,\rho}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\upsilon n+\rho)},\left(\upsilon,\rho\in\mathrm{C},\Re(\upsilon)>0,\Re(\rho)>0,z\in\mathrm{C}\right). \tag{2}$$

In 1971, Prabhakar introduced the function $E_{\upsilon,\rho}^{\delta}(z)$ in the following form (see also, Kilbas et al. [4]):

$$E_{\upsilon,\rho}^{\delta}(z)=\sum_{n=0}^{\infty}\frac{(\delta)_n z^n}{\Gamma(\upsilon n+\rho)n!}, \tag{3}$$

where $\upsilon,\rho,\delta,z\in\mathbb{C},\Re(\upsilon)>0,\Re(\rho)>0,\Re(\delta)>0$, and

$$(\delta)_n=\frac{\Gamma(\delta+n)}{\Gamma(\delta)}, \tag{4}$$

here, $(\delta)_n$ denotes the Pochhammer symbol.

For $\delta=1$ and $\rho=\delta=1$, equation (3) reduces to the generalized Mittag-Leffler function $E_{\upsilon,\rho}(z)$ and Mittag-Leffler function $E_{\upsilon}(z)$ respectively. The generalized Wright hypergeometric function introduced by Wright [5,18] and represented by

$${}_p\psi_q(z)={}_p\psi_q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\frac{z^n}{n!}, \tag{5}$$

where $z,a_i,b_j\in\mathbb{C}$ and $\alpha_i,\beta_j\in\mathbb{R}-\{0\},(i=1,\ldots,p;j=1,\ldots,q)$. Wright proved several theorems on the asymptotic expansion of generalized Wright function ${}_p\Psi_q(z)$ for all values of the argument z under the condition

$$\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i>-1. \tag{6}$$

2. Generalized fractional calculus operators

Let $\alpha,\beta,\gamma\in\mathbb{C}$ and $x\in\mathbb{R}_+$, then the generalized fractional integral and differential operators associated with Gauss hypergeometric function are defined by Saigo [12] for $\Re(\alpha)>0$, as follows:

$$\left(I_{0+}^{\alpha,\beta,\gamma}f\right)(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{t}{x}\right)f(t)dt, \tag{7}$$

$$\left(I_{-}^{\alpha,\beta,\gamma}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{x}{t}\right)f(t)dt, \tag{8}$$

and

$$\left(D_{0+}^{\alpha,\beta,\gamma} f\right)(x) = \left(I_{0+}^{-\alpha,-\beta,\alpha+\gamma} f\right)(x) = \left(\frac{d}{dx}\right)^k \left(I_{0+}^{-\alpha+k,-\beta-k,\alpha+\gamma-k} f\right)(x), \quad (9)$$

$$\left(D_{-}^{\alpha,\beta,\gamma} f\right)(x) = \left(I_{-}^{-\alpha,-\beta,\alpha+\gamma} f\right)(x) = \left(-\frac{d}{dx}\right)^k \left(I_{-}^{-\alpha+k,-\beta-k,\alpha+\gamma} f\right)(x), \quad (10)$$

here $k = [\Re(\alpha)]+1$. If we set $\beta = -\alpha$, then operators (7)-(10) reduce to Riemann-Liouville fractional calculus operators [3].

Let $\alpha,\alpha',\beta,\beta',\gamma \in \mathrm{C}, x > 0$ and $\Re(\gamma) > 0$, then the generalized fractional integral operators involving Appell's function (Horn's function) F_3 are introduced by Saigo and Maeda, as follows (see [13], p.393, eq.(4.12) and (4.13)):

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}\, t^{-\alpha'}\, F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\mathrm{d}t, \quad (11)$$

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma-1}\, t^{-\alpha}\, F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\mathrm{d}t. \quad (12)$$

The generalized fractional differentiation operators [13] involving the Appell's function F_3 as a kernel are defined by

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \quad (13)$$

$$= \left(\frac{d}{dx}\right)^k \left(I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k} f\right)(x), \left(Re(\gamma) > 0; k = [Re(\gamma)]+1\right), \quad (14)$$

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \quad (15)$$

$$= \left(-\frac{d}{dx}\right)^k \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k} f\right)(x), \left(Re(\gamma) > 0; k = [Re(\gamma)]+1\right). \quad (16)$$

If we take $\alpha = \alpha+\beta, \alpha' = \beta' = 0, \beta = -\gamma$ and $\gamma = \alpha$, then operators given in (11)-(16) reduce to the Saigo fractional calculus operators as given by (7)-(10).

