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# ON GENERALIZED FRACTIONAL CALCULUS OF THE GENERALIZED K-BESSEL FUNCTION

**Bhupender Singh Shaktawat, Devendra Singh Rawat and Rajeev Kumar Gupta**
Department of Mathematics and Statistics
Jai Narain Vyas University, Jodhpur – 342005, India
Email: b.s.shaktawatmath@gmail.com, dev.rawatjnvu@gmail.com, drrkbgupta@yahoo.co.in

**Abstract:** The object of this paper is to establish four theorems for the generalized fractional integration and differentiation of the generalized k-Bessel function. It has been shown that the generalized fractional integrals and derivatives of the generalized k-Bessel function are established in terms of generalized Wright hypergeometric function. Some elegant results obtained by Purohit et al.[8] and Agarwal et al.[1] are the special cases of the main results are also pointed out.

**Keywords and Phrases**: Generalized fractional calculus operators, Generalized k-Bassel function, Generalized Wright hypergeometric function.

**2010 Mathematics Subject Classification**: 33C10, 33C20, 33B15, 26A33, 33C65

## 1. Introduction and Preliminaries

Recently, a new generalization of Bessel function called generalized k-Bessel function is defined by Mondal [6] in the series as

$$W_{v,c}^{k}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(n\mathrm{k}+v+\mathrm{k})} \frac{(\frac{z}{2})^{2n+\frac{v}{k}}}{n!} \tag{1}$$

where $k > 0$, $Re(v) > -1, c \in R$ and $\Gamma_k(x)$ is the k-gamma function [2] defined as

$$\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{k}} t^{x-1}\, dt \; (x \in C, k \in R, Re(x) > 0) \tag{2}$$

and it follows easily that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\, \Gamma(\frac{x}{k}) \tag{3}$$

If we take $k = 1$ $and$ $c = 1$ in (1), then $W_{v,c}^{k}$ reduce to the Classical Bessel function $J_v$ ([3], 7.2 (2)) as (see also [7], [12])

$$J_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+v}}{\Gamma(n+v+1)n!} \tag{4}$$

The generalized Wright hypergeometric function ${}_p\psi_q(\mathrm{z})$, for z ∈ C, Complex $a_i$, $b_j$ ∈ C and Real $\alpha_i$, $\beta_j$ ∈ R ($\alpha_i$, $\beta_j \neq 0$, $i$ = 1, 2..., p; j = 1, 2..., q) is defined by

$${}_p\psi_q(\mathrm{z}) = {}_p\psi_q\left[\begin{matrix}(a_i\,,\alpha_i)_{1,p}\\ (b_j\,,\beta_j)_{1,q}\end{matrix}\,;z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q} \Gamma(b_j+\beta_j\, n)}\ \frac{z^n}{n!} \tag{5}$$

Wright [13] introduced the generalized Wright function (5) and proved several theorems on the asymptotic expansion of ${}_p\psi_q(\mathrm{z})$ (for instance, see [13], [14]) for all values of the argument z, under the condition:

$$\sum_{j=1}^{q} \beta_j \ -\sum_{i=1}^{P} \alpha_i \ > -1 \tag{6}$$

An useful generalization of the hypergeometric fractional operators including the Saigo operators [9], has been introduced by Marichev [4] (see details in Samko et al. [11], p – 194, (10.47) and whole section 10.3) and later extended and studied by Saigo and Maeda [10] in terms of any complex order with Appell function $F_3(.)$ in the Kernel as follows:

Let $\alpha$, $\alpha'$, $\beta$, $\beta'$, $\gamma$ ∈C and $x > 0$, then the generalized fractional calculus operators involving the Appell function $F_3(\cdot)$ are defined as follows:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1}\ t^{-\alpha'}\, F_3\left(\alpha, \alpha'\, \beta\,, \beta', \gamma\,; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt$$

$$(Re(\gamma) > 0) \tag{7}$$

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^{\infty} (t-x)^{\gamma-1}\ t^{-\alpha}\, F_3\left(\alpha, \alpha', \beta\,, \beta', \gamma\,; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)dt$$

$$(Re(\gamma) > 0) \quad (8)$$

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \quad (Re(\gamma) > 0) \tag{9}$$

$$= \left(\frac{d}{dx}\right)^k \left(I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k} f\right)(x)$$

