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# CERTAIN CONDITION OF SASAKIAN MANIFOLDS WITH QUARTER-SYMMETRIC NON-METRIC CONNECTION

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**Abstract:** The objective of this paper is to investigate the Sasakian manifolds with quarter-symmetric non-metric connection. We have investigate a Sasakian manifolds admitting the quarter-symmetric non-metric connections satisfying certain conditions. Further, we have proved that a Ricci soliton on a Sasakian manifold equipped with quarter-symmetric non-metric connection to be steady. We have also investigated $\xi$-conharmonically flat, Globally $\varphi$-conharmonically symmetric, $\eta$-parallel Ricci tensor and some interesting results. Finally, we have given an example of 3-dimensional Sasakian manifolds with respect to quarter-symmetric non-metric connection.

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## 1. Introduction

In 1950, an algebraic frame work for a connection as a differential operator was define by Koszul. Carton also characterised connection as a specific type of differential form. Torsion and curvature are two major components of an affine connection. On a Riemannian manifold, the concept of metric connection with torsion was introduced by Hayden [15]. In 1922, Agashe and Chafle [2] developed the concept of semi-symmetric non-metric connection. Golab [13] explored a quarter-symmetric connection in a differentiable manifold with affine connection. The quarter-symmetric non-metric connection is the most recognized connection and the study on Kenmotsu manifold with quarter-symmetric non-metric $\varphi$-connection was studied by several authors [3, 5, 11, 18, 20, 21].

The concept of contact geometry has involved for the mathematical formalism of classical mechanics [12]. Two important classes of contact manifolds are K-contact manifolds and Sasakian manifolds [4]. An odd dimensional analogue of Kaehler geometry is the Sasakian geometry. Sasakian manifolds were firstly studied by the famous geometer Sasaki [25] in 1960. Sasakian manifold have been studied by several authors [6, 8, 16, 19, 22, 24, 28]. Takahashi [29] introduced the notion of locally $\varphi$-symmetric Sasakian manifolds as a weaker version of local symmetry of such manifolds.

Adati and Motsumoto [1] defined para-Sasakian and special para-Sasakian manifolds which are considered as a special cases of an almost para contact manifold introduced by Sato and Matsumoto [26]. On a type of para-Sasakian manifolds have been studied by De and Tarafdar [9]. De and Guha [10] showed that an n-dimensional Weyl-semisymmetric para-Sasakian manifold is conformally flat. In [7], authors studied the properties of $\eta$-parallel Ricci tensor and provided several results. Analogous to the definition of $\eta$-parallelism given by Kon [17] on Sasakian manifolds.

Hamilton introduce the theory of Ricci flow to establish a canonical metric on a smooth manifold in 1982. The Ricci flow is an evolution equation for metrics on a Riemannian manifold is defined as follows:

$$\frac{\partial}{\partial t} g(t) = -R(t)g(t)$$

A Ricci soliton $(g, V, \lambda)$ on a Riemannian manifold $(M, g)$ is generalization of an Einstein metric such that it satisfies the following condition [14, 23]

$$L_V g + 2S + 2\lambda g = 0, \tag{1}$$

where S is the Ricci tensor, $L_V$ is the Lie derivative operator along the vector field V on $(M, g)$ and $\lambda$ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according as $\lambda$ is negative, zero or positive.

The present work is organized as follows: In section-1 introduction is discussed. Section-2 is equipped with some prerequisites about Sasakian manifolds. Section-3, deals with Sasakian manifold admitting the quarter-symmetric non-metric connection (QSNMC) with some interesting results. Section-4 concerned with the study of Ricci soliton on a Sasakian manifold admitting the QSNMC. In section-5, we study the $\xi$-conharmonically flat Sasakian manifold admitting the QSNMC. Section-6 is devoted to the study of Globally $\varphi$-conharmonically symmetric Sasakian manifold with respect to the connection $\tilde{\nabla}$. In section-7, we study the $\eta$-parallel Ricci tensor admitting the QSNMC. In the last section, we have given an example of 3-dimensional Sasakian manifold in the support of our results.