Further ([13], p.394, eq.(4.18) and (4.19)), we also have

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\, t^{\rho-1}\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1}\, \Gamma\left[\begin{matrix}\rho,\ \rho+\gamma-\alpha-\alpha'-\beta, \rho+\beta'-\alpha' \\ \rho+\gamma-\alpha-\alpha', \rho+\gamma-\alpha'-\beta, \rho+\beta'\end{matrix}\right], \quad (17)$$

where $Re(\gamma) > 0,\ Re(\rho) > \max[0, Re(\alpha+\alpha'+\beta-\gamma), Re(\alpha'-\beta')]$, and

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\, t^{\rho-1}\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1}\,\Gamma\left[\begin{matrix}1+\alpha+\alpha'-\gamma-\rho, 1+\alpha+\beta'-\gamma-\rho, 1-\beta-\rho\\ 1-\rho, 1+\alpha+\alpha'+\beta'-\gamma-\rho, 1+\alpha-\beta-\rho\end{matrix}\right], \tag{18}$$

where $Re(\gamma) > 0, Re(\rho) < 1+\min\left[Re(-\beta), Re(\alpha+\alpha'-\gamma), Re(\alpha+\beta'-\gamma)\right]$.

Here, the symbol $\Gamma\left[\begin{matrix}a,b,c\\ d,e,f\end{matrix}\right]$ will be used to represent the ratio of product of gamma functions as $\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)}$.

3. Main Results

In this section, we establish the left-sided and right-sided generalized fractional integral and differential formulas of the generalized Mittag-Leffler function. These formulas are given by the following theorems:

3.1. Left-sided generalized fractional integration of generalized Mittag-Leffler function

Theorem 1 *Let* $\alpha,\alpha',\beta,\beta',\mu,\delta,\rho\in \mathbb{C}, \Re(\mu)>0, x>0, \nu>0, \lambda>0$ *and* $a\in\mathbb{R}$. *If the condition (6) is satisfied and* $I_{0+}^{\alpha,\alpha',\beta,\beta',\mu}$ *be the left-sided operator of generalized fractional integration associated with Appell's function, then there holds the formula*

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\, E_{\nu,\rho}^{\delta}\left(at^{\lambda}\right)\right]\right\}(x) = \frac{x^{\rho-\alpha-\alpha'+\mu-1}}{\Gamma(\delta)}$$

$$\times\, {}_4\psi_4\left[ax^{\lambda}\left|\begin{matrix}(\delta,1),(\rho,\lambda),(\rho-\alpha-\alpha'-\beta+\mu,\lambda),(\rho-\alpha'+\beta',\lambda)\\ (\rho,\nu),(\rho-\alpha-\alpha'+\mu,\lambda),(\rho-\alpha'-\beta+\mu,\lambda),(\rho+\beta',\lambda)\end{matrix}\right.\right]. \tag{19}$$

ProofBy using series representation of generalized Mittag-Leffler function [6-10] and left-sided Saigo-Maeda fractional integration power function formula (17), we have

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\, E_{\nu,\rho}^{\delta}\left(at^{\lambda}\right)\right]\right\}(x) = \left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\sum_{n=0}^{\infty}\frac{(\delta)_n\left(at^{\lambda}\right)^n}{\Gamma(\nu n+\rho)n!}\right]\right\}(x), \tag{20}$$

by interchanging the order of integration and summation, we have

$$=\sum_{n=0}^{\infty}\frac{(\delta)_n(a)^n}{\Gamma(\nu n+\rho)n!}\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\mu}t^{(\lambda n+\rho)-1}\right)(x)$$

$$=\sum_{n=0}^{\infty}\frac{(\delta)_n(a)^n}{\Gamma(\nu n+\rho)n!}x^{\lambda n+\rho-\alpha-\alpha'+\mu-1}$$