$$(Re(\gamma) > 0\,; k = [Re\,(\gamma)] + 1)$$

and

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \quad (Re(\gamma) > 0) \tag{10}$$

$$= \left(\frac{-d}{dx}\right)^k \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k} f\right)(x)$$

$$(Re(\gamma) > 0\,, k = [Re\,(\gamma)] + 1)$$

These operators (7) – (10) reduce to the Saigo fractional calculus operators [9] as

$$\left(I_{0+}^{\alpha,0,\beta,\beta',\gamma} f\right)(x) = \left(I_{0+}^{\gamma,\alpha-\gamma,-\beta}\, f\right)\,(x) \quad (\gamma \in C) \tag{11}$$

$$\left(I_{-}^{\alpha,0,\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{\gamma,\alpha\,-\gamma,-\beta} f\right)\,(x) \quad (\gamma \in C) \tag{12}$$

$$\left(D_{0+}^{0,\alpha',\beta,\beta',\gamma} f\right)\,(x) = \left(D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma}\, f\right)\,(x) \quad (Re\,(\gamma) > 0) \tag{13}$$

$$\left(D_{-}^{\,0,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(D_{-}^{\,\gamma,\alpha'-\gamma,\beta'-\,\gamma} f\right)(x) \qquad (Re\,(\gamma) > 0) \tag{14}$$

The left-hand sided and right-hand sided generalized integration of the type (7) and (8) for a power function formulas (see [10], P-394, eq.(4.18) and (4.19) are given by $\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\; t^{\sigma-1}\right)(x)$

$$= \frac{\Gamma(\sigma)\,\Gamma(\sigma+\gamma-\alpha-\alpha'-\beta)\,\Gamma(\sigma+\beta'-\,\alpha')}{\Gamma(\sigma+\beta')\,\Gamma(\sigma+\gamma-\alpha-\alpha')\;\Gamma(\sigma+\gamma-\alpha'-\beta)}\; x^{\sigma-\alpha-\alpha'+\gamma-1}\;, \tag{15}$$

$$where\; Re\,(\gamma) > 0, Re\,(\sigma) > \max\{0, Re\,(\alpha + \alpha' + \beta - \gamma), Re\,(\alpha' - \beta')\}$$

and $\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1}\right)(x)$

$$= \frac{\Gamma\,(1-\sigma-\gamma+\alpha+\alpha')\;\;\Gamma(1-\sigma+\alpha+\beta'-\gamma)\,\Gamma(1-\sigma-\beta)}{\Gamma(1-\sigma))\,\Gamma(1-\sigma+\alpha+\alpha'+\beta'-\gamma)\;\;\Gamma(1-\sigma+\alpha-\beta)}\; x^{\sigma-\alpha-\alpha'+\gamma-1}, \tag{16}$$

$where\; Re\,(\gamma) > 0, Re\,(\sigma) < 1 + \min\,\{Re\,(-\beta), Re\;\,(\alpha + \alpha' - \gamma), Re\,(\alpha + \beta' - \gamma)\}.$

## 2. Generalized Fractional Integration of generalized k-Bessel function

In this section, we establish two image formulas for the generalized k-Bessel function involving Saigo-Maeda fractional integral operators (7) and (8), in terms of the generalized Wright function. These formulas are given by the following theorems.

**Theorem 1** Let $\alpha, \alpha', \beta, \beta', \gamma, \upsilon, \sigma \in C, k > 0, c \in R$ and $x > 0$ *be* such that $Re(\upsilon) > -1, Re\,(\gamma) > 0$ and

$$Re\left(\frac{\sigma+\upsilon}{k}\right) > \max[0, Re\,(\alpha +\; \alpha' +\; \beta -\; \gamma), Re(\alpha' -\; \beta')]. \tag{17}$$

Then there holds the following formula

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\frac{\sigma}{k}-1}\; W_{\upsilon,c}^{k}\;(t)\right]\right)\;(x)$$

$$= \frac{x^{\frac{\sigma+\upsilon}{k}+\gamma-\alpha-\alpha'-1}}{(2k)^{\frac{\upsilon}{k}}}\quad {}_3\psi_4\left[\begin{array}{l}\left(\dfrac{\sigma+\upsilon}{k}, 2\right), \left(\dfrac{\sigma+\upsilon}{k} + \gamma - \alpha -\; \alpha' -\; \beta, 2\right),\\ \left(\dfrac{\upsilon}{k} + 1{,}1\right), \left(\dfrac{\sigma+\upsilon}{k} + \gamma - \alpha -\; \alpha', 2\right),\end{array}\right.$$

$$\left.\begin{matrix}\left(\frac{\sigma+\upsilon}{k}+\beta'-\alpha',2\right)\\ \left(\frac{\sigma+\upsilon}{k}+\gamma-\alpha'-\beta,2\right),\left(\frac{\sigma+\upsilon}{k}+\beta',2\right)\end{matrix};\frac{-cx^2}{4k}\right] \tag{18}$$

