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GENERALIZED FRACTIONAL KINETIC EQUATIONS AND ITS SOLUTIONS INVOLVING GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract: The aim of this paper is to establish the solution of the fractional kinetic equation involving generalized Mittag-Leffler function given by Salim and Faraj [14].The results obtained in terms of Wright hypergeometric function [23] are rather general in nature and can easily construct various known and new kinetic equations.

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1. Introduction and Preliminaries

The Mittag-Leffler function (M-L function) and its various generalizations have been investigated by many researchers in both mathematics and engineering. Yet, during the twentieth century, they were practically unknown to the majority of scientists since they were ignored in the common books on special functions. An extremely growing interest in the study of their diverse properties is due mainly to the close connection of the Mittag-Leffler function to fractional calculus, its application to the study of mathematical physics, differential and integral equations of fractional orders.

The function $E_\alpha(z)$ was introduced by the Swedish mathematician Gosta Mittag-Leffler [9, 10], and defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \quad (\alpha \in C, \mathrm{Re}(\alpha) > 0), \tag{1}$$

the Mittag-Leffler function is a direct generalization of $\exp(z)$ in which $\alpha = 1$.

A generalization of $E_{\alpha}(z)$ was given and studied by Wiman [22], defined as

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\alpha n+\beta)},\quad (\alpha,\beta\in C,\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0). \tag{2}$$

In 1971, the generalization of (2) was introduced by Prabhakar [11] in terms of the series representation as given follows (see also, [5])

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n}}{\Gamma(\alpha n+\beta)}\frac{z^{n}}{n!},\quad (\alpha,\beta,\gamma\in C,\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0), \tag{3}$$

where $(\gamma)_{n}=\dfrac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$ is the Pochhammer symbol.

Further generalization of Mittag-Leffler function (Four parameter Mittag-Leffler function) was defined by Salim [13], as follows

$$E_{\alpha,\beta}^{\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n}z^{n}}{\Gamma(\alpha n+\beta)(\delta)_{n}},\ (\alpha,\beta,\gamma\in C,\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0,\mathrm{Re}(\delta)>0). \tag{4}$$

Recently, a new generalization of Mittag-Leffler function introduced by Salim and Faraj [14] in the following manner (see also, [3]):

$$E_{\alpha,\beta,p}^{\delta,\xi,q}(z)=\sum_{n=0}^{\infty}\frac{(\delta)_{qn}z^{n}}{\Gamma(\alpha n+\beta)(\xi)_{pn}}, \tag{5}$$

where $\alpha,\beta,\delta,\xi\in C,\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta),\mathrm{Re}(\xi)>0$; $p,q>0,q\leq\mathrm{Re}(\alpha)+p$; and

$$(\gamma)_{qn}=\frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}, \tag{6}$$

here, $(\gamma)_{qn}$ denotes the generalized Pochhammer symbol.

The generalized Fox-Wright hypergeometric function was introduced by Wright [23] and given as

$${}_{p}\Psi_{q}(z)={}_{p}\Psi_{q}\left[\begin{matrix}(a_{i},\alpha_{i})_{1,p}\\(b_{j},\beta_{j})_{1,q}\end{matrix};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_{i}+\alpha_{i}n)}{\prod_{j=1}^{q}\Gamma(b_{j}+\beta_{j}n)}\frac{z^{n}}{n!}, \tag{7}$$

where $z,a_{i},b_{j}\in C$ and $\alpha_{i},\beta_{j}\in R-\{0\}$ $(i=1,2,\cdots,p;j=1,2,\cdots,q)$.

2. Generalized Fractional Kinetic Equation

The fractional differential equation between rate of change of reaction was established by Haubold and Mathai [4], the destruction rate and the production rate are calculated as follows:

$$\frac{dN}{dt} = -d(N_t) + p(N_t), \tag{8}$$

where $N = N(t)$ is the rate of reaction, $d = d(N)$ is the rate of destruction, $p = p(N)$ is the rate of production and N_t denotes the function defined by $N_t(t^*) = N(t - t^*), t^* > 0$.