$$\times \frac{\Gamma(\lambda n+\rho)\Gamma(\lambda n+\rho-\alpha-\alpha'-\beta+\mu)\Gamma(\lambda n+\rho-\alpha'+\beta')}{\Gamma(\lambda n+\rho-\alpha-\alpha'+\mu)\Gamma(\lambda n+\rho-\alpha'-\beta+\mu)\Gamma(\lambda n+\rho+\beta')},$$

next, using (4), (5) and rearranging the terms, we get

$$=\frac{x^{\rho-\alpha-\alpha'+\mu-1}}{\Gamma(\delta)}\,{}_4\psi_4\left[ax^{\lambda}\left|\begin{matrix}(\delta,1),(\rho,\lambda),(\rho-\alpha-\alpha'-\beta+\mu,\lambda),(\rho-\alpha'+\beta',\lambda)\\(\rho,\nu),(\rho-\alpha-\alpha'+\mu,\lambda),(\rho-\alpha'-\beta+\mu,\lambda),(\rho+\beta',\lambda)\end{matrix}\right.\right],$$

which completes the proof of (19).

If we take $\alpha=\alpha+\beta,\alpha'=\beta'=0,\beta=-\gamma$ and $\mu=\alpha$ then Theorem 1 reduces to the following result given by Ahmed ([1], eq.(3.1)).

Corollary 1.1 *Let* $\alpha,\beta,\gamma,\rho,\delta\in\mathrm{C},x>0,\Re(\alpha)>0,\Re(\rho+\gamma-\beta)>0,\nu>0,\lambda>0$ *and* $a\in\mathrm{R}$ *. If the condition (6) is satisfied and* $I_{0+}^{\alpha,\beta,\gamma}$ *be the left-sided operator of generalized fractional integration associated with Gauss hypergeometric function, then there holds the following formula:*

$$\left\{I_{0+}^{\alpha,\beta,\gamma}\left[t^{\rho-1}\,E_{\nu,\rho}^{\delta}\left(at^{\lambda}\right)\right]\right\}(x)=\frac{x^{\rho-\beta-1}}{\Gamma(\delta)}$$

$$\times\,{}_3\psi_3\left[ax^{\lambda}\left|\begin{matrix}(\rho,\lambda),(\rho-\beta+\gamma,\lambda),(\delta,1)\\(\rho-\beta,\lambda),(\alpha+\rho+\gamma,\lambda),(\rho,\nu)\end{matrix}\right.\right]. \tag{21}$$

Remark 3.1 *If we take* $\lambda=\nu$ *in above result (21), then we can easily obtain the known result given by Chaurasia and Pandey ([2], p. 116, eq. (3.1)); further if we set* $\beta=-\alpha$ *then (21) reduces to a known result given by Saxena and Saigo([16], p.145, eq.(14)).*

3.2. Right-sided generalized fractional integration of generalized Mittag-Leffler function

Theorem 2 *Let* $\alpha,\alpha',\beta,\beta',\mu,\delta,\rho\in\mathrm{C},\Re(\mu)>0,x>0,\nu>0,\lambda>0$ *and* $a\in\mathrm{R}$ *. If the condition (6) is satisfied and* $I_{-}^{\alpha,\alpha',\beta,\beta',\mu}$ *be the right-sided operator of generalized fractional integration associated with Appell's function, then there holds the following formula:*

$$\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{-\mu-\rho}\,E_{\nu,\rho}^{\delta}\left(at^{-\lambda}\right)\right]\right\}(x)=\frac{x^{-\rho-\alpha-\alpha'}}{\Gamma(\delta)}$$

$$\times\,{}_4\psi_4\left[ax^{-\lambda}\left|\begin{matrix}(\delta,1),(\rho+\alpha+\alpha',\lambda),(\rho+\alpha+\beta',\lambda),(\rho-\beta+\mu,\lambda)\\(\rho,\nu),(\rho+\mu,\lambda),(\rho+\alpha+\alpha'+\beta',\lambda),(\rho+\alpha-\beta+\mu,\lambda)\end{matrix}\right.\right]. \tag{22}$$