**Proof :** By using (1), (7) and changing the order of integration and summation, we have

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(t)\right]\right)(x)$$

$$=\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\frac{\sigma}{k}-1}\sum_{n=0}^{\infty}\frac{(-c)^n\left(\frac{t}{2}\right)^{2n+\frac{\upsilon}{k}}}{\Gamma_k(n\mathrm{k}+\upsilon+\mathrm{k})\,n!}\right]\right)(x)$$

$$=\sum_{n=0}^{\infty}\frac{(-c)^n\left(\frac{1}{2}\right)^{2n+\frac{\upsilon}{k}}}{\Gamma_k(n\mathrm{k}+\upsilon+\mathrm{k})\,n!}\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\,t^{\frac{\sigma+\upsilon}{k}+2n-1}\right)(x)$$

Now, using (1.3) and (1.15), we get

$$=\frac{x^{\frac{\sigma+\upsilon}{k}+\gamma-\alpha-\alpha'-1}}{(2k)^{\frac{\upsilon}{k}}}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\sigma+\upsilon}{k}+2n\right)\Gamma\left(\frac{\sigma+\upsilon}{k}+\gamma-\alpha-\alpha'-\beta+2n\right)}{\Gamma\left(\frac{\upsilon}{k}+1+n\right)\Gamma\left(\frac{\sigma+\upsilon}{k}+\gamma-\alpha-\alpha'+2n\right)}$$

$$\times\frac{\Gamma\left(\frac{\sigma+\upsilon}{k}-\alpha'+\beta'+2n\right)}{\Gamma\left(\frac{\sigma+\upsilon}{k}+\gamma-\alpha'-\beta+2n\right)\Gamma\left(\frac{\sigma+\upsilon}{k}+\beta'+2n\right)}\quad\frac{\left(\frac{-cx^2}{4k}\right)^n}{n!} \tag{19}$$

Interpreting the right-hand side of the above equation (19), in view of the definition (5), we arrive at the result (18). This completes the proof of theorem 1.

If we take $k=1$ and $c=1$ in (18), then generalized k-Bessel function $W_{\upsilon,c}^{k}$ reduce to the Classical Bessel function $J_{\upsilon}$, we obtain the following known result given by Purohit et a1.([8], P-24, eq.(10)).

**Corollary 1.1** Let $\alpha,\alpha',\beta,\beta',\gamma,\upsilon,\sigma\in C$ and $x>0$ *be* such that $Re(\upsilon)>-1, Re(\gamma)>0$ and $Re(\sigma+\upsilon)>\max[0, Re(\alpha+\alpha'+\beta-\gamma), Re(\alpha'-\beta')]$, then the following formula holds true:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}J_{\upsilon}(t)\right]\right)(x)$$

$$=\frac{x^{\sigma+\upsilon+\gamma-\alpha-\alpha'-1}}{2^{\upsilon}}\;{}_3\psi_4\left[\begin{matrix}(\sigma+\upsilon,2),(\sigma+\upsilon+\gamma-\alpha-\alpha'-\beta,2),\\(\upsilon+1,1),(\sigma+\upsilon+\gamma-\alpha-\alpha',2),\end{matrix}\right.$$

$$\left.\begin{matrix}(\sigma+\upsilon+\beta'-\alpha',2)\\(\sigma+\upsilon+\gamma-\alpha'-\beta,2),(\sigma+\upsilon+\beta',2)\end{matrix};\frac{-x^2}{4}\right]. \tag{20}$$

In view of the relation (11) into account, then we get a formula concerning the left-sided Saigo fractional integral operator (see [9], [10]) in the following corollary.