The special case of (8), for spatial fluctuations or inhomogeneities in $N(t)$ the quantity are neglected, that is the equation

$$\frac{dN_i}{dt} = -c_i N_i(t), \tag{9}$$

with the initial condition that $N_i(t = 0) = N_0$ is the number density of species i at time $t = 0$ and constant $c_i > 0$. If we remove the index i and integrate the standard kinetic equation (2.2), we have

$$N(t) - N_0 = -c\, {}_0D_t^{-1} N(t), \tag{10}$$

where ${}_0D_t^{-1}$ is the special case of the Riemann-Liouville fractional integral operator ${}_0D_t^{-\nu}$ defined as

$${}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s) \mathrm{d}s, \qquad (t > 0, \mathrm{Re}(\nu) > 0). \tag{11}$$

The fractional generalization of the standard kinetic equation (10) is given by Haubold and Mathai [4], given as

$$N(t) - N_0 = -c^\nu\, {}_0D_t^{-\nu} N(t), \tag{12}$$

and obtained the solution of (12) as

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu k + 1)} (ct)^{\nu k}. \tag{13}$$

Further, Saxena and Kalla [15] considered the following fractional kinetic equation

$$N(t) - N_0 f(t) = -c^\nu\, {}_0D_t^{-\nu} N(t), \quad (\mathrm{Re}(\nu) > 0), \tag{14}$$

where, $N(t)$ denotes the number density of a given species at time t, $N_0 = N(0)$ is the number density of that species at time $t = 0$ and c is a constant.

Remark 2.1 Recently, Kumar et al. [8] have studied the generalized fractional kinetic equation in terms of the generalized Bessel function $\omega_{l,b,c}(t)$ of the first kind. In 2013, Saxena et al. [18] have given the solution of fractional kinetic equation in terms of generalized Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}$ by using Sumudu transform technique. Moreover, Choi and Kumar [1] have provided the solutions of generalized fractional kinetic equations involving Aleph function [12] (see also, [6, 7]).

3. Solution of Generalized Fractional Kinetic Equations by using Laplace Transform

In this section, we will investigate the solution of the generalized kinetic equations involving the generalized Mittag-Leffler function (5) by applying the Laplace transform technique.

The Laplace transform of the function $f(t)$, denoted by $F(s)$ is defined as following manner [19, 20]:

$$F(s) = L\{f(t) : s\} = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t, \ \mathrm{Re}(s) > 0. \tag{15}$$

and convolution theorem is given by

$$L\{f * g\}(s) = L\left\{\int_0^t f(t-\xi) g(\xi)\mathrm{d}\xi\right\} = L\{f(s)\}.L\{g(s)\}. \tag{16}$$

The following Lemmas are required to prove our main results, are given by

Lemma 1 Let $\mathrm{Re}(\lambda) > 0, \mathrm{Re}(\rho) > 0$ and $\mathrm{Re}(s) > 0$. Then, the following Laplace transform of generalized Mittag-Leffler function $E^{\delta,\xi,q}_{\alpha,\beta,p}(z)$ holds true:

$$L\left\{t^{\lambda-1} E^{\delta,\xi,q}_{\alpha,\beta,p}\left(t^{\rho}\right) : s\right\} = s^{-\lambda} \frac{\Gamma(\xi)}{\Gamma(\delta)} {}_3\Psi_2\left[\begin{matrix}(\delta, q), (\lambda, \rho), (1,1) \\ (\beta, \alpha), (\xi, p)\end{matrix}; s^{-\rho}\right], \tag{17}$$

where ${}_3\Psi_2(.)$ is the Fox-Wright hypergeometric function given by (7).

Proof By using (5) and (15), we can easily obtain the formula (17) after a little simplification. We, therefore, omit the details involved.

If we take $\lambda = \beta$ and $\rho = \alpha$ in (17), then a special case of (17) is given by

Lemma 2 Let $\min\{\mathrm{Re}(s), \mathrm{Re}(\alpha), \mathrm{Re}(\beta)\} > 0$. Then, the Laplace transform of (5) is given by

$$L\left\{t^{\beta-1}E_{\alpha,\beta,p}^{\delta,\xi,q}\left(t^{\alpha}\right):s\right\}=s^{-\beta}\frac{\Gamma(\xi)}{\Gamma(\delta)}\,{}_{2}\Psi_{1}\left[\begin{matrix}(\delta,q),(1,1)\\(\xi,p)\end{matrix};s^{-\alpha}\right]. \tag{18}$$

