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ALMOST C(λ) MANIFOLDS ADMITTING W_2 CURVATURE TENSOR

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Abstract : The object of the present paper is to study almost C(λ) manifolds admitting W_2 -curvature tensor. We consider almost C(λ) manifolds satisfying some curvature conditions such as $W_2 = 0$, $W_2.S=0$, $R(\xi ,U). W_2 = 0$, $R (\xi ,Y).S = 0$ and $R(X,Y). W_2 =0$.

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1. Introduction

The notion of almost C(λ) manifolds was introduced by Janssen and Vanhecke [5]. Further Olszak and Rosca [8] investigated such manifolds. Again Kharitonova [6] studied conformally flat almost C(λ) manifolds. In the paper [1] the first author of this paper studied Ricci tensor and quasi-conformal curvature tensor of almost C(λ) manifolds. On the other hand in 1970 Pokhariyal and Mishra [10] have introduced new curvature tensor called W_2-curvature tensor in a Riemannian manifold and studied their properties. Further Pokhariyal [9] has studied some properties of this curvature tensor in Sasakian manifold. Matsumoto, Ianus and Mihai [7], Yildiz and De [11] have studied W_2-curvature tensor in P-Sasakian and Kenmotsu manifolds respectively. Recently in the paper [4] Hui and Sarkar studied W_2-curvature tensor on generalized Sasakian-Space-Forms. In the present paper we study some curvature conditions on almost C(λ) manifolds. The present paper is organized as follows: After introduction in Sec. 1 we give some preliminaries in Sec. 2. In Sec.3 we study W_2-flat almost C(λ) manifolds. Sec. 4 is devoted to the study of almost C(λ) manifolds satisfying $W_2.S = 0$. In 5 we study almost C (λ) manifolds satisfying $W_2. R = 0$. In Sections 6 and 7 we have studied almost C(λ) manifolds satisfying $R(\xi, Y).S = 0$ and $R(X, Y).W_2 = 0$ respectively.

2. Preliminaries

Let M be a (2n+1)-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is a tensor field of type (1, 1), ξ is a vector field, η is an 1-form and g is a Riemannian metric on M such that [2]

$$\phi^2 (X) = - X + \eta(X)\xi \;, \eta(\xi) = 1 \qquad ..... (1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in TM. \qquad ..... (2)$$

Then also, $\phi\xi = 0, \eta(\phi X) = 0,$ .... (3)

$$g(\phi X, X) = 0, \qquad .... (4)$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y). \qquad .... (5)$$

If an almost contact Riemannian manifold M satisfies the condition

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \;, \qquad .... (6)$$

for some smooth functions a and b on M, then M is said to be an η-Einstein manifold. If b = 0, then the manifold is called Einstein.

Definition 2.1. *An almost C(λ) manifold M is an almost co-Hermitian manifold such that the Riemannian curvature tensor satisfies the following property [5]; there exist a real number λ such that for all X, Y, Z € TM,*

$$R(X, Y, Z, W) = R(X, Y, \phi Z, \phi W) + \lambda\{-g(Y, W)g(X, Z)+g(X, W)g(Y, Z) +g(X, \phi Z)g(Y, \phi W) - g(X, \phi W)g(Y, \phi Z)\} \quad .... (7)$$

From (7) we have the following

$$R(X, Y)Z = R(\phi X, \phi Y)Z - \lambda\{g(Y, Z)X - g(X, Z)Y - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y\}. \quad ... (8)$$

From (8) we have the following:

$$R(X, Y)\xi = R(\phi X, \phi Y)\xi - \lambda\{\eta(Y)X - \eta(X)Y\}, \quad ... (9)$$

$$R(\xi, Y)Z = -\lambda\{g(Y, Z)\xi - \eta(Z)Y\}, \quad ... (10)$$

$$R(\xi, Y)\xi = -\lambda\{\eta(Y)\xi - Y\}, \quad ... (11)$$

$$R(\xi, \xi)Z = 0, \quad ... (12)$$

On an almost C(λ) manifold, we also have [1]

$$QX = AX + B\eta(X)\xi \quad ... (13)$$

where, $-\lambda(2n - 1) = A$, $-\lambda = B$ and Q is the Ricci-operator .

$$\eta(QX) = (A + B)\eta(X), \quad ... (14)$$

$$S(X, Y) = Ag(X, Y) + B\eta(X)\eta(Y), \quad ... (15)$$

$$r = -4n2\lambda , \quad ... (16)$$

$$S(X, \xi) = (A + B)\eta(X), \quad ... (17)$$

$$S(\xi, \xi) = (A + B), \quad ... (18)$$

$$g(QX, Y) = S(X, Y) \quad ... (19)$$

In [10], Pokhariyal and Mishra have defined the curvature tensor W_2, given by

$$W_2(X, Y)Z = R(X, Y)Z + (1/2n)\{g(X, Z)QY - g(Y, Z)QX\} \quad (20)$$

where Q is the Ricci operator, i.e. , $g(QX, Y) = S(X, Y)$ for all vector fields on the tangent space of M.