## 2. Preliminaries

An $(2n + 1)$-dimensional smooth manifold $M$ together with a $(1,1)$-tensor field $\varphi$, a vector field $\xi$, $\eta$ is 1-form and Riemannian metric $g$ is called an almost contact metric manifold if

$$\varphi^2(X) = -X + \eta(X)\xi, \tag{2}$$

$$\eta(\xi) = 1, \ \varphi\,\xi = 0, \ \ \eta(\varphi X) = 0, \tag{3}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \quad g(\varphi X, Y) = -g(X, \varphi Y). \tag{4}$$

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{5}$$

where $\nabla$ denotes the Riemannian connection of $g$, then manifold $(M, \varphi, \xi, \eta, g)$ is called a Sasakian manifold. It can be shown that

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{6}$$

$$(\nabla_X \xi) = -\varphi(X). \tag{7}$$

In a Sasakian manifold, we have [4]

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{8}$$

$$R(\xi,Y)X = g(X,Y)\xi - \eta(X)Y, \tag{9}$$

$$R(\xi,X)\xi = \eta(X)\xi - X, \tag{10}$$

$$S(X,\xi) = 2n\eta(X), \tag{11}$$

$$Q\xi = 2n\xi, \tag{12}$$

$$S(\varphi X, \varphi Y) = S(X,Y) - 2n\eta(X)\eta(Y), \tag{13}$$

for any vector fields $X, Y, Z$ on M, where R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator.

The notion of $\eta$-parallelism on a Sasakian manifold was introduced by Kon [17]. A Ricci tensor S of an n-dimensional Kenmotsu manifold M is said to be $\eta$-parallel if it satisfies the tensorial relation

$$(\nabla_X S)(\varphi Y, \varphi Z) = 0, \tag{14}$$

for all $X, Y, Z$ in $\chi(M)$.

**3. Quarter-Symmetric Non-Metric Connection (QSNMC)**

Let $M$ be a Sasakian manifold with Levi-Civita connection $\nabla$. We define a linear connection $\tilde{\nabla}$ on $M$ [30] as

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\varphi X. \tag{15}$$

Then a linear connection $\tilde{\nabla}$ is said to be a QSNMC if the torsion tensor $\tilde{T}$ with respect to connection $\tilde{\nabla}$ satisfy

$$\tilde{T}(X,Y) = \eta(Y)\varphi X - \eta(X)\varphi Y, \tag{16}$$

where linear connection $\tilde{\nabla}$ is said to be a non-metric connection if $\tilde{\nabla}_g \neq 0$ .

Using (4), we have

$$(\tilde{\nabla}_X g)(Y,Z) = -[\eta(Y)g(\varphi X, Z) + \eta(Z)g(\varphi X, Y)] \neq 0. \tag{17}$$

A linear connection $\tilde{\nabla}$ is said to be a QSNMC, if satisfies (15), (16) and (17).

Now from equation (15), we have

$$(\tilde{\nabla}_X \varphi)Y = g(X,Y)\xi - \eta(X)\eta(Y)\xi, \tag{18}$$

$$(\tilde{\nabla}_X \eta)Y = (\nabla_X \eta)Y = g(X, \varphi Y), \tag{19}$$

$$(\tilde{\nabla}_X g)(Y, \varphi Z) = -\eta(Y) g(\varphi X, \varphi Z), \tag{20}$$

On replacing Y by $\xi$ in (15), we have

$$\tilde{\nabla}_X \xi = 0. \tag{21}$$

Thus a relation between the curvature tensor $\tilde{R}$ and $R$ of $M$ is given as :

$$\tilde{R}(X,Y)Z = R(X,Y)Z + g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X. \tag{22}$$

Also, from equation (18), we obtain

$$\tilde{S}(X,Y) = S(X,Y) + g(Y,Z) - (2n+1)\eta(Y)\eta(Z), \tag{23}$$

$$\tilde{R}(\xi, X)Y = g(X,Y)\xi - \eta(X)\eta(Y)\xi, \tag{24}$$

$$\eta(\tilde{R}(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{25}$$

$$R(X,Y)\xi = 0. \tag{26}$$

**Theorem 3.1.** Every (2n+1)-dimensional Sasakian manifold equipped with QSNMC is irregular with respect to QSNMC.

Now, from (23), we get

$$\tilde{S}(Y, \xi) = 0, \tag{27}$$

$$\tilde{Q}(X) = Q(X) + X - (2n+1)\eta(Y)\xi, \tag{28}$$

$$\tilde{r} = r. \tag{29}$$

**Theorem 3.2.** In a Sasakian manifolds admitting QSNMC $\tilde{\nabla}$, the scalar curvature tensor is invariant with respect to $\nabla$ and $\tilde{\nabla}$ .