**Corollary 1.2** Let $\alpha, \beta, \gamma, \upsilon, \sigma \in C, k > 0,\ c \in R$ and $x > 0$ *be* such that $Re(\upsilon) > -1, Re\,(\alpha) > 0$ and $Re\left(\frac{\sigma+\upsilon}{k}\right) > \max[0, Re\,(\beta - \gamma)]$, then the following result holds true:

$$\left(I_{0+}^{\alpha,\beta,\gamma}\left[t^{\frac{\sigma}{k}-1}\, W_{\upsilon,c}^{k}(t)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma+\upsilon}{k}-\beta-1}}{(2k)^{\frac{\upsilon}{k}}}\ {}_2\psi_3\left[\begin{matrix}\left(\frac{\sigma+\upsilon}{k},2\right),\left(\frac{\sigma+\upsilon}{k}-\beta+\gamma,2\right)\\ \left(\frac{\upsilon}{k}+1,1\right),\left(\frac{\sigma+\upsilon}{k}-\beta,2\right),\left(\frac{\sigma+\upsilon}{k}+\alpha+\gamma,2\right)\end{matrix};\ \frac{-cx^2}{4k}\right]. \tag{21}$$

Further, if we set $\beta = -\alpha$ in above Corollary 1.2, then we obtain a formula regarding the left-sided Riemann-Liouville fractional integral operator (see [5], [9]) asserted by the following corollary.

**Corollary 1.3** Let $\alpha, \upsilon, \sigma \in C,\ k > 0,\ c \in R$ and $x > 0$ be such that $Re(\upsilon) > -1$ *and* $Re(\alpha) > 0$, then there holds the following formula:

$$\left(I_{0+}^{\alpha}\left[t^{\frac{\sigma}{k}-1}\, W_{\upsilon,c}^{k}(t)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma+\upsilon}{k}+\alpha-1}}{(2k)^{\frac{\upsilon}{k}}}\ {}_1\psi_2\left[\begin{matrix}\left(\frac{\sigma+\upsilon}{k},2\right)\\ \left(\frac{\upsilon}{k}+1,1\right),\left(\frac{\sigma+\upsilon}{k}+\alpha,2\right)\end{matrix};\ \frac{-cx^2}{4k}\right]. \tag{22}$$

**Theorems 2** Let $\alpha, \alpha', \beta, \beta', \gamma, \upsilon, \sigma \in C, k > 0, c \in R$ and $x > 0$ *be* such that $Re(\upsilon) > -1, Re\,(\gamma) > 0$ and

$$Re\left(\frac{\sigma-\upsilon}{k}\right) < 1 + \min\left[\,Re(-\beta), Re\,(\alpha+\alpha'-\gamma), Re(\alpha+\beta'-\gamma)\right]. \tag{23}$$

Then the following formula holds true:

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\frac{\sigma}{k}-1}\, W_{\upsilon,c}^{k}\left(\frac{1}{t}\right)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma-\upsilon}{k}+\gamma-\alpha-\alpha'-1}}{(2k)^{\frac{\upsilon}{k}}}\ {}_3\psi_4\left[\begin{matrix}\left(1+\alpha+\alpha'-\gamma-\frac{\sigma-\upsilon}{k},2\right),\left(1+\alpha+\beta'-\gamma-\frac{\sigma-\upsilon}{k},2\right),\left(1-\beta-\frac{\sigma-\upsilon}{k},2\right)\\ \left(\frac{\upsilon}{k}+1,1\right),\left(1-\frac{\sigma-\upsilon}{k},2\right),\left(1+\alpha+\alpha'+\beta'-\gamma-\frac{\sigma-\upsilon}{k},2\right),\left(1+\alpha-\beta-\frac{\sigma-\upsilon}{k},2\right)\end{matrix};\ \frac{-c}{4kx^2}\right]. \tag{24}$$

**Proof :** We can establish (24) by similar argument as in the proof of Theorem 1, using (16) instead of (15). Therefore we omit the details.

If we take $k = 1$ and $c = 1$ in (24), then generalized k-Bessel function $W_{\upsilon,c}^{k}$ reduce to the Classical Bessel function $J_{\upsilon}$, we obtain the following known result given by Purohit et a1. ([8], P-24, eq.(13)).