Theorem 1 Let $c,\omega,\nu,\lambda,\rho\in R^{+}$; $\alpha,\beta,\delta,\xi\in C$; $\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta),\mathrm{Re}(\xi)>0$; $p,q>0,\quad q\le \mathrm{Re}(\alpha)+p$. Then, the solution of the following generalized fractional kinetic equation

$$N(t)-N_0\,t^{\lambda-1}E_{\alpha,\beta,p}^{\delta,\xi,q}\left(\omega t^{\rho}\right)=-c^{\nu}\,{}_{0}D_{t}^{-\nu}N(t) \tag{19}$$

is given by

$$N(t)=N_0t^{\lambda-1}\frac{\Gamma(\xi)}{\Gamma(\delta)}\sum_{r=0}^{\infty}\left(-c^{\nu}t^{\nu}\right)^{r}\,{}_{3}\Psi_{3}\left[\begin{matrix}(\delta,q),(\lambda,\rho),(1,1)\\(\beta,\alpha),(\xi,p),(\lambda+\nu r,\rho)\end{matrix};\omega t^{\rho}\right]. \tag{20}$$

Proof By applying the Laplace transform to the both sides of (19) and using the Laplace transform of the Riemann-Liouville fractional integral operator given by Erdélyi et al. [2] as

$$L\left\{{}_{0}D_{t}^{-\nu}N(t):s\right\}=s^{-\nu}N(s), \tag{21}$$

and also

$$L\left\{N(t):s\right\}=N(s), \tag{22}$$

we obtain

$$N(s)=\frac{N_0}{1+\left(\frac{c}{s}\right)^{\nu}}L\left\{t^{\lambda-1}E_{\alpha,\beta,p}^{\delta,\xi,q}\left(\omega t^{\rho}\right):s\right\},$$

now, by using (17) and the binomial expansion given as

$$\left[1+\left(\frac{c}{s}\right)^{\nu}\right]^{-1}=\sum_{r=0}^{\infty}(-1)^{r}\left(\frac{c}{s}\right)^{\nu r}\quad\left(c<|s|\right),$$

we have

$$N(s)=N_0\sum_{r=0}^{\infty}(-1)^{r}\left(\frac{c}{s}\right)^{\nu r}\sum_{n=0}^{\infty}\frac{(\delta)_{qn}}{\Gamma(\alpha n+\beta)(\xi)_{pn}}\frac{\Gamma(\lambda+\rho n)\Gamma(n+1)}{s^{\lambda+\rho n}}\frac{\omega^{n}}{n!}$$

$$=N_0\sum_{r=0}^{\infty}\left(-c^{\nu}\right)^{r}\sum_{n=0}^{\infty}\frac{(\delta)_{qn}}{\Gamma(\alpha n+\beta)(\xi)_{pn}}\frac{\Gamma(\lambda+\rho n)\Gamma(n+1)}{s^{\lambda+\nu r+\rho n}}\frac{\omega^{n}}{n!}. \tag{23}$$

Finally, taking Laplace inverse transform of (23) and make use of the following formula $L^{-1}\left\{s^{-\nu}:t\right\}=\frac{t^{\nu-1}}{\Gamma(\nu)},\ \left(\mathrm{Re}(\nu)>0\right)$ and $L^{-1}\left\{N(s):t\right\}=N(t),$ then we arrive at

$$L^{-1}\left\{N(s):t\right\}=N_0\sum_{r=0}^{\infty}\left(-c^{\nu}\right)^r\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\Gamma(\lambda+\rho n)\Gamma(n+1)}{\Gamma(\alpha n+\beta)(\xi)_{pn}}\frac{\omega^n}{n!}L^{-1}\left\{\frac{1}{s^{\lambda+\nu r+\rho n}}:t\right\}$$

$$N(t)=N_0\sum_{r=0}^{\infty}\left(-c^{\nu}\right)^r\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\Gamma(\lambda+\rho n)\Gamma(n+1)}{\Gamma(\alpha n+\beta)(\xi)_{pn}}\frac{\omega^n}{n!}\frac{t^{\lambda+\nu r+\rho n-1}}{\Gamma(\lambda+\nu r+\rho n)}$$

$$=N_0 t^{\lambda-1}\frac{\Gamma(\xi)}{\Gamma(\delta)}\sum_{r=0}^{\infty}\left(-c^{\nu}t^{\nu}\right)^r\sum_{n=0}^{\infty}\frac{\Gamma(\delta+qn)\Gamma(\lambda+\rho n)\Gamma(n+1)}{\Gamma(\alpha n+\beta)\Gamma(\xi+pn)\Gamma(\lambda+\nu r+\rho n)}\frac{\left(\omega t^{\rho}\right)^n}{n!},$$

by using the definition of Fox-Wright function (7), we easily get the desired result (20). This complete the proof of Theorem 1.

