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UNIFIED PROBABILITY DENSITY FUNCTION INVOLVING T-CONFLUENT HYPERGEOMETRIC FUNCTION AND T-GAUSS HYPERGEOMETRIC FUNCTIONS

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Abstract: In the present paper, we establish a new probability density function involving product of τ-confluent hypergeometric function and τ-Gauss hypergeometric function which unifies and extends the probability density functions studied by various authors namely Al-Zamel et al. [7], Kalla et al [12,13], Ali et al. [2], Al-Zamel [6]. To explore the statistical properties of this distribution, the paper introduces a generalized gamma function along with its incomplete forms. Several fundamental statistical functions associated with this proposed probability density function such as moments, mean, Mellin transform, moment generating function, characteristic function, the distribution function, the hazard rate function and mean residue life function are derived.

Keywords: Probability density function (p.d.f.), Confluent hypergeometric function, Gauss hypergeometric function.

1. Introduction & Definitions

The generalized inverse Gaussian distribution has been introduced by Good [9] in the following form.

$$f(x) = \frac{1}{A(\alpha;a,b)} x^{\alpha-1} e^{-ax-b/x},\ a,b,x > 0, -\infty < \alpha < \infty \tag{1}$$

where $A(\alpha;a,b) = 2\left(\frac{b}{a}\right)^{\alpha/2} K_\alpha(2\sqrt{ba})$

and $K_\alpha(z)$ is the modified Bessel function of second kind [8,14]. Such distributions have several applications in brownian motion, reliability theory and theory of demographic rates [10,11].

Ali et. at [2] have further generalized this p.d.f. and studied another generalized inverse Gaussian distribution involving τ-confluent hypergeometric function in the following form

$$f(x) = Ax^{\lambda-1} e^{-px^\delta} \phi^\tau\left(a;c;-bx^{-\delta}\right),\ x > 0 \tag{2}$$

where

$A = \delta\ [\ \ {}_{\tau}\Gamma_b(a; c; \lambda\ \delta^{-1}; p)]^{-1} \qquad\qquad \delta, \tau, \lambda, p > 0; b \geq 0; c \neq 0, -1, -2, ...$

Here ${}_{\tau}\Gamma_b$ denotes the generalized gamma function defined as:

$$ {}_{\tau}\Gamma_b(a; c; \lambda; p) = \int_0^{\infty} t^{\lambda-1} e^{-pt} \phi^{\tau}\left(a; c; -\frac{b}{t}\right) dt \tag{3}$$

and $\phi^{\tau}(a; c; -bx^{-\delta})$ is the τ-confluent hypergeometric function defined by Virchenko [15]

$$\phi^{\tau}(\mathrm{a; c; z}) = \frac{\Gamma(\mathrm{c})}{\Gamma(\mathrm{a})} \sum_{\mathrm{k}=0}^{\infty} \frac{\Gamma(\mathrm{a}+\tau\mathrm{k})}{\Gamma(\mathrm{c}+\tau\mathrm{k})} \frac{\mathrm{z}^{\mathrm{k}}}{\mathrm{k}!} \tag{4}$$

On the other hand, Kalla and Al-Saqabi [12] have introduced a gamma type of p.d.f. which involves the τ-Gauss hypergeometric function and is given by

$$f(x) = \frac{\beta \alpha^{\frac{u}{\beta}+1} x^{u+\beta-1} e^{-\delta x^{\beta}}}{v^{\lambda}\ \Gamma\left(\begin{matrix}\lambda, b; c \\ \frac{u}{\beta}+1, v\end{matrix}; \frac{\delta}{\alpha}, \tau\right)}\ {}_2R_1\left(\lambda, b\ ; c\ ; \tau; -\frac{\alpha x^{\beta}}{v}\right),\ x > 0 \tag{5}$$

$\beta, \mathrm{u}, \delta, \mathrm{v}, \mathrm{p}, \tau > 0, |\arg \mathrm{v}| < \pi$ and $\mathrm{c} \neq 0, -1, -2, ...$

where

$$\Gamma\left(\begin{matrix}a, b; c \\ \mathrm{u}, v\end{matrix};\ \mathrm{p}\ , \tau\right) = \mathrm{v}^{-\mathrm{a}} \int_0^{\infty} \mathrm{t}^{\mathrm{u}-1} \mathrm{e}^{-\mathrm{pt}}\ {}_2\mathrm{R}_1\left(\mathrm{a, b; c}; \tau; -\frac{\mathrm{t}}{\mathrm{v}}\right) \mathrm{dt} \tag{6}$$

is a generalization of gamma function studied by Virchenko et al [16] and

${}_2R_1(\mathrm{a, b; c}; \tau; \mathrm{z})$ is the τ-Gauss hypergeometric function [15,16] defined by

$$R_2(a, b; c; \tau; z) = \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+\tau k)}{\Gamma(c+\tau k)} \frac{z^k}{k!} \tag{7}$$

Further generalizations of these distributions have been given by Al Saqabi, Kalla and Tuan [3] and Al-Zamel, Ali and Kalla [7].

