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DOUBLE DIRICHLET AVERAGE OF GENERALIZED MITTAG-LEFFLER FUNCTION VIA FRACTIONAL DERIVATIVE

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Abstract: The aim of the present paper is to establish the results of Dirichlet average of generalized Mittag-Leffler function, using fractional derivative. In this paper the solution is obtained in compact form of double Dirichlet average of generalized Mittag-Leffler function. At the end of this paper, several special cases have also been obtained.

Keywords: Dirichlet average, generalized Mittag-Leffler function, fractional derivative.

Subject Classification: 26A33, 33C65, 33E12.

1. Introduction

The Dirichlet average of functions is introduced by Carlson [1-5], which represents certain type of integral average with respect to Dirichlet measure. The various types of Dirichlet average has been given by Carlson. Also, Dirichlet average of function has been studied by Deora and Banerji [6], Gupta and Agarwal [8, 9] and Ram et al. [11,12 &13] by using the fractional derivative. In this paper, the Dirichlet average of generalized Mittag-Leffler function has been established.

2. Definitions

Some definitions which are required in the preparation of this paper:

2.1 Standard Simplex in R^n, $n \geq 1$

The standard simplex in R^n, $n \geq 1$ is defined by Carlson [1, p.62].

$$E = E_n = \{S\ (u_1, u_2, ..., u_n) : u_1 \geq 0, ..., u_n \geq 0, u_1 + u_2 + ... + u_n \leq 1\} \tag{1}$$

2.2 Dirichlet Measure

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The Dirichlet measure $d\mu_b$ is defined by

$$d\,\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1},...,u_{k-1}^{b_{k-1}-1}(1-u_1-...u_{k-1})\,b_k^{-1}\,du_1...du_{k-1} \tag{2}$$

where B(b) is the multivariable Beta function which is defined as

$$B(b) = B(b_1,...,b_k) = \frac{\Gamma(b_1)...\Gamma(b_k)}{\Gamma(b_1+b_2+...+b_k)};(Re(b_j)>0;\, j=1,2,..., k)$$

2.3 Dirichlet Average [1, p.75]

The general Dirichlet average function is defined by Carlson [1] in the form

$$F(b,z) = \int_E f(u.z)\,d\,\mu_b(u) \tag{3}$$

where $d\mu_b$ is defined by (2) and

$$u.z = \sum_{i=1}^{k} u_i z_i \text{ and } u_k = 1-u_1-...-u_{k-1}.$$

2.4 Double Averages of Functions of One Variable (from [1, 2])

Let z be a $k \times x$ matrix with complex elements z_{ij}, let u = ($u_1,u_2,\ldots,u_k$) and v = ($v_1,v_2,\ldots,v_x$) be an ordered k-tuple and x-tuple of real non-negative weights $\sum u_i=1$ and $\sum v_j = 1$ respectively. Now, we define

$$u.z.v = \sum_{i=1}^{k} \sum_{j=1}^{x} u_i\, z_{ij}\, v_j$$

If z_{ij} is regarded as a point of the complex plane, all these convex combinations are points in the convex hull of ($z_{11},\ldots,z_{kx}$) denote by H(z).

Let b= ($b_1,b_2,\ldots,b_k$) be an ordered k-tuple of complex numbers with positive real part Re(b) > 0 and similarly for β = ($\beta_1,\beta_2,\ldots, \beta_x$) then define $d\mu_b(u)$ and $d\mu_b(v)$.

Let f be the holomorphic on a domain D in the complex plane, if Re(b) > 0, Re(β) > 0 and H(z) ⊂ D, we define

$$F(b,z,\beta) = \int \int f(u.z.v)\; d\mu_b(u)\; d\mu_\beta(v)$$

Double average for (k=x=2) of $(u.z.v)^t$ is the S function is defined by Gupta and Agrawal [9].