**Corollary 2.1** Let $\alpha, \alpha', \beta, \beta', \gamma, \upsilon, \sigma \in C$ and $x > 0$ *be* such that $Re(\upsilon) > -1, Re\,(\gamma) > 0$ and $Re(\sigma - \upsilon) < 1 + \min\,[\,Re(-\beta), Re\,(\alpha + \alpha' - \gamma), Re(\alpha + \beta' - \gamma)]$, then there holds the following result:

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1} J_{\upsilon}\left(\frac{1}{t}\right)\right]\right)(x)$$

$$= \frac{x^{\sigma-\upsilon+\gamma-\alpha-\alpha'-1}}{2^{\upsilon}}\ {}_{3}\psi_{4}\left[\begin{matrix}(1+\alpha+\alpha'-\gamma-\sigma+\upsilon,2),(1+\alpha+\beta'-\gamma-\sigma+\upsilon,2), (1-\beta-\sigma+\upsilon,2) \\ (\upsilon+1,1),(1-\sigma+\upsilon,2), (1+\alpha+\alpha'+\beta'-\gamma-\sigma+\upsilon,2),(1+\alpha-\beta-\sigma+\upsilon,2)\end{matrix};\frac{-1}{4x^{2}}\right]. \quad (25)$$

By using the relation (12) into account, then we get an integral formula concerning the right-sided Saigo fractional integral operator (see [9], [10]) in the following corollary.

**Corollary 2.2** Let $\alpha, \beta, \gamma, \upsilon, \sigma \in C, k > 0, c \in R$ and $x > 0$ *be* such that $Re(\upsilon) > -1, Re\,(\alpha) > 0$ and $Re\left(\frac{\sigma-\upsilon}{k}\right) < 1 + \min\,[\,Re(\beta), Re\,(\gamma)]$, then the following result holds true:

$$\left(I_{-}^{\alpha,\beta,\gamma}\left[t^{\frac{\sigma}{k}-1}\, W_{\upsilon,c}^{k}\left(\frac{1}{t}\right)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma-\upsilon}{k}-\beta-1}}{(2k)^{\frac{\upsilon}{k}}}\ {}_{2}\psi_{3}\left[\begin{matrix}\left(1+\beta-\frac{\sigma-\upsilon}{k},2\right),\left(1+\gamma-\frac{\sigma-\upsilon}{k},2\right) \\ \left(\frac{\upsilon}{k}+1,1\right),\left(1-\frac{\sigma-\upsilon}{k},2\right),\left(1+\alpha+\beta+\gamma-\frac{\sigma-\upsilon}{k},2\right)\end{matrix};\frac{-c}{4kx^{2}}\right]. \quad (26)$$

Further, if we put $\beta = -\alpha$ in (26), we obtain a formula concering the right-sided Riemann-Liouville fractional integral operator (see [5], [9]) as in the following corollary.

**Corollary 2.3** Let $\alpha, \upsilon, \sigma \in C, k > 0, c \in R$ and $x > 0$ *be* such that $Re(\upsilon) > -1$ *and* $Re\,(\alpha) > 0$, then there holds the following formula:

$$\left(I_{-}^{\alpha}\left[t^{\frac{\sigma}{k}-1}\, W_{\upsilon,c}^{k}\left(\frac{1}{t}\right)\right](x)\right)$$

$$= \frac{x^{\frac{\sigma-\upsilon}{k}+\alpha-1}}{(2k)^{\frac{\upsilon}{k}}}\ {}_{1}\psi_{2}\left[\begin{matrix}\left(1-\alpha-\frac{\sigma-\upsilon}{k},2\right) \\ \left(\frac{\upsilon}{k}+1,1\right),\left(1-\frac{\sigma-\upsilon}{k},2\right)\end{matrix};\frac{-c}{4kx^{2}}\right]. \quad (27)$$

## 3. Generalized fractional Differentiation of generalized k-Bessel function

In this section, we establish two image formulas for the generalized k-Bessel function involving Saigo-Maeda fractional differential operators (9) and (10), in terms of the Wright hypergeometric function. These formulas are given by the following theorems.