Proof By using (3) and right-sided Saigo-Maeda fractional integration power function formula (18), we get

$$\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{-\mu-\rho}\;E_{\nu,\rho}^{\delta}\left(at^{-\lambda}\right)\right]\right\}(x)=\left(I_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{-\mu-\rho}\sum_{n=0}^{\infty}\frac{(\delta)_n\left(at^{-\lambda}\right)^n}{\Gamma(\nu n+\rho)n!}\right]\right)(x), \tag{23}$$

by interchanging the order of integration and summation, we have

$$=\sum_{n=0}^{\infty}\frac{(\delta)_n(a)^n}{\Gamma(\nu n+\rho)n!}\left(I_{-}^{\alpha,\alpha',\beta,\beta',\mu}t^{(1-\lambda n-\rho-\mu)-1}\right)(x)$$

$$=\sum_{n=0}^{\infty}\frac{(\delta)_n(a)^n}{\Gamma(\nu n+\rho)n!}x^{-\lambda n-\rho-\alpha-\alpha'}$$

$$\times\frac{\Gamma(\lambda n+\rho+\alpha+\alpha')\Gamma(\lambda n+\rho+\alpha+\beta')\Gamma(\lambda n+\rho-\beta+\mu)}{\Gamma(\lambda n+\rho+\mu)\Gamma(\lambda n+\rho+\alpha+\alpha'+\beta')\Gamma(\lambda n+\rho+\alpha-\beta+\mu)},$$

next, using (4), (5) and rearranging the terms, we obtain

$$=\frac{x^{-\rho-\alpha-\alpha'}}{\Gamma(\delta)}\,{}_4\psi_4\left[ax^{-\lambda}\left|\begin{matrix}(\delta,1),(\rho+\alpha+\alpha',\lambda),(\rho+\alpha+\beta',\lambda),(\rho-\beta+\mu,\lambda)\\(\rho,\nu),(\rho+\mu,\lambda),(\rho+\alpha+\alpha'+\beta',\lambda),(\rho+\alpha-\beta+\mu,\lambda)\end{matrix}\right.\right],$$

which completes the proof of the Theorem 2.

If we take $\alpha=\alpha+\beta,\alpha'=\beta'=0,\beta=-\gamma$ and $\mu=\alpha$ then (22) reduces to the following result given by Ahmed ([1], eq.(4.1)).

Corollary 2.1 *Let* $\alpha,\beta,\gamma,\rho,\delta\in\mathrm{C},x>0,\Re(\alpha)>0,\Re(\rho+\alpha)>\max\left[-\Re(\beta),-\Re(\gamma)\right]$, $\Re(\beta)\neq\Re(\gamma),\nu>0,\lambda>0$ *and* $a\in\mathrm{R}$. *If the condition (6) is satisfied and* $I_{-}^{\alpha,\beta,\gamma}$ *be the right-sided Saigo fractional integral operator, then there holds the following corollary:*

$$\left\{I_{-}^{\alpha,\beta,\gamma}\left[t^{-\alpha-\rho}\;E_{\nu,\rho}^{\delta}\left(at^{-\lambda}\right)\right]\right\}(x)=\frac{x^{-\rho-\alpha-\beta}}{\Gamma(\delta)}$$

$$\times\,{}_3\psi_3\left[ax^{-\lambda}\left|\begin{matrix}(\alpha+\beta+\rho,\lambda),(\alpha+\rho+\gamma,\lambda),(\delta,1)\\(\alpha+\rho,\lambda),(2\alpha+\beta+\gamma+\rho,\lambda),(\rho,\nu)\end{matrix}\right.\right]. \tag{24}$$

Remark 3.2 *If we take* $\lambda=\nu$ *in above result (24), then we obtain the known result given by Chaurasia and Pandey([2], p. 117, eq. (4.1)); further if we set* $\beta=-\alpha$ *then (24) reduces to another known result given by Saxena and Saigo([16], p.147, eq.(23)).*

3.3. Left-sided generalized fractional derivative of generalized Mittag-Leffler function

Theorem 3 *Let* $\alpha,\alpha',\beta,\beta',\mu,\delta,\rho\in\mathrm{C},\Re(\mu)>0,x>0,\nu>0,\lambda>0$ *and* $a\in\mathrm{R}$. *If the condition (6) is satisfied and* $D_{0+}^{\alpha,\alpha',\beta,\beta',\mu}$ *be the left-sided Saigo-Maeda fractional derivative operator associated with Appell's function, then there holds the formula*