In view of (8) we get from above,

$$W_2(X, Y)Z=R(\phi X, \phi Y)Z-\lambda\{g(Y, Z)X-g(X, Z)Y-g(\phi Y, Z)\phi X+g(\phi X, Z)\phi Y\}$$
$$+(1/2n)\{g(X, Z)QY-g(Y, Z)QX\} \quad (21)$$

We also have the following from the above,

$$W_2(X, Y)\xi = R(\phi X, \phi Y)\xi + (\lambda/2n)\{\eta(X)Y-\eta(Y)X\} \quad .... (22)$$

$$W_2(\xi, Y)Z = (\lambda/2n)\{\eta(Z)Y-\eta(Y)\eta(Z) \quad .... (23)$$

$$W_2(X, \xi)Z = (\lambda/2n)\{-\eta(Z)X + \eta(X)\eta(Z)\xi\} \quad ... (24)$$

$$W_2(\xi, Y)\xi = (\lambda/2n)\{Y - \eta(Y)\xi\} \quad ... (25)$$

$$W_2(\xi, \xi)Z = 0, \quad ... (26)$$

$$\eta\{W_2(\xi, Y)Z\} = \eta\{W_2(X, \xi)Z\} = \eta\{W_2(\xi, Y)\xi\}=0. \quad ... (27)$$

3. W_2-flat almost C(λ) manifolds

Let us consider an almost C(λ) manifold which is W_2-flat, i.e. $W_2 = 0$, then we get from (21)

$$R(\phi X, \phi Y)Z =\lambda\{ g(Y, Z)X - g(X, Z)Y +\phi X g(\phi Y, Z) +g(\phi X, Z)\phi Y\}$$
$$-(1/2n)\{g(X, Z)QY-g(Y, Z)QX\}. \quad ... (28)$$

In view of (13) we get from above

$$R(\phi X, \phi Y)Z =\lambda\{ g(Y, Z)X - g(X, Z)Y +\phi X g(\phi Y, Z) +g(\phi X, Z)\phi Y\}$$
$$-(1/2n)\{g(X, Z)(AY+B\eta(Y)\xi)-g(Y, Z)(AX+B\eta(X)\xi\} \quad ... (29)$$

Putting $X = \phi X$ in above we get

$$R(\phi^2X, \phi Y)Z =\lambda\{ g(Y, Z)\phi X - g(\phi X, Z)Y + g(\phi Y, Z)\phi^2X + g(\phi^2X, Z)\phi Y\}$$
$$-(1/2n)\{g(\phi X, Z)(AY+B\eta(Y)\xi)-g(Y, Z)A\phi X. \quad ... (30)$$

Setting Y = Z = ξ in above we get,

$\lambda(\phi X)-(1/2n)(A\phi)=0.$ ... (31)

Putting A = -λ(2n - 1) in above we get,

$$\lambda\left\{1+\frac{2n-1}{2n}\right\}A\phi=0 \quad \text{... (32)}$$

Since $X\phi \neq 0$, in general, therefore we get from above, λ= 0.

Thus we have the following result:

Theorem 3.1. Every W_2-flat almost C(λ) manifold is almost C(0)-manifold.

4. Almost C(λ) manifolds satisfying W_2.S=0

Let us consider an almost C(λ) manifold M with W_2 .S = 0. Then we get

$S(W_2(X,Y)Z,\xi)+S(Z,W_2(X,Y)\xi)=0.$ ... (33)

Using (22) we get from above

$S(W_2(X,Y)Z,\xi)+S(Z,R(\phi X,\phi Y)\xi)+(\lambda/2n)\{\eta(X)S(Y,Z)$

$-\eta(Y)S(Z,X)\}=0$ .... (34)

Putting X = ξ in above we get,

$S(W_2(\xi,Y)Z,\xi)+(\lambda/2n)\{S(Y,Z)-\eta(Y)S(Z,\xi)\}=0$ .... (35)

Using (23) in above we get,

$(\lambda/2n)\{\eta(Z)S(\xi,Y)-\eta(Y)\eta(Z)S(\xi,\xi)\}+(\lambda/2n)\{S(Y,Z)-\eta(Y)S(Z,\xi)\}=0.$ ...(36)