**4. Ricci soliton on Sasakian manifolds equipped with the quarter-symmetric non-metric connection**

Let $(g, \xi, \lambda)$ be the Ricci soliton on Sasakian manifold admitting $\tilde{\nabla}$, then from (1), we have

$$(\tilde{L}_\xi g)(X,Y) + 2\tilde{S}(X,Y) + 2\lambda g(X,Y) = 0. \tag{30}$$

With the help of (3), (4), (7) and (15), we get

$$(\tilde{L}_\xi g)(X,Y) = 0. \tag{31}$$

Using (23) and (31) in (30), we obtain

$$S(X,Y) = -(1+\lambda) g(X,Y) + (2n+1)\eta(X)\eta(Y). \tag{32}$$

Putting $Y = \xi$ in (32) and (27), we get

$$\lambda = 0. \tag{33}$$

Hence, we have the following theorem:

**Theorem 4.1.** A Ricci soliton on a Sasakian manifold $M$ with respect to QSNMC $\tilde{\nabla}$ is always steady.

**Corollary 4.1.** A Ricci soliton on an $\eta$-Einstein Sasakian manifold $M$ in terms of QSNMC $\tilde{\nabla}$ is always steady.

Let $(g, V, \lambda)$ be the Ricci soliton on a Sasakian manifold $M(\varphi, \xi, \eta, g)$ admitting QSNMC $\tilde{\nabla}$ such that V is pointwise collinear with $\xi$ i.e. $V = b\xi$, where b is a function. Then (1) implies that

$$bg(\tilde{\nabla}_X\xi, Y) + (Xb)\eta(Y) + bg(X, \tilde{\nabla}_Y\xi) + (Yb)\eta(X) + 2\tilde{S}(X, Y) + 2\lambda g(X, Y) = 0. \quad (34)$$

Replacing $Y$ by $\xi$ in (34) and using (7), (15) in (27), we get

$$(Xb) + (\xi\, b)\eta(X) + 2\lambda\eta(X) = 0. \quad (35)$$

Again replacing X by $\xi$ in (35), we have

$$\xi\, b = -\lambda. \quad (36)$$

By virtue of (35) and (36), takes the form

$$(db) = \lambda n. \quad (37)$$

Applying $d$ on (37), we yield

$$\lambda = 0. \quad (38)$$

Consequently from (37) we obtain $db = 0$, i.e., $b$ is constant.

Hence we have the following theorem:

**Theorem 4.2.** If $(g, V, \lambda)$ be a Ricci soliton on a Sasakian manifold $M$ in reference to QSNMC $\tilde{\nabla}$ such that $V = b\xi$, then V is a constant multiple of $\xi$ and the Ricci soliton is always steady.

## 5. $\xi$-conharmonically flat Sasakian manifolds with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$

**Definition 5.1.** A rank tensor $\tilde{C}$ that remains invariant under Conharmonic transformation for an $(2n+1)$-dimensional Riemannian manifold $M$ is given as

$$\tilde{C}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{2n-1}\left[g(Y, Z)\tilde{Q}X - g(X, Z)\tilde{Q}Y + \tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y\right]. \quad (39)$$

Where $\tilde{R}$ and $\tilde{S}$ are the curvature tensor and Ricci tensor equipped with QSNMC $\tilde{\nabla}$ respectively and $\tilde{S}(Y, Z) = g(\tilde{Q}Y, Z)$.

Using (22) in (39), we get

$$\tilde{C}(X, Y)Z = R(X, Y)Z + g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + \eta(X)\eta(Z)Y$$
$$- \eta(Y)\eta(Z)X - \frac{1}{2n-1}\left[g(Y, Z)\tilde{Q}X - g(X, Z)\tilde{Q}Y + \tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y\right]. \quad (40)$$

Using equations (23) and (28) in (40), we have

$$\tilde{C}(X,Y)Z = C(X,Y)Z + g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X$$
$$-\frac{2}{2n-1}[g(Y,Z)X - g(X,Z)Y - \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X]$$
$$-\frac{2n+1}{2n-1}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi], \quad (41)$$

where

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y]. \quad (42)$$

Putting $Z = \xi$ in (41) and using equations (3) and (4), it follows that

$$\tilde{C}(X,Y)\xi = C(X,Y)\xi - \frac{4}{2n-1}[\eta(Y)X - \eta(X)Y]. \quad (43)$$

Suppose $X$ and $Y$ are orthogonal to $\xi$, then (43) becomes

$$\tilde{C}(X,Y)\xi = C(X,Y)\xi. \quad (44)$$

In view of the above discussion we can state the following theorem:

**Theorem 5.1.** An $(2n+1)$-dimensional Sasakian manifold is $\xi$-conharmonically flat with respect to QSNMC $\tilde{\nabla}$ iff the manifold is also $\xi$-conharmonically flat with respect to the Levi-Civita connection $\nabla$ provided the vector fields $X$ and $Y$ are horizontal vector fields.