If we take $\xi=p=q=1$, then $E_{\alpha,\beta,p}^{\delta,\xi,q}(z)$ reduces to the generalized Mittag-Leffler function $E_{\alpha,\beta}^{\delta}(z)$ studied and given by Prabhakar [11], we get generalized fractional kinetic equation with its solution given by the following Corollary:

Corollary 1.1 Let $c,\omega,\nu,\lambda,\rho\in R^{+};\alpha,\beta,\delta\in C;\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0$.Then, the following generalized fractional kinetic equation

$$N(t)-N_0\,t^{\lambda-1}E_{\alpha,\beta}^{\delta}\left(\omega t^{\rho}\right)=-c^{\nu}\,{}_0D_t^{-\nu}N(t) \tag{24}$$

has the solution as given by

$$N(t)=\frac{N_0 t^{\lambda-1}}{\Gamma(\delta)}\sum_{r=0}^{\infty}\left(-c^{\nu}t^{\nu}\right)^r{}_2\Psi_2\left[\begin{matrix}(\delta,1),(\lambda,\rho)\\(\beta,\alpha),(\lambda+\nu r,\rho)\end{matrix};\omega t^{\rho}\right]. \tag{25}$$

Further, if we set $\delta=1$ in (24), then generalized M-L function (Prabhakar function) reduces to the Mittag-Leffler function $E_{\alpha,\beta}(z)$ given and studied by Wiman [22], we arrive at

Corollary 1.2 Let $c,\omega,\nu,\lambda,\rho\in R^{+};\alpha,\beta\in C;\ \mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0$.Then, the solution of the following generalized fractional kinetic equation

$$N(t)-N_0\,t^{\lambda-1}E_{\alpha,\beta}\left(\omega t^{\rho}\right)=-c^{\nu}\,{}_0D_t^{-\nu}N(t) \tag{26}$$

is given by

$$N(t) = N_0 t^{\lambda-1} \sum_{r=0}^{\infty} \left(-c^{\nu} t^{\nu}\right)^r {}_2\Psi_2 \left[\begin{matrix} (1,1),(\lambda,\rho) \\ (\beta,\alpha),(\lambda+\nu r,\rho) \end{matrix} ; \omega t^{\rho} \right]. \tag{27}$$

Theorem 2 Let $c,\omega,\nu \in R^{+}; \alpha,\beta,\delta,\xi \in C; \mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\delta), \mathrm{Re}(\xi) > 0; p,q > 0,$ $q \le \mathrm{Re}(\alpha) + p$. Then, the solution of the following generalized fractional kinetic equation

$$N(t) - N_0\, t^{\beta-1} E_{\alpha,\beta,p}^{\delta,\xi,q}\left(\omega t^{\alpha}\right) = -c^{\nu}\, {}_0D_t^{-\nu} N(t) \tag{28}$$

is given by

$$N(t) = N_0 t^{\beta-1} \frac{\Gamma(\xi)}{\Gamma(\delta)} \sum_{r=0}^{\infty} \left(-c^{\nu} t^{\nu}\right)^r {}_2\Psi_2 \left[\begin{matrix} (\delta,q),(1,1) \\ (\xi,p),(\beta+\nu r,\alpha) \end{matrix} ; \omega t^{\alpha} \right]. \tag{29}$$

Proof As in the proof of the Theorem 1, we make use (18) instead of (17), the Theorem 2 can be proved easily. Therefore, we omit the details of the proof.

If we set $\xi = p = q = 1$ in (28), then we have the solution of generalized fractional kinetic equation involving generalized M-L function (Prabhakar function) as follows.

Corollary 2.1 Let $c,\omega,\nu \in R^{+}; \alpha,\beta,\delta \in C;$ $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$ .Then, the generalized fractional kinetic equation

$$N(t) - N_0\, t^{\beta-1} E_{\alpha,\beta}^{\delta}\left(\omega t^{\alpha}\right) = -c^{\nu}\, {}_0D_t^{-\nu} N(t) \tag{30}$$

has solution as

$$N(t) = \frac{N_0 t^{\beta-1}}{\Gamma(\delta)} \sum_{r=0}^{\infty} \left(-c^{\nu} t^{\nu}\right)^r {}_1\Psi_1 \left[\begin{matrix} (\delta,1) \\ (\beta+\nu r,\alpha) \end{matrix} ; \omega t^{\alpha} \right]. \tag{31}$$