**Theorems 3** Let $\alpha, \alpha', \beta, \beta', \gamma, \upsilon, \sigma \in C, k > 0, c \in R$ and $x > 0$ *be* such that $Re(\upsilon) > -1, Re\,(\gamma) > 0$ and

$$Re\left(\frac{\sigma+\upsilon}{k}\right) > \max[0, Re\,(\gamma - \alpha - \alpha' - \beta), Re(\beta - \alpha\,)]. \tag{28}$$

Then the following formula holds true:

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\frac{\sigma}{k}-1}\,W_{\upsilon,c}^{k}\,(t)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma+\upsilon}{k}-\gamma+\alpha+\alpha'-1}}{(2k)^{\frac{\upsilon}{k}}}\;{}_3\psi_4\left[\begin{matrix}\left(\frac{\sigma+\upsilon}{k},2\right),\left(\frac{\sigma+\upsilon}{k}-\gamma+\alpha+\alpha'+\beta',2\right),\left(\frac{\sigma+\upsilon}{k}+\alpha-\beta,2\right)\\ \left(\frac{\upsilon}{k}+1,1\right),\left(\frac{\sigma+\upsilon}{k}-\beta,2\right),\left(\frac{\sigma+\upsilon}{k}-\gamma+\alpha+\alpha',2\right),\left(\frac{\sigma+\upsilon}{k}-\gamma+\alpha+\beta',2\right)\end{matrix};\frac{-cx^2}{4k}\right]. \tag{29}$$

**Proof:** By using (1), (9) and changing the order of differentiation and summation, we have

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\frac{\sigma}{k}}\,W_{\upsilon,c}^{k}\,(t)\right]\right)(x)$$

$$= \left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\frac{\sigma}{k}}\sum_{n=0}^{\infty}\frac{(-c)^n\,\left(\frac{t}{2}\right)^{2n+\frac{\upsilon}{k}}}{\Gamma_k\,(nk+\upsilon+k)\,n!}\right]\right)(x)$$

$$= \sum_{n=0}^{\infty}\frac{(-c)^n\,\left(\frac{1}{2}\right)^{2n+\frac{\upsilon}{k}}}{\Gamma_k\,(nk+\upsilon+k)\,n!}\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\,t^{\frac{\sigma+\upsilon}{k}+2n-1}\right)(x)$$

$$= \sum_{n=0}^{\infty}\frac{(-c)^n\,\left(\frac{1}{2}\right)^{2n+\frac{\upsilon}{k}}}{\Gamma_k\,(nk+\upsilon+k)\,n!}\left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}\,t^{\frac{\sigma+\upsilon}{k}+2n-1}\right)(x)$$

Now, using (3) and (15), we get

$$= \frac{x^{\frac{\sigma+\upsilon}{k}-\gamma+\alpha+\alpha'-1}}{(2k)^{\frac{\upsilon}{k}}}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\sigma+\upsilon}{k}+2n\right)\Gamma\left(\frac{\sigma+\upsilon}{k}-\gamma+\alpha+\alpha'+\beta'+2n\right)}{\Gamma\left(\frac{\upsilon}{k}+1+n\right)\Gamma\left(\frac{\sigma+\upsilon}{k}-\beta+2n\right)}$$

$$\times\frac{\Gamma\left(\frac{\sigma+\upsilon}{k}+\alpha-\beta+2n\right)}{\Gamma\left(\frac{\sigma+\upsilon}{k}-\gamma+\alpha+\alpha'+2n\right)\Gamma\left(\frac{\sigma+\upsilon}{k}-\gamma+\alpha+\beta'+2n\right)}\;\frac{\left(\frac{-cx^2}{4k}\right)^n}{n!} \tag{30}$$

Finally, using the definition of the generalized Wright hypergeometric function given by (5), the above equation (30) leads to the result (29). This completes the proof of Theorem 3.

If we take $k = 1$ *and* $c = 1$ in (29), then generalized k-Bessel function $W_{\upsilon,c}^{k}$ reduce to the Classical Bessel function $J_\upsilon$, we get the following known result given by Agarwal et a1.([1], P-105, eq.(14)).

**Corollary 3.1** Let $\alpha, \alpha', \beta, \beta', \gamma, \upsilon, \sigma \in C$ and $x > 0$ *be* such that $Re(\upsilon) > -1, Re(\gamma) > 0$ and $Re(\sigma + \upsilon) > \max[0, Re(\gamma - \alpha - \alpha' - \beta), Re(\beta - \alpha)]$, then there holds the following result:

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\,[t^{\sigma-1}J_{\upsilon}(t)]\right)(x)$$

$$= \frac{x^{\sigma+\upsilon-\gamma+\alpha+\alpha'-1}}{2^{\upsilon}}\; {}_3\psi_4\left[\begin{matrix}(\sigma+\upsilon,2),(\sigma+\upsilon-\gamma+\alpha+\alpha'+\beta',2),\\ (\upsilon+1,1),(\sigma+\upsilon-\beta,2),\end{matrix}\right.$$

$$\left.\begin{matrix}(\sigma+\upsilon+\alpha-\beta,2)\\ (\sigma+\upsilon-\gamma+\alpha+\alpha',2),(\sigma+\upsilon-\gamma+\alpha+\beta',2)\end{matrix};\frac{-x^2}{4}\right] \tag{31}$$

In view of the relation (13), we get a formula concerning the left-sided Saigo fractional derivative operator (see [9], [10]) in the following corollary.