In the present paper, we study and develop a unified probability density function which involves the product of τ-confluent hypergeometric function and τ-Gauss hypergeometric function.

2. A Generalized Gamma-Type Function

In this section we define a new generalization of the gamma-type function in the following form

$$\begin{aligned} E &= E_{\alpha,\beta}^{\tau,\tau'}(a, c; a', b'; c'; \lambda, \delta, p) \\ &= \int_0^{\infty}\ t^{\lambda-1} e^{-pt^{\delta}} \phi^{\tau}(a; c; -\alpha t^{-\delta})\ {}_2R_1^{\tau'}(a', b'; c'; -\beta t^{\delta}) dt \end{aligned} \tag{8}$$

Where $\mathrm{Re}(\lambda), \mathrm{Re}(\mathrm{p}) > 0, \quad \delta, \tau, \tau' > 0, \quad \alpha \geq 0, \quad |\arg \beta| < \pi, \ \ \lambda\tau + \delta \mathrm{a} > 0$

Taking $\beta = 0$ and $\delta = 1$ in (28), we get the generalized gamma function defined by Ali et al [2] as given by equation (3).

For $\alpha = 0$ and $\delta = 1$, we obtain the generalized gamma function defined by Virchenko et al [16] as given by equation (6). This function further reduces to the D-function given by Al-Musallam and Kalla [4,5].

If we take $\alpha = 0, \beta = 0, \delta = 1$ and $p = 1$ the function given by (8) reduces to Gamma function.

3. The Generalized Incomplete Gamma Function

We define the generalized incomplete gamma function as follows:

$$\gamma_{\alpha,\beta}^{\tau,r'}(a;c;a',b';c';\lambda,\delta,p;x) = \int_0^x t^{\lambda-1} e^{-pt^{\delta}} \phi^{\tau}\left(a;c;-\alpha t^{-\delta}\right) {}_2R_1^{\tau'}\left(a',b';c';-\beta t^{\delta}\right) dt \tag{9}$$

with Re $(\lambda) > 0, \delta, \tau, \tau', x > 0, \alpha \geq 0, \lambda\tau + \delta a > 0, |\arg \beta| < \pi$

and Complementary incomplete gamma function as follows:

$$\Gamma_{\alpha,\beta}^{\tau,r'}(a:c\,;\mathrm{a}',\mathrm{b}';\mathrm{c}';\lambda,\delta,\mathrm{p};\mathrm{x}) = \int_x^{\infty} \mathrm{t}^{\lambda-1} \mathrm{e}^{-pt^{\delta}} \phi^{\tau}\left(\mathrm{a};\mathrm{c};-\alpha \mathrm{t}^{-\delta}\right) \; {}_2\mathrm{R}_1^{\mathrm{r}'}\left(\mathrm{a}',\mathrm{b}';\mathrm{c}';-\beta \mathrm{t}^{\delta}\right)\mathrm{dt} \tag{10}$$

With Re(p) $> 0, \delta, \tau, \tau', x > 0, \alpha \geq 0, |\arg \beta| < \pi$.

We thus have

$$\mathrm{E}_{\alpha,\beta}^{\tau,r'}(a;c;a',b';c';\lambda,\delta,p) = \gamma_{\alpha,\beta}^{\tau,\tau'}(a,c;a',b';c';\lambda,\delta,p;x) + \Gamma_{\alpha,\beta}^{\tau,\tau'}(a,c;a',b';c';\lambda,\delta,p;x) \tag{11}$$