In view of (17) and (18) we get from above

$(\lambda/2n)\{(A+B)\eta(Y)\eta(Z)-(A+B)\eta(Y)\eta(Z)\}$

$+(\lambda/2n)\{S(Y,Z)-(A+B)\eta(Y)\eta(Z)\}=0$ .... (37)

Putting A=-λ(2n - 1) and B= -λ in the above equation we get

$(\lambda/2n)\{S(Y,Z)+2n\lambda\eta(Y)\eta(Z)\}=0$ .... (38)

Therefore, either $\lambda=0$ or $S(Y, Z)= - 2n\lambda\eta(Y)\eta(Z)$.

If $S(Y, Z)= -2n\lambda\eta(Y)\eta(Z)$, then $r= - 2n\lambda$.

Thus we have the following result:

Theorem 4.1. *Every almost C(λ) manifold satisfying $W_2.S= 0$ is C(0)-manifold or a manifold of constant scalar curvature.*

5. Almost C(λ) manifolds satisfying $W_2.R= 0$

Let us consider an almost C(λ) manifold M with $W_2.R= 0$. Then we get

$$W_2(X,Y)R(U,V)Z - R(W_2(X, Y)U, V)Z - R(U, (X, Y)V)Z$$
$$- R(U,V)W_2(X,Y)Z = 0. \quad .... (39)$$

Putting $Y = U = \xi$ in (39) and using (9) and (22) we get

$$R(X, V)Z = \lambda[\{g(X,Z)\eta(V) + g(X, V)\eta(Z)g\}\xi + \eta(X)\eta(Z)V$$
$$-g(V,Z)X - 2\eta(X)\eta(Z)\eta(V). \quad .... (40)$$

Thus we are in a position to state the following result:

Theorem 5.1. *In an almost C(λ) manifold satisfying W_2. $R = 0$, the curvature tensor satisfies the relation (40).*

6. Almost C(λ) manifolds satisfying $R(\xi, Y).S = 0$

Let us consider an almost C(λ) manifold M with $R(\xi, Y).S= 0$. Then we get

$$S(R(\xi, Y)Z, U) + S(Z, R(\xi, Y)U) =0. \quad ..... (41)$$

Using (10) we get from above,

$$\lambda\{-g(Y, Z)S(U, \xi) + \eta(Z)S(Y, U) + \eta(U)S(Z, Y) - g(Y, V)S(Z, \xi)\} = 0. \quad (42)$$

In view of (10) we obtain from (42)

$$\lambda\{-(A + B)g(Y, Z)\eta(U) + \eta(Z)S(Y, U) + \eta(U)S(Z, Y) - (A + B)\eta(Z)g(Y, V)\}=0. \quad ..... (43)$$

Putting $U = \xi$ in above we get,

$\lambda\{-(A + B)g(Y, Z) + S(Z, Y)\} = 0.$ .... (44)

Therefore, ether $\lambda = 0$ or, $S(Z, Y) = (A + B)g(Y, Z)$.

If $S(Z, Y) = (A + B)g(Y, Z)$, then r is constant.

Thus we have the following result:

Theorem 6.1. *Every almost C(λ) manifolds satisfying R(ξ, Y).S = 0, is either an almost C(0)-manifold or a manifold of constant scalar curvature.*

7. W_2-semiymmetric almost C(λ) manifolds

Definition 7.1. *An almost C(λ) manifold will be called W_2-semiymmetric [3] if $R(X, Y). W_2 = 0$, where R(X, Y) is to be considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. Let us consider*

$(R(\xi, U). W_2)(X, \xi)Y = 0$ ..... (45)

Or, $R(\xi, U) W_2 (X, \xi)Y - W_2 (R(\xi, U)X, \xi)Y - W_2 (X, R(\xi, U)\xi)Y$

$- W_2 (X, \xi)R(\xi, U)Y = 0$ ....(44)

Using (9), (10) and (24) in the above equation and after straightforward calculation we get,

$$W_2 (X, U)Y = (\lambda / 2n) [\eta(Y)\{g(U, X) - \eta(X)\eta(U)\}\xi + \eta(X)\eta(Y)\{U - \eta(U)\xi\} - g(U, Y)\{X - \eta(X)\xi\}]. \quad ..... (45)$$

Thus we have the following result:

Theorem 7.1. *For a W_2-semiymmetric almost C(λ) manifolds the W_2 curvature tensor satisfies the relation (45).*

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