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\,E_{\nu,\rho}^{\delta}\left(at^{\lambda}\right)\right]\right\}(x)=\frac{x^{\rho+\alpha+\alpha'-\mu-1}}{\Gamma(\delta)}$$

$$\times\,{}_4\psi_4\left[ax^{\lambda}\left|\begin{matrix}(\delta,1),(\rho,\lambda),(\rho+\alpha+\alpha'+\beta'-\mu,\lambda),(\rho+\alpha-\beta,\lambda)\\(\rho,\nu),(\rho+\alpha+\alpha'-\mu,\lambda),(\rho+\alpha+\beta'-\mu,\lambda),(\rho-\beta,\lambda)\end{matrix}\right.\right]. \qquad (25)$$

ProofBy using series representation of generalized Mittag-Leffler function as defined by (3) and using (13) also taking (17) into account, we have

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\,E_{\nu,\rho}^{\delta}\left(at^{\lambda}\right)\right]\right\}(x)=\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\sum_{n=0}^{\infty}\frac{(\delta)_n\left(at^{\lambda}\right)^n}{\Gamma(\nu n+\rho)n!}\right]\right\}(x), \qquad (26)$$

by interchanging the order of differentiation and summation, we have

$$=\sum_{n=0}^{\infty}\frac{(\delta)_n(a)^n}{\Gamma(\nu n+\rho)n!}\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\mu}t^{(\lambda n+\rho)-1}\right)(x)$$

$$=\sum_{n=0}^{\infty}\frac{(\delta)_n(a)^n}{\Gamma(\nu n+\rho)n!}\left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\mu}t^{(\lambda n+\rho)-1}\right)(x)$$

$$=\sum_{n=0}^{\infty}\frac{(\delta)_n(a)^n}{\Gamma(\nu n+\rho)n!}x^{\lambda n+\rho+\alpha+\alpha'-\mu-1}$$

$$\times\frac{\Gamma(\lambda n+\rho)\Gamma(\lambda n+\rho+\alpha+\alpha'+\beta'-\mu)\Gamma(\lambda n+\rho+\alpha-\beta)}{\Gamma(\lambda n+\rho+\alpha+\alpha'-\mu)\Gamma(\lambda n+\rho+\alpha+\beta'-\mu)\Gamma(\lambda n+\rho-\beta)},$$

next, using (4), (5) and rearranging the terms, we easily get the desired result of (25). This is complete proof of the Theorem 3.

If we take $\alpha=\alpha+\beta,\alpha'=\beta'=0,\beta=-\gamma$ and $\mu=\alpha$ then (25) reduces to the following result given by Ahmed ([1], eq.(5.1)).

Corollary 3.1 *Let* $\alpha,\beta,\gamma,\rho,\delta\in\mathrm{C},x>0,\Re(\alpha)>0,\Re(\alpha+\beta+\gamma)>0,\nu>0,\lambda>0$ *and* $a\in\mathrm{R}$. *The condition (6) is also satisfied and* $D_{0+}^{\alpha,\beta,\gamma}$ *be the left-sided Saigo* *fractional derivative operator, then there holds the following corollary:*

$$\left\{D_{0+}^{\alpha,\beta,\gamma}\left[t^{\rho-1}\,E_{\nu,\rho}^{\delta}\left(at^{\lambda}\right)\right]\right\}(x)=\frac{x^{\rho+\beta-1}}{\Gamma(\delta)}$$

$$\times\,{}_3\psi_3\left[ax^{\lambda}\left|\begin{matrix}(\rho,\lambda),(\alpha+\beta+\gamma+\rho,\lambda),(\delta,1)\\(\rho+\beta,\lambda),(\rho+\gamma,\lambda),(\rho,\nu)\end{matrix}\right.\right]. \tag{27}$$

Remark 3.3 *If we take* $\lambda=\nu$ *in above result (27), then we can easily obtain the known result given by Chaurasia and Pandey ([2], p. 118, eq. (5.1)); further if we set* $\beta=-\alpha$ *then we arrive at the result given by Saxena and Saigo([16], p.149, eq.(29)).*