Further, If we take $\delta = 1$ in (30), then generalized M-L function reduces to the M-L function (Wiman function) $E_{\alpha,\beta}(z)$, and we have the following result:

Corollary 2.2 Let $c,\omega,\nu \in R^{+}; \alpha,\beta \in C; \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$ .Then, the solution of the following generalized fractional kinetic equation

$$N(t) - N_0\, t^{\beta-1} E_{\alpha,\beta}\left(\omega t^{\alpha}\right) = -c^{\nu}\, {}_0D_t^{-\nu} N(t) \tag{32}$$

is given as follows

$$N(t) = N_0 t^{\beta-1} \sum_{r=0}^{\infty} \left(-c^{\nu} t^{\nu}\right)^r {}_1\Psi_1 \left[\begin{matrix} (1,1) \\ (\beta+\nu r,\alpha) \end{matrix} ; \omega t^{\alpha} \right]. \tag{33}$$

4. Solution of Generalized Fractional Kinetic Equations by using Sumudu Transform

In this section, we discuss the solution of the generalized fractional kinetic equations (19) and (28) involving the generalized Mittag-Leffler function [14] by applying the Sumudu transform technique.

An integral transform, called the Sumudu transform was defined and studied by Watugala [21] to facilitate the process of solving differential and integral equations in the time domain, and for the use in various applications of system engineering and applied physics. It turns out that the Sumudu transform has very special and useful properties and it is useful in solving problems of science and engineering governing kinetic equations.The Sumudu transform has been shown to be the theoretical dual of the Laplace transform.

The Sumudu transform is derived from the classical Fourier integral and defined over the set of the functions

$$A=\left\{f(t)\,|\,\exists M,\tau_1,\tau_2>0,\left|f(t)\right|<M\,e^{|t|/\tau_j}\ if\ t\in(-1)^j\times[0,\infty)\right\},\tag{34}$$

by the following formula

$$G(u)=S\{f(t):u\}=\int_0^\infty e^{-t}f(ut)\mathrm{d}t,\ \ (-\tau_1<u<\tau_2),\tag{35}$$

where M is a real finite number and τ_1 and τ_2 can be finite or infinite (see [21]).

Hence, $G(u)$ is called as the Sumudu transform of $f(t)$. It is obvious that this is a linear operator. It can be easily verified that in (35) the function $G(u)$ keeps the same units as $f(t)$,for any real or complex number λ it gives that $S[f(\lambda t)]=G(\lambda u)$.

The Sumudu and Laplace transforms exhibit a duality relation that may be expressed either as

$$G\left(\frac{1}{u}\right)=u\,F(u)\ \text{ or }\ G(u)=\frac{1}{u}F\left(\frac{1}{u}\right),\tag{36}$$

$$F\left(\frac{1}{p}\right)=p\,G(p)\ \text{ or }\ F(p)=\frac{1}{p}G\left(\frac{1}{p}\right).\tag{37}$$

The convolution theorem for Sumudu transform is given by

$$S\{f*g;u\}=u\,S\{f;u\}\,S\{g;u\},\tag{38}$$

and if we apply convolution theorem for Sumudu transform, we observe that (2.4) can be written in the following form:

$$S\left\{{}_0D_t^{-\nu} f(t);u\right\} = u\,S\left\{\frac{t^{\nu-1}}{\Gamma(\nu)}\right\}.S\left\{f(t);u\right\} = u^{\nu}G(u). \tag{39}$$

Next, we begin by stating and proving the following lemma:

Lemma 3 Let $\min\left\{\mathrm{Re}(\lambda),\mathrm{Re}(\rho),\mathrm{Re}(u)\right\} > 0$. Then, the Sumudu transform of the generalized Mittag-Leffler function (5) is given by

$$S\left\{t^{\lambda-1}E_{\alpha,\beta,p}^{\delta,\xi,q}\left(t^{\rho}\right):u\right\} = u^{\lambda-1}\frac{\Gamma(\xi)}{\Gamma(\delta)}\,{}_3\Psi_2\left[\begin{matrix}(\delta,q),(\lambda,\rho),(1,1)\\ (\beta,\alpha),(\xi,p)\end{matrix};u^{\rho}\right], \tag{40}$$

where ${}_3\Psi_2(.)$ is defined by (7).