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2) = \int_0^1 \int_0^1 (u.z.v)^t\; dm_{\mu_1\mu_2}(u)\; dm_{\rho_1\rho_2}(v) \tag{4}$$

where $\text{Re}(\mu_1) > 0, \text{Re}(\mu_2) > 0, \text{Re}(\rho_1) > 0, \text{Re}(\rho_2) > 0$

$$dm_{\mu_1\mu_2}(u) = \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} u^{\mu_1-1}(1-u)^{\mu_2-1}\, du \tag{5}$$

$$dm_{\rho_1\rho_2}(v) = \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)} v^{\rho_1-1}(1-v)^{\rho_2-1}\, dv \tag{6}$$

and

$$u.z.v = \sum_{i=1}^{2}\sum_{j=1}^{2}(u_i z_{ij} v_j) = \sum_{i=1}^{2}[u_1(z_{i1}v_1 + z_{i2}v_2)]$$

$$= [u_1 z_{11} v_1 + u_1 z_{12} v_2 + u_2 z_{21} v_1 + u_2 z_{22} v_2]$$

Let $z_{11} = a, z_{12} = b, z_{21} = c, z_{22} = d$ and $\begin{cases} u_1 = u & u_2 = 1-u \\ v_1 = v & v_2 = 1-v \end{cases}$

thus $z = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Therefore

$$u.z.v = uva + ub(1-v) + (1-u)cv + (1-u)d(1-v)$$

$$= uv(a-b-c+d) + u(b-d) + v(c-d) + d \tag{7}$$

2.5 Fractional Derivative [7, p.181]

The theory of fractional derivative with respect to an arbitrary function has been used by Erdélyi et al. [7]. The fractional derivative is obtained by proceeding via fractional integral. The Riemann-Liouville fractional integral of order α is defined by

$$D_z^{\alpha} F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t)(z-t)^{\alpha-1}\, dt \tag{8}$$

where $\text{Re}(\alpha) < 0$ and $F(x)$ is the form of $x^p f(x)$; $f(x)$ is analytic at $x = 0$.

2.6 The Generalized Mittag-Leffler Function[10]

The Generalized Mittag-Leffler function is defined by Khan and Shakeel [10] as follows

$$E_{\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn}\, z^n}{\Gamma(\alpha n+\beta)(\upsilon)_{\sigma n}(\delta)_{pn}} \tag{9}$$

where $\alpha, \beta, \gamma, \delta, \mu, \upsilon, \rho, \sigma \in \mathbb{C}$; $p, q > 0$, $q \leq \operatorname{Re}(\alpha) + p$ and

$\min\{\operatorname{Re}(\alpha), \operatorname{Re}(\beta), \operatorname{Re}(\gamma), \operatorname{Re}(\delta), \operatorname{Re}(\mu), \operatorname{Re}(\upsilon), \operatorname{Re}(\rho), \operatorname{Re}(\sigma)\} > 0$.

3. Main Result and Proof

Theorem 1 The equivalence relation of double Dirichlet average of generalized Mittag-Leffler function $E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\upsilon,\sigma,\delta,\rho}(u.z.v)$ with the fractional derivative for (k = x = 2) is

$$S(\mu_1, \mu_2; z; \rho_1, \rho_2) = \frac{\Gamma(\rho_1 + \rho_2)}{\Gamma(\rho_1)} (x-y)^{1-\rho_1-\rho_2} D^{-\rho_2}_{x-y} E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\upsilon,\sigma,\delta,\rho}(x)(x-y)^{\rho_1 - 1} \tag{10}$$