Differentiation Formulas

Performing differentiation under the sign of integration we obtain following relations for the generalized incomplete gamma functions defined by equations (9) and (10) as :

$$\frac{d}{dx}\gamma_{\alpha,\beta}^{\tau,\tau'}(a,c;a',b';c';\lambda,\delta,p;x) = -\frac{d}{dx}\Gamma_{\alpha,\beta}^{\tau,\tau'}(a,c;a',b';c';\lambda,\delta,p;x) = \mathrm{x}^{\hat{\lambda}-1}\mathrm{e}^{-\mathrm{px}^{\delta}} \phi^{\tau}\left(\mathrm{a};\mathrm{c};-\alpha \mathrm{x}^{-\delta}\right) {}_2\mathrm{R}_1^{\tau'}\left(\mathrm{a}',\mathrm{b}';\mathrm{c}';-\beta \mathrm{x}^{\delta}\right) \tag{12}$$

For $\lambda = a, p = \delta = 1, \alpha = \beta = 0$, (3.4) reduces to the classical formulas for the incomplete gamma function [1] as

$$\frac{d}{dx}\gamma(a,x) = -\frac{d}{dx}\Gamma(a,x) = x^{a-1}e^{-x}$$

Recurrence Relations

$$\delta p\ \Gamma_{\alpha,\beta}^{\tau,\tau'}(a,c;a',b';c';\lambda+\delta,\delta,p;x)=\lambda\ \Gamma_{\alpha,\beta}^{\tau,\tau'}(a,c;a',b';c';\lambda,\delta,p;x)$$
$$+x^{\lambda}e^{-px^{\delta}}\phi^{\tau}\left(a;c;-\alpha x^{-\delta}\right){}_2R_1^{\tau'}\left(a',b';c';-\beta x^{\delta}\right)$$
$$+\frac{\Gamma(c)\Gamma(a+\tau)}{\Gamma(a)\Gamma(c+\tau)}(\alpha\delta)\ \Gamma_{\alpha,\beta}^{\tau,\tau'}(a+\tau,c+\tau;a',b';c',\lambda-\delta,\delta,p;x)$$
$$+\frac{a'\Gamma(c')\Gamma(b'+c')}{\Gamma(b')\Gamma(c'+\tau')}(-\beta\delta)\ \Gamma_{\alpha,\beta}^{\tau,\tau'}(a,c;a'+1,b'+\tau';c'+\tau',\lambda+\delta,\delta,p;x) \quad (13)$$

$$\delta p\ \gamma_{\alpha,\beta}^{\tau,\tau'}(a,c;a',b';c';\lambda+\delta,\delta,p;x)=\lambda\ \ \gamma_{\alpha,\beta}^{\tau,\tau'}(a;c;a',b';c';\lambda,\delta,p;x)$$
$$-x^{\lambda}e^{-px^{\delta}}\phi^{\tau}\left(a,c;-\alpha x^{-\delta}\right){}_2R_1^{\tau'}\left(a',b';c';-\beta x^{\delta}\right)$$
$$+\frac{\Gamma(c)\Gamma(a+\tau)}{\Gamma(a)\Gamma(c+\tau)}(\alpha\delta)\ \gamma_{\alpha,\beta}^{\tau,\tau'}(a+\tau,c+\tau;a',b';c';\lambda-\delta,\delta,p;x)$$
$$+\frac{a'\Gamma(c')\Gamma(b'+\tau')}{\Gamma(b')\Gamma(c'+\tau')}(-\beta\delta)\,\gamma_{\alpha,\beta}^{\tau,\tau'}(a,c;a'+1,b'+\tau';c'+\tau';\lambda+\delta,\delta,p;x) \quad (14)$$

Proof:

To derive the recurrence relation (3.5), we use the following differential properties of ϕ^{τ} and ${}_2R_1^{\tau'}$ [2, 16]

$$\frac{d}{dz}\phi^{\tau}(a;c;z)=\frac{\Gamma(c)\Gamma(a+\tau)}{\Gamma(a)\Gamma(c+\tau)}\phi^{\tau}(a+\tau;c+\tau;z)$$

$$\frac{d}{dz}\ {}_2R_1^{\tau}(a,b;c;z)=a\ \ \frac{\Gamma(c)\Gamma(b+\tau)}{\Gamma(b)\Gamma(c+\tau)}\ \ {}_2R_1(a+1,b+\tau,c+\tau;z)$$