**Corollary 3.2** Let $\alpha, \beta, \gamma, \upsilon, \sigma \in C, k > 0,\ c \in R$ and $x > 0$ *be* such that $Re(\upsilon) > -1, Re(\alpha) > 0$ and $Re\left(\frac{\sigma+\upsilon}{k}\right) > \max[0, Re(\gamma - \beta), Re(-\alpha - \beta - \gamma)]$, then the following formula holds true:

$$\left(D_{0+}^{\alpha,\beta,\gamma}\left[t^{\frac{\sigma}{k}-1}\,W_{\upsilon,c}^{k}(t)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma+\upsilon}{k}+\beta-1}}{(2k)^{\frac{\upsilon}{k}}}\; {}_2\psi_3\left[\begin{matrix}\left(\frac{\sigma+\upsilon}{k},2\right),\left(\frac{\sigma+\upsilon}{k}+\alpha+\beta+\gamma,2\right)\\ \left(\frac{\upsilon}{k}+1,1\right),\left(\frac{\sigma+\upsilon}{k}+\gamma,2\right),\left(\frac{\sigma+\upsilon}{k}+\beta,2\right)\end{matrix};\frac{-cx^2}{4k}\right] \tag{32}$$

Further, if we set $\beta = -\alpha$ in above Corollary 3.2, we obtain a formula concerning the left-sided Riemann-Liouville fractional derivative operator (see [5], [9]) asserted in the following corollary.

**Corollary 3.3** Let $\alpha, \upsilon, \sigma \in C, k > 0,\ c \in R$ and $x > 0$ be such that $(\upsilon) > -1, and$ $Re(\alpha) > 0$, then there holds the following result:

$$\left(D_{0+}^{\alpha}\left[t^{\frac{\sigma}{k}-1}\,W_{\upsilon,c}^{k}(t)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma+\upsilon}{k}-\alpha-1}}{(2k)^{\frac{\upsilon}{k}}}\; {}_1\psi_2\left[\begin{matrix}\left(\frac{\sigma+\upsilon}{k},2\right)\\ \left(\frac{\upsilon}{k}+1,1\right),\left(\frac{\sigma+\upsilon}{k}-\alpha,2\right)\end{matrix};\frac{-cx^2}{4k}\right] \tag{33}$$

**Theorems 4** Let $\alpha, \alpha', \beta, \beta', \gamma, \upsilon, \sigma \in C, k > 0, c \in R$ and $x > 0$ *be* such that $Re(\upsilon) > -1, Re(\gamma) > 0$ and

$$Re\left(\frac{\sigma-\upsilon}{k}\right) < 1 + \min[Re(\beta'), Re(\gamma - \alpha - \alpha'), Re(\gamma - \alpha' - \beta)]. \tag{34}$$

Then the following formula holds true:

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\frac{\sigma}{k}-1}\,W_{\upsilon,c}^{k}\left(\frac{1}{t}\right)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma-\upsilon}{k}-\gamma+\alpha+\alpha'-1}}{(2k)^{\frac{\upsilon}{k}}} \; {}_3\psi_4 \left[ \begin{matrix} \left(1-\alpha-\alpha'+\gamma-\frac{\sigma-\upsilon}{k},2\right),\left(1-\alpha'-\beta+\gamma-\frac{\sigma-\upsilon}{k},2\right), \\ \left(\frac{\upsilon}{k}+1,1\right),\left(1-\frac{\sigma-\upsilon}{k},2\right), \end{matrix} \right.$$

$$\left. \begin{matrix} \left(1+\beta'-\frac{\sigma-\upsilon}{k},2\right) \\ \left(1-\alpha-\alpha'-\beta+\gamma-\frac{\sigma-\upsilon}{k},2\right),\left(1-\alpha'+\beta'-\frac{\sigma-\upsilon}{k},2\right) \end{matrix} ; \frac{-c}{4kx^2} \right] \tag{35}$$

**Proof :** A similar argument as in the proof of Theorem 3 will establish the result here. So we choose to skip the detailed account of its proof.