## 6. Globally $\varphi$-conharmonically symmetric Sasakian manifolds equipped with the quarter-symmetric non-metric connection $\tilde{\nabla}$

**Definition 6.1.** A Sasakian manifold $M$ with respect to QSNMC $\tilde{\nabla}$ is called to be globally $\varphi$-conharmonically symmetric if

$$\varphi^2\big((\nabla_W C)(X,Y)Z\big) = 0. \quad (45)$$

for all vector fields $X, Y, Z, W$ $in$ $\chi$(M).

From equation (42), we have

$$\eta(C(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X)$$
$$-\frac{1}{2n-1}[g(X,Z)\eta(QX) - g(X,Z)\eta(QY) + S(Y,Z)\eta(X) - S(X,Z)\eta(Y)]. \quad (46)$$

Contracting (11) and $S(X,Y) = g(QX,Y)$,yields

$$\eta(QX) = 2n\eta(X). \quad (47)$$

Moreover, combining (15), (46) and (47) and taking $X, Y, Z, W$ are orthogonal to $\xi$, it follows that

$$\big(\tilde{\nabla}_W C\big)(X,Y)Z = \tilde{\nabla}_W C(X,Y)Z - C\big(\tilde{\nabla}_W X, Y\big)Z - C\big(X, \tilde{\nabla}_W Y\big)Z - \tilde{C}(X,Y)\tilde{\nabla}_W Z$$

$$= (\nabla_W C)(X,Y)Z. \tag{48}$$

Taking covariant differentiation of (41) with respect to W and using (17), (18), (19), (20) and (48) also taking X, Y, Z, W orthogonal to $\xi$, we have

$$\big(\tilde{\nabla}_W \tilde{C}\big)(X,Y)Z = (\nabla_W C)(X,Y)Z + g(X,\varphi Z)g(Y,W)\xi - g(Y,\varphi Z)g(X,W)\xi$$

$$-\frac{2n+1}{2n-1}[g(X,Z)g(W,\varphi Y)\xi + g(Y,Z)g(W,\varphi X)\xi]. \tag{49}$$

Now, applying $\varphi^2$on both sides of (49) and using (2), it follows that

$$\varphi^2\left(\big(\tilde{\nabla}_W \tilde{C}\big)(X,Y)Z\right) = \varphi^2((\nabla_W C)(X,Y)Z). \tag{50}$$

**Theorem 6.1.** An $(2n+1$-dimensional Sasakian manifold is globally $\varphi$-conharmonically symmetric with respect to QSNMC $\tilde{\nabla}$ iff the manifold is also globally $\varphi$-conharmonically symmetric with the Levi-Civita connection $\nabla$ provided the vector fields $X, Y, Z, W$ are orthogonal to $\xi$.

## 7. $\eta$-parallel Ricci tensor equipped with the quarter-symmetric non-metric connection $\tilde{\nabla}$

In this section, we study $\eta$-parallel Ricci tensor with respect to QSNMC $\tilde{\nabla}$.

**Definition 7.1.** A Ricci tensor $\tilde{S}$ of an n-dimensinal Kenmotsu manifold M endowed with a semi-symmetric non-metric connection $\tilde{\nabla}$ is said to be $\eta$-parallel for $\tilde{\nabla}$ if it satisfies the relation $\big(\tilde{\nabla}_X S\big)(\varphi Y,\varphi Z) = 0$, for arbitrary vector fields $X, Y$ and $Z$.