3.4. Right-sided generalized fractional derivative of generalized Mittag-Leffler function

Theorem 4 *Let* $\alpha,\alpha',\beta,\beta',\mu,\delta,\rho\in\mathbb{C},\Re(\mu)>0,x>0,\nu>0,\lambda>0$ *and* $a\in\mathbb{R}$ *. If the condition (6) is also satisfied and* $D_{-}^{\alpha,\alpha',\beta,\beta',\mu}$ *be the right-sided operator of generalized fractional derivative associated with Appell's function, then there holds the following formula:*

$$\left\{D_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\mu-\rho}\,E_{\nu,\rho}^{\delta}\left(at^{-\lambda}\right)\right]\right\}(x)=\frac{x^{-\rho+\alpha+\alpha'}}{\Gamma(\delta)}$$

$$\times\,{}_4\psi_4\left[ax^{-\lambda}\left|\begin{matrix}(\delta,1),(\rho-\alpha-\alpha',\lambda),(\rho-\alpha'-\beta,\lambda),(\rho+\beta'-\mu,\lambda)\\(\rho,\nu),(\rho-\mu,\lambda),(\rho-\alpha-\alpha'-\beta,\lambda),(\rho-\alpha'+\beta'-\mu,\lambda)\end{matrix}\right.\right]. \tag{28}$$

Proof By using (3), (15) and taking (18) into account, we have

$$\left\{D_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\mu-\rho}\,E_{\nu,\rho}^{\delta}\left(at^{-\lambda}\right)\right]\right\}(x)=\left(D_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\mu-\rho}\sum_{n=0}^{\infty}\frac{(\delta)_n\left(at^{-\lambda}\right)^n}{\Gamma(\nu n+\rho)n!}\right]\right)(x), \tag{29}$$

by interchanging the order of differentiation and summation, we have

$$=\sum_{n=0}^{\infty}\frac{(\delta)_n(a)^n}{\Gamma(\nu n+\rho)n!}\left(D_{-}^{\alpha,\alpha',\beta,\beta',\mu}t^{-\lambda n+\mu-\rho}\right)(x)$$

$$=\sum_{n=0}^{\infty}\frac{(\delta)_n(a)^n}{\Gamma(\nu n+\rho)n!}\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\mu}t^{(1-\lambda n-\rho+\mu)-1}\right)(x)$$

$$=\sum_{n=0}^{\infty}\frac{(\delta)_n(a)^n}{\Gamma(\nu n+\rho)n!}x^{-\lambda n-\rho+\alpha+\alpha'}$$

$$\times \frac{\Gamma(\rho-\alpha-\alpha'+\lambda n)\Gamma(\rho-\alpha'-\beta+\lambda n)\Gamma(\rho+\beta'-\mu+\lambda n)}{\Gamma(\rho-\mu+\lambda n)\Gamma(\rho-\alpha-\alpha'-\beta+\lambda n)\Gamma(\rho-\alpha'+\beta'-\mu+\lambda n)},$$

by using (4), (5) and rearranging the terms, we easily obtain the desired result of (28). This completes the proof of the Theorem 4.

If we take $\alpha=\alpha+\beta, \alpha'=\beta'=0, \beta=-\gamma$ and $\mu=\alpha$ then (28) reduces to the following result given by Ahmed ([1], eq.(6.1)).

Corollary 4.1 *Let* $\alpha,\beta,\gamma,\rho,\delta\in \mathrm{C}, x>0, \Re(\alpha)>0, \Re(\rho)>\max[\Re(\alpha+\beta+k), -\Re(\gamma)]$, $\nu>0, \lambda>0$ *and* $a\in \mathrm{R}$ *with* $\Re(\alpha+\beta+\gamma)+k\neq 0$ *(where* $k=[\Re(\alpha)]+1$*). If the condition (6) is satisfied and* $D_{-}^{\alpha,\beta,\gamma}$ *be the right-sided Saigo fractional derivative operator, then there holds the following corollary:*

$$\left\{D_{-}^{\alpha,\beta,\gamma}\left[t^{\alpha-\rho}\, E_{\nu,\rho}^{\delta}\left(at^{-\lambda}\right)\right]\right\}(x)=\frac{x^{\alpha+\beta-\rho}}{\Gamma(\delta)}$$

$$\times\, {}_3\psi_3\left[ax^{-\lambda}\middle| \begin{matrix}(\rho+\gamma,\lambda),(\rho-\alpha-\beta,\lambda),(\delta,1)\\ (\rho-\alpha,\lambda),(\rho+\gamma-\alpha-\beta,\lambda),(\rho,\nu)\end{matrix}\right]. \tag{30}$$