Proof: Let us consider the double average for (k = x = 2) of generalized Mittag-Leffler function $E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\upsilon,\sigma,\delta,\rho}(u.z.v)$

$$S(\mu_1, \mu_2; z; \rho_1, \rho_2) = \int_0^1 \int_0^1 E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\upsilon,\sigma,\delta,\rho}(u.z.v)\, dm_{\mu_1\mu_2}(u)\, dm_{\rho_1\rho_2}(v)$$

$$= \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n} (\gamma)_{qn}}{\Gamma(\alpha n + \beta)(\upsilon)_{\sigma n} (\delta)_{pn}} \int_0^1 \int_0^1 [u.z.v]^n dm_{\mu_1\mu_2}(u) dm_{\rho_1\rho_2}(v) \tag{11}$$

where $\operatorname{Re}(\mu_1) = 0, \operatorname{Re}(\mu_2) = 0, \operatorname{Re}(\rho_1) > 0, \operatorname{Re}(\rho_2) > 0$

Using equation (5), (6) and (7) in (11), we have

$$S(\mu_1, \mu_2; z; \rho_1, \rho_2) = \frac{\Gamma(\mu_1 + \mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} \frac{\Gamma(\rho_1 + \rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)} \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n} (\gamma)_{qn}}{\Gamma(\alpha n + \beta)(\upsilon)_{\sigma n} (\delta)_{pn}}$$

$$\times \int_0^1 \int_0^1 [uv(a-b-c+d) + u(b-d) + v(c\text{-}d) + d]^n u^{\mu_1 - 1}(1-u)^{\mu_2 - 1} v^{\rho_1 - 1}(1-v)^{\rho_2 - 1} du\, dv \tag{12}$$

To obtain the fractional derivative, we assume a = c = x; b = d = y then

$$S(\mu_1, \mu_2; z; \rho_1, \rho_2) = \frac{\Gamma(\mu_1 + \mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} \frac{\Gamma(\rho_1 + \rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)} \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n} (\gamma)_{qn}}{\Gamma(\alpha n + \beta)(\upsilon)_{\sigma n} (\delta)_{pn}}$$

$$\times \int_0^1 \int_0^1 [v(x-y) + y]^n u^{\mu_1 - 1}(1-u)^{\mu_2 - 1} v^{\rho_1 - 1}(1-v)^{\rho_2 - 1} du\, dv$$

Now, using the definition of Beta and Gamma function and due to suitable adjustment, we arrive at

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\upsilon)_{\sigma n}(\delta)_{pn}}$$

$$\times\int_0^1 [v(x-y)+y]^n\, v^{\rho_1-1}(1-v)^{\rho_2-1}\,dv$$

On putting v(x – y) = t in above equation, we get

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\upsilon)_{\sigma n}(\delta)_{pn}}$$

$$\times\int_0^{x-y}[t+y]^n\left(\frac{t}{x-y}\right)^{\rho_1-1}\left(1-\frac{t}{x-y}\right)^{\rho_2-1}\frac{1}{(x-y)}\,dt$$

$$=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\upsilon)_{\sigma n}(\delta)_{pn}}(x-y)^{1-\rho_1-\rho_2}\int_0^{x-y}[(y+t)]^n t^{\rho_1-1}(x-y-t)^{\rho_2-1}dt$$

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}(x-y)^{1-\rho_1-\rho_2}\int_0^{x-y}E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\upsilon,\sigma,\delta,p}(y+t)t^{\rho_1-1}(x-y-t)^{\rho_2-1}dt$$

by using the definition of fractional derivative from equation (8), we get

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2}\,D^{-\rho_2}_{x-y}E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\upsilon,\sigma,\delta,p}(x)(x-y)^{\rho_1-1}$$

This completes the proof of Theorem 1.