If we put $k=1$ and $c=1$ in (35), then generalized k-Bessel function $W_{\upsilon,c}^{k}$ reduce to the Classical Bessel function $J_\upsilon$, we obtain the following known result given by Agarwal et a1. ([1], P-105, eq.(18)).

**Corollary 4.1** Let $\alpha,\alpha',\beta,\beta',\gamma,\upsilon,\sigma\in C$ and $x>0$ *be* such that $Re(\upsilon)>-1, Re(\gamma)>0$ and $Re(\sigma-\upsilon)<1+\min[Re(\beta'), Re(\gamma-\alpha-\alpha'), Re(\gamma-\alpha'-\beta)]$, then there holds the following result:

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}J_\upsilon\left(\frac{1}{t}\right)\right]\right)(x)$$

$$= \frac{x^{\sigma-\upsilon-\gamma+\alpha+\alpha'-1}}{2^\upsilon} \; {}_3\psi_4 \left[ \begin{matrix} (1-\alpha-\alpha'+\gamma-\sigma+\upsilon,2),(1-\alpha'-\beta+\gamma-\sigma+\upsilon,2), \\ (\upsilon+1,1),(1-\sigma+\upsilon,2), \end{matrix} \right.$$

$$\left. \begin{matrix} (1+\beta'-\sigma+\upsilon,2) \\ (1-\alpha-\alpha'-\beta+\gamma-\sigma+\upsilon,2),(1-\alpha'+\beta'-\sigma+\upsilon,2) \end{matrix} ; \frac{-1}{4x^2} \right] \tag{36}$$

In view of the relation (14), we get a formula concerning the right-sided Saigo fractional derivative operator (see [9], [10]) given in the following corollary.

**Corollary 4.2** Let $\alpha,\beta,\gamma,\upsilon,\sigma\in C, k>0, c\in R$ and $x>0$ *be* such that $Re(\upsilon)>-1, Re(\alpha)>0$ and $Re(\sigma-\upsilon)<1+\min[0, Re(-\beta), Re(\alpha+\gamma)]$, then there holds the following formula :

$$\left(D_{-}^{\alpha,\beta,\gamma}\left[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}\left(\frac{1}{t}\right)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma-\upsilon}{k}+\beta-1}}{(2k)^{\frac{\upsilon}{k}}} \; {}_2\psi_3 \left[ \begin{matrix} \left(1-\beta-\frac{\sigma-\upsilon}{k},2\right),\left(1+\alpha+\gamma-\frac{\sigma-\upsilon}{k},2\right) \\ \left(\frac{\upsilon}{k}+1,1\right),\left(1-\frac{\sigma-\upsilon}{k},2\right),\left(1-\beta+\gamma-\frac{\sigma-\upsilon}{k},2\right) \end{matrix} ; \frac{-c}{4kx^2} \right] \tag{37}$$

Further, if we set $\beta=-\alpha$ in (37), we obtain a formula concering the right-sided Riemann-Liouville fractional derivative operator (see [5], [9]) stated in the following corollary.

**Corollary 4.3** Let $\alpha,\upsilon,\sigma\in C, k>0, c\in R$ and $x>0$ *be* such that $Re(\upsilon)>-1$ *and* $Re(\alpha)>0$, then the following results holds true :

$$\left(D_{-}^{\alpha}\left[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}\left(\frac{1}{t}\right)\right]\right)(x)$$
$$=\frac{x^{\frac{\sigma-\upsilon}{k}-\alpha-1}}{(2k)^{\frac{\upsilon}{k}}}\;{}_1\psi_2\left[\begin{matrix}\left(1+\alpha-\frac{\sigma-\upsilon}{k},2\right)\\ \left(\frac{\upsilon}{k}+1,1\right),\left(1-\frac{\sigma-\upsilon}{k},2\right)\end{matrix};\frac{-c}{4kx^2}\right] \tag{38}$$

## 4. Concluding Remarks

In the present paper, we have studied and obtain generalized fractional integral and derivative formulas of the generalized k-Bessel function. The results have been developed in terms of generalized Wright hypergeometric function with the help of Saigo-Maeda fractional power function formulas. We can easily obtain many results of earlier work done by Purohit et al.[8] and Agarwal et al.[1] as special cases of our main results.

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