Proof By using (5) and taking (35) into account, we can easily obtain the formula (40) after a little simplification. Therefore, we omit the details involved.

If we put $\lambda = \beta$ and $\rho = \alpha$ in (40), then a special case of the above Lemma 3 is given by

Lemma 4 Let $\min\left\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(u)\right\} > 0$.Then, the following Sumudu transform of the generalized M-L function (5) holds true:

$$S\left\{t^{\beta-1}E_{\alpha,\beta,p}^{\delta,\xi,q}\left(t^{\alpha}\right):u\right\} = u^{\beta-1}\frac{\Gamma(\xi)}{\Gamma(\delta)}\,{}_2\Psi_1\left[\begin{matrix}(\delta,q),(1,1)\\ (\xi,p)\end{matrix};u^{\alpha}\right]. \tag{41}$$

Discussion I Let $c,\omega,\nu,\lambda,\rho \in R^{+}$ and $\mathrm{Re}(u) > 0$ with $|u| < c^{-1}\left(c \neq \omega\right)$. Also $\alpha,\beta,\delta,\xi \in C$; $\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta),\mathrm{Re}(\xi) > 0$; $p,q > 0, q \le \mathrm{Re}(\alpha) + p$.Then, the solution of the generalized fractional kinetic equation (19) is ginen by (20).

Applying the Sumudu transform to the both sides of (19) and using the relation (39),we get

$$N(u) = N_0\sum_{r=0}^{\infty}\left(-c^{\nu}\right)^{r}\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\Gamma(\rho n+\lambda)\Gamma(n+1)}{\Gamma(\alpha n+\beta)(\xi)_{pn}}\frac{u^{\rho n+\nu r+\lambda-1}\omega^{n}}{n!}. \tag{42}$$

Next, by taking Sumudu inverse transform of (42) and make use of the following formula

$S^{-1}\left\{u^{\nu-1}:t\right\} = \dfrac{t^{\nu-1}}{\Gamma(\nu)}$, $\left(\min\left\{\mathrm{Re}(\nu),\mathrm{Re}(u)\right\} > 0\right)$ and, also $S^{-1}\left\{N(u):t\right\} = N(t)$, then we get

$$S^{-1}\left\{N(u):t\right\} = N_0\sum_{r=0}^{\infty}\left(-c^{\nu}\right)^{r}\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\Gamma(\rho n+\lambda)\Gamma(n+1)}{\Gamma(\alpha n+\beta)(\xi)_{pn}}\frac{\omega^{n}}{n!}\,S^{-1}\left\{u^{\rho n+\nu r+\lambda-1}:t\right\}$$

$$N(t) = N_0 \sum_{r=0}^{\infty} \left(-c^{\nu}\right)^r \sum_{n=0}^{\infty} \frac{(\delta)_{qn} \Gamma(\rho n + \lambda) \Gamma(n+1)}{\Gamma(\alpha n + \beta)(\xi)_{pn}} \frac{\omega^n}{n!} \frac{t^{\rho n + \nu r + \lambda - 1}}{\Gamma(\rho n + \nu r + \lambda)},$$

by using the definition of Fox-Wright function (7), we easily arrive the desired result (20).

Discussion II Let $c, \omega, \nu \in R^+$ and $\mathrm{Re}(u) > 0$ with $|u| < c^{-1}\,(c \neq \omega)$.Let also $\alpha, \beta, \delta, \xi \in C$; $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\delta), \mathrm{Re}(\xi) > 0$; $p, q > 0, q \leq \mathrm{Re}(\alpha) + p$.Then, the solution of the generalized fractional kinetic equation (28) is given by (29).

By following Theorem 2 and applying Sumudu transform instead of Laplace transform, we easily obtain the result (29) which is the solution of (28).

5. Conclusion

In this paper we have introduced an extended fractional generalization of the standard kinetic equation and established solution for the same. Fractional kinetic equation can be used to compute the particle reaction rate and describes the statistical mechanics associated with the particle distribution function. The generalized fractional kinetic equation discussed in this paper, involving generalized Mittag-Leffler function contains a number of known fractional kinetic equations involving various other special functions. We also conclude this paper by remarking that the results presented here are general enough to yield, as their special cases, solutions of a number of known or new fractional kinetic equations involving such other special functions as (for example) those considered by Haubold and Mathai [4] and Saxena et al. [16, 17, 18](see, also [1, 6, 8, 15]).

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