4. Special Cases

1. If we put $\mu=\upsilon,\rho=\sigma$ in Theorem 1, we find a new result after a little simplification:

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2}\,D^{-\rho_2}_{x-y}\,E^{\gamma,\delta,q}_{\alpha,\beta,p}(x)(x-y)^{\rho_1-1}$$

2. If we substitute $\mu=\upsilon,\rho=\sigma$ and p = q = 1 in Theorem 1, we arrive at the following result:

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2}\,D^{-\rho_2}_{x-y}\,E^{\gamma,\delta}_{\alpha,\beta}(x)(x-y)^{\rho_1-1}$$

3. Again, if we take $\mu = \upsilon, \rho = \sigma$ and p = δ = 1 in Theorem 1, we get a new result

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2}\, D_{x-y}^{-\rho_2}\, E_{\alpha,\beta}^{\gamma,q}(x)(x-y)^{\rho_1-1}$$

4. If we take $\mu = \upsilon, \rho = \sigma$ and p = q = δ = 1 in the Theorem 1, we have

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2}\, D_{x-y}^{-\rho_2}\, E_{\alpha,\beta}^{\gamma}(x)(x-y)^{\rho_1-1}$$

Theorem 2 The equivalence relation of double Dirichlet average of generalized Mittag-Leffler function $E_{\alpha,\beta,\upsilon,\sigma,\delta,\rho}^{\mu,\rho,\gamma,q}(u.z.v)$ with the fractional derivative for (k = x = 2) is

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{(\mu_1)_n}{(\mu_1+\mu_2)_n}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2} D_{x-y}^{-\rho_2} E_{\alpha,\beta,\upsilon,\sigma,\delta,\rho}^{\mu,\rho,\gamma,q}(x)(x-y)^{\rho_1-1}$$

Proof: Using equation (12)

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\upsilon)_{\sigma n}(\delta)_{pn}}$$

$$\times\int_0^1\int_0^1 [uv(a-b-c+d)+u(b-d)+v(c-d)+d]^n u^{\mu_1-1}(1-u)^{\mu_2-1}v^{\rho_1-1}(1-v)^{\rho_2-1}du\,dv$$

If we set a = x, b = y, c = d = 0, then we have

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\upsilon)_{\sigma n}(\delta)_{pn}}$$

$$\times\int_0^1\int_0^1 [uv(x-y)+uy]^n\, u^{\mu_1-1}(1-u)^{\mu_2-1}\, v^{\rho_1-1}(1-v)^{\rho_2-1}du\,dv$$

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\upsilon)_{\sigma n}(\delta)_{pn}}$$

$$\times\int_0^1\int_0^1 [vx+y(1\text{-}v)]^n\, u^{\mu_1+n-1}(1-u)^{\mu_2-1}\, v^{\rho_1-1}(1-v)^{\rho_2-1}du\,dv$$

Now, by using the definition of Beta and Gamma function, we get

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}\frac{\Gamma(\mu_1+n)\Gamma(\mu_2)}{\Gamma(\mu_1+\mu_2+n)}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\upsilon)_{\sigma n}(\delta)_{pn}}$$

$$\times\int_0^1 [vx+y(1-v)]^n v^{\rho_1-1}(1-v)^{\rho_2-1}\,dv$$

$$=\frac{(\mu_1)_n}{(\mu_1+\mu_2)_n}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\upsilon)_{\sigma n}(\delta)_{pn}}\int_0^1 [vx+y(1-v)]^n v^{\rho_1-1}(1-v)^{\rho_2-1}\,dv$$

On putting v(x – y) = t in above equation, we get

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\mu_1)_n}{\Gamma(\mu_1+\mu_2)_n}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\upsilon)_{\sigma n}(\delta)_{pn}}$$

$$\times\int_0^{x-y} [t+y]^n\left(\frac{t}{x-y}\right)^{\rho_1-1}\left(1-\frac{t}{x-y}\right)^{\rho_2-1}\frac{1}{(x-y)}\,dt$$

Now, using the definition of fractional derivative, we have

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{(\mu_1)_n}{(\mu_1+\mu_2)_n}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}(x-y)^{1-\rho_1-\rho_2}\,D_{x-y}^{-\rho_2}E_{\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(x)(x-y)^{\rho_1-1}$$

A number of several special cases of Theorem 2 can also be obtained.

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