From (15), we have

$$\tilde{\nabla}_X\big(\tilde{Q}Y\big) = \nabla_X\big(\tilde{Q}Y\big) - \eta\big(\tilde{Q}Y\big)\varphi X. \tag{51}$$

With the help of (3), (6), (15) and (28), we find

$$\tilde{\nabla}_X\big(\tilde{Q}Y\big) = \big(\tilde{\nabla}_X\tilde{Q}\big)Y + Q(\nabla_X Y) + \nabla_X Y - (2n+1)\eta(\nabla_X Y)\xi + \eta(Y)Q(\varphi X) + \eta(Y)\varphi X, \tag{52}$$

and

$$\nabla_X\big(\tilde{Q}Y\big) = (\nabla_X Q)Y + Q(\nabla_X Y) + \nabla_X Y - (2n+1)[g(X,\varphi Y)\xi + \eta(\nabla_X Y)\xi - \eta(Y)\varphi X]. \tag{53}$$

From the equation (52) and (53), we obtain

$$\big(\tilde{\nabla}_X\tilde{Q}\big)Y = (\nabla_X Q)Y - \eta(Y)Q\varphi X - (2n+1)g(X,\varphi Y)\xi - 2(2n+1)\eta(Y)\varphi X. \tag{54}$$

In view of (3), (11) and $\big(\tilde{\nabla}_X\tilde{S}\big)(Y,Z) = g\left(\big(\tilde{\nabla}_X\tilde{Q}\big)Y,Z\right)$, the above relation becomes

$$\big(\tilde{\nabla}_X\tilde{S}\big)(Y,Z) = (\nabla_X S)(Y,Z) - \eta(Y)S(\varphi X,Z) - (2n+1)[\eta(Z)g(X,\varphi Y) + \eta(Y)g(\varphi X,Z)]. \tag{55}$$

Replacing the vector fields $Y$ by $\varphi Y$ and $Z$ by $\varphi Z$ in (55) and using (3), we find

$$(\tilde{\nabla}_X \tilde{S})(\varphi Y, \varphi Z) = (\nabla_X S)(\varphi Y, \varphi Z). \tag{56}$$

In view of (14), (56) and definition (7.1), we state the following:

**Theorem 7.1.** Let $M$ be an $(2n+1)$-dimensional Sasakian manifold is equipped with a QSNMC $\tilde{\nabla}$, then the Ricci tensor $\tilde{S}$ on $M$ is $\eta$-parallel with respect to connection $\tilde{\nabla}$ iff the manifold has $\eta$-parallel Ricci tensor $S$ for the Levi-Civita connection $\nabla$.

**8. Example of Sasakian Manifolds.**

Let $M$ = $[(x, y, z)$ in $\mathfrak{R}^3 : (z)>0]$ be a 3-dimensional manifold, where $(x, y, z)$ are the standard coordinates in $\mathfrak{R}^3$. Choosing vector fields [27]

$$e_1 = x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) - 2y\frac{\partial}{\partial z}, e_2 = \frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z} = \xi,$$

are linearly independent at every point of $M$.

The Riemannian metric $g$ is defined by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1. \tag{57}$$

Suppose $\eta$ be the 1-form defined as $\eta(X) = g(X, \xi)$.

Let $\varphi$ be the (1,1)-tensor field defined by

$$\varphi(e_1) = e_2, \ \varphi(e_2) = -e_1, \ \varphi(e_3) = 0. \tag{58}$$

By linearity property of $\varphi$ and $g$, we have

$$\eta(e_3) = \eta(\xi), \qquad \varphi^2 X = -X + \eta(X)e_3, \ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

Consider $\nabla$ be the Levi-Civita connection with Riemannian metric $g$, then we have

$$[e_1, e_2] = 2e_3, \ [e_1, e_3] = 0, \ [e_2, e_3] = 0. \tag{59}$$

Koszul's formula is given as

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$$

$$+g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y). \tag{60}$$

for arbitrary vector fields $X, Y, Z$ in $\chi(M)$.

By virtue of (60), we have

$$\nabla_{e_1} e_1 = 0, \ \nabla_{e_1} e_2 = e_3, \ \ \nabla_{e_1} e_3 = -e_2, \ \nabla_{e_2} e_1 = -e_3, \ \ \nabla_{e_2} e_2 = 0,$$

$$\nabla_{e_2} e_3 = e_1, \quad \nabla_{e_3} e_1 = -e_2, \quad \nabla_{e_3} e_2 = e_1, \ \nabla_{e_3} e_3 = 0. \tag{61}$$

Let us take a vector field $X = X^1 e_1 + X^2 e_2 + X^3 e_3$ and the structure vector field $\xi = e_3$, then we have

$\tilde{\nabla}_X \xi = -X^1 e_2 + X^2 e_1,$ (62)

and

$\nabla_X \xi = -\varphi X = -X^1 e_2 + X^2 e_1,$ (63)

where $X^1$, $X^2$, $X^3$ are scalars.