Remark 3.4 *If we set* $\lambda=\nu$ *in above result (30), then we obtain the known result given by Chaurasia and Pandey ([2], p. 119, eq. (6.1)); further if we take* $\beta=-\alpha$ *then (30) reduces to another known result given by Saxena and Saigo ([16], p.150, eq.(35)).*

4. Concluding Remarks

In the present paper we study the generalized M-L-function and Saigo-Maeda fractional calculus operators. The obtained results are extension of work done by many authors, for example Ahmed [1], Chaurasia and Pandey [2], Saxena and Saigo [16], and many more. The provided results are new and have uniqueness identity in the literature. On account of the general nature of the generalized Mittag-Leffler function and generalized Wright function, a number of known results can easily be found as special cases of our main results.

Acknowledgments: The authors are thankful to the Referee for valuable suggestions and one of the author (Dinesh Kumar) thanks NBHM (National Board of Higher Mathematics), India, for granting a Post-Doctoral Fellowship (sanction no.2/40(37)/2014/R&D-II/14131).

References

[1] Ahmed, S. (2014). *On the generalized fractional integrals of the generalized Mittag-Lefflerfunction*, Springer Plus, 2014, **3**:198, 5 pages.

[2] Chaurasia, V.B.L. and Pandey, S.C. (2010). *On the fractional calculus of generalized Mittag-Leffler function*, Scientia, Ser. A; Math Sci.**20**, 113–122.

[3] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G. (1953). *Higher transcendental functions*, McGraw-Hill, New York, 1.

[4] Kilbas, A.A., Saigo, M. and Saxena, R.K. (2004). *Generalized Mittag-Leffler function and generalized fractional calculus operators*, Integral Transform Special Function, **15**, 31–49.

[5] Kilbas, A.A., Saigo, M. and Trujillo, J.J. (2002). *On the generalized Wright function*, Fract. Calc. Appl. Anal., **5**(4), 437–460.

[6] Kumar, D. and Kumar, S. (2014). F*ractional Calculus of the Generalized Mittag-Leffler Type Function*, International Scholarly Research Notices 2014, Article ID 907432, 6 pages.

[7] Kumar, D. and Purohit, S.D. (2014). *Fractional differintegral operators of the generalized Mittag-Leffler type function*, Malaya J. Mat., **2**(4), 419–425.

[8] Kumar, D. and Saxena, R.K. (2015). *Generalized fractional calculus of the M-Series involving F_3 hypergeometric function*, Sohag J. Math., **2**(1), 17–22.

[9] Mittag-Leffler, G.M. (1903). *Sur la nouvelle fonction* $E_\alpha\left(x\right)$, C.R. Acad. Sci. Paris **137**, 554-558.

[10] Mittag-Leffler, G.M. (1905). *Sur la representation analytique d'une branche uniforme d'une function* monogene, Acta Math. **29**, 101-181.

[11] Prabhakar, T.R. (1971). *A singular integral equation with a generalized Mittag-Lefflerfunction in the Kernel*, Yokohama Math. J. **19**, 7-15.

[12] Saigo, M. (1978). *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep., College General Ed. Kyushu Univ. **11**, 135-143.

[13] Saigo, M. and Maeda, N. (1996). *More generalization of fractional calculus Transform Methods and Special Functions*, Varna, Bulgaria, 386-400.

[14] Samko, S.G., Kilbas, A.A. and Marichev, O.I. (1993). *Fractional Integrals and Derivatives*, Theory and Applications, Gordon and Breach, Yverdon et alibi.

[15] Saxena, R.K., Ram, J. and Kumar, D. (2013). *On the Two-Dimensional Saigo-Maeda fractional calculus associated with Two-Dimensional Aleph Transform*, Le Matematiche, **68**, 267-281.

[16] Saxena, R.K., and Saigo, M. (2005). *Certain properties of the fractional calculus operators associated with generalized Mittag-Leffler function*, Fract. Calc. Appl. Anal., **8**(2), 141–154.

[17] Wiman, A. (1905). *Uber de fundamental satz in der theorie der funktionen* $E_\alpha(x)$, Acta Math. **29**, 191–201.

[18] Wright, E.M. (1935). *The asymptotic expansion of generalized hypergeometric function*, J. London Math. Soc.**10**, 286–293.