Hence for $\xi$=$e_3$ the manifold $(M, \varphi, \xi, \eta, g)$ under consideration example is a Sasakian manifolds.

By virtue of (3), (15), (58) and (61), we have

$\tilde{\nabla}_{e_1} e_1 = 0,\ \tilde{\nabla}_{e_1} e_2 = e_3,\ \tilde{\nabla}_{e_1} e_3 = 0,\ \tilde{\nabla}_{e_2} e_1 = -e_3,\ \tilde{\nabla}_{e_2} e_2 = 0,$

$\tilde{\nabla}_{e_2} e_3 = 0\ ,\ \tilde{\nabla}_{e_3} e_1 = -e_2\ ,\ \tilde{\nabla}_{e_3} e_2 = e_1,\ \tilde{\nabla}_{e_3} e_3 = 0.$ (64)

In view of (16), the torsion tensor $\tilde{T}$ with respect to $\tilde{\nabla}$ as follows:

$\tilde{T}(e_i, e_j) = 1,2,3.\ \tilde{T}(e_1, e_3) = e_2 \neq 0,\ \tilde{T}(e_2, e_3) = -e_{1,} \neq 0.$

Also we have

$(\tilde{\nabla}_{e_1}, g)(e_{2,}, e_3) = -1 \neq 0,$

Hence the manifold is a Sasakian manifold with respect to the connection $\tilde{\nabla}$.

The curvature tensor $(e_i, e_j)e_k$ ; i, j, k = 1, 2, 3 of $\nabla$ can be calculated as follows:

$R(e_1, e_2)e_3 = 0,\quad R(e_1, e_3)e_3 = e_1,\ R(e_2, e_3)e_2 = -e_3,$

$R(e_3, e_1)e_1 = e_3,\ R(e_2\ , e_1)e_1 = -3e_2,\ R(e_2, e_3)e_3 = -e_2,$

$R(e_2, e_3)e_1 = 0,\ R(e_1, e_2)e_2 = -3e_1,\quad R(e_3, e_1)e_2 = 0.$ (65)

Along with R($e_i$,$e_i$) $e_i = 0$; $\forall$ i =1, 2, 3.

From the above calculation, we also calculate

$\tilde{R}(e_1, e_2)e_3 = 0\ ,\quad \tilde{R}(e_1, e_3)e_3 = 0,\ \tilde{R}(e_2, e_3)e_2 = -e_3,$

$\tilde{R}(e_3, e_1)e_1 = e_3,\ \tilde{R}(e_2\ , e_1)e_1 = -2e_2,\ \tilde{R}(e_2, e_3)e_3 = 0,$

$\tilde{R}(e_2, e_3)e_1 = 0,\ \tilde{R}(e_1, e_2)e_2 = 0,\quad \tilde{R}(e_3, e_1)e_2 = 0.$ (66)

Along with $\tilde{R}$ ($e_i$,$e_i$) $e_i$=0;$\forall$i =1, 2, 3.

The Ricci tensor S ($e_j$,$e_k$) ;$\forall$ j, k =1, 2, 3 of $\nabla$ is given by using (65) in the equation

$S(e_j, e_k) = \sum_{i=1}^{3} g(R(e_i, e_j)e_k, e_i).$

It follows that :

$S(e_1, e_1) = -2,\ S(e_2, e_2) = -2,\ S(e_3, e_3) = -1.$ (67)

The $\tilde{S}$ ($e_j$,$e_k$) ;$\forall$ j, k =1, 2, 3 of $\tilde{\nabla}$ can also be calculated by using (66) in the equation :

$\tilde{S}(e_j, e_k) = \sum_{i=1}^{3} g(\tilde{R}(e_i, e_j)e_k, e_i)$.

It follows that :

$\tilde{S}(e_1, e_1) = -1,\ \tilde{S}(e_2, e_2) = 1,\ \tilde{S}(e_3, e_3) = 0.$ (68)

Along with $\tilde{S}(e_j, e_k) = 0; \forall$ j, k =1, 2, 3 (j≠k).

By virtue of (33) and (67), we have

$\lambda = 0.$ (69)

Thus the Ricci soliton $(g, \xi, \lambda)$ on manifold admitting the connection $\tilde{\nabla}$ is always steady. Hence theorem (4.1) is verified.

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