To the following expression

$$\frac{d}{dt}\left[t^{\lambda-1}e^{-pt^{\delta}}\phi^{\tau}\left(a,c;-\alpha t^{-\delta}\right){}_2R_1^{\tau'}\left(a',b';c';-\beta t^{\delta}\right)\right]$$
$$=\mathrm{t}^{\lambda-2}\ e^{-pt^{\delta}}\ \left[(\lambda-1)\ \phi^{\tau}\left(a,c;-\alpha t^{-\delta}\right)\ {}_2R_1^{\tau'}\left(a',b';c';-\beta t^{\delta}\right)\right.$$
$$-p\ \delta\ t^{\delta}\ \phi^{\tau}\left(a,c;-\alpha t^{-\delta}\right){}_2R_1^{\tau'}\left(a',b';c';-\beta t^{\delta}\right)$$
$$+\frac{\Gamma(c)\Gamma(a+\tau)}{\Gamma(a)\Gamma(c+\tau)}\frac{\alpha\delta}{t^{\delta}}\ \ \phi^{\tau}\left(a+\tau,c+\tau;-\alpha t^{-\delta}\right)\ {}_2R_1^{\tau'}\left(a',b';c';-\beta t^{\delta}\right)$$
$$\left.+\frac{a'\Gamma(c')\Gamma(b'+\tau')}{\Gamma(b')\Gamma(c'+\tau')}\left(-\beta\delta t^{\delta}\right)\phi^{\tau}\left(a,c;-\alpha t^{-\delta}\right){}_2R_1^{\tau'}\left(a'+1,b'+\tau';c'+\tau';-\beta t^{\delta}\right)\right] \quad (15)$$

Integrating the above equation (15) with respect to t from x to ∞ and interpreting the resulting expression in terms of the complementary form of incomplete gamma function by using definition (10), we easily arrive at the desired result (13) after a little simplification.

To obtain the recurrence relation (14), we proceed similar to the proof of the recurrence relation (13) except that in this case we integrate the equation (15) with respect to t from 0 to x and use the definition (9) in place of (10).

4. The Probability Density Function

In this section, we define probability density function of a random variable x as follows:

$$f(x) = \begin{cases} \frac{1}{E} x^{\lambda-1} e^{-px^{\delta}} \phi^{\tau}\left(a; c; -\alpha x^{-\delta}\right) \quad {}_2R_1^{\tau'}\left(a', b'; c'; -\beta x^{\delta}\right), & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

$\lambda, \delta, p, \tau, \tau' > 0, \lambda\tau + \delta a > 0, \alpha \geq 0, |\arg\beta| < \pi.$

Where $E = E_{\alpha,\beta}^{\tau,r'}(a, c; a', b'; c'; \lambda, \delta, p)$ is the generalized gamma type function as given by equation (9).

The parameters involving in (16) are so restricted that $f(x)$ remains positive for $x > 0$. Here, p, α, β are scale parameters whereas λ, δ represent shape parameters.

It is easy to verify that (4.1) represent a p.d.f. i.e.

$$\int_{-\infty}^{\infty} f(x)dx = 1 \tag{17}$$

We observe that the behavior of $f(x)$ at $x = 0$ depends on $\lambda\tau + \delta a$, i.e.

$$f(0) = \begin{cases} 0 & \lambda\tau + \delta a > 1 \\ \left[E_{\alpha,\beta}^{\tau,\tau'}(a, c; a', b'; c'; \lambda, \delta, p)\right]^{-1} & \lambda\tau + \delta a = 1 \end{cases} \tag{18}$$

and $\lim_{x \to o^+} f(x) = \infty, \quad 0 < (\lambda\tau + \delta a) < 1$

Also,

$$\lim_{x \to \infty} f(x) = 0, \quad \delta > 0, \qquad p > 0$$

The probability density function defined by (16) is quite general in nature and several other probability density functions follow as its particular cases as given below :

(i) If we take $\beta = 0$ in (16), we obtain the density function defined by Ali 2], as given by equation (2)

(ii) which on further taking a $=$ c and $\delta = 1$ reduces to the density function defined by Good [9] given by equation (1).

(iii) On taking $\alpha = 0$ in (16), we get the density function defined by Kalla and Al-Saqabi [12] given by equation (5) which can further be reduced to p.d.f. considered by Kalla et al [13] and Al-Zamel [3] on proper choice of parameters.

(iv) Several other probability density functions can be derived from (16) by restricting the parameters. On proper choice of parameters this density (16) reduces to the generatized form of the p.d.f. of the Weibull distribution, gamma density function.

By logarithmic differentiation of (16), we get.

$$f'(x) = f(x)\left[\frac{\lambda-1}{x} - p\delta x^{\delta-1} + \frac{\alpha\delta\Gamma(c)\Gamma(a+\tau)}{\Gamma(a)\Gamma(c+\tau)}\,\frac{\phi^{\tau}\left(a+\tau;c+\tau;-\alpha x^{-\delta}\right)}{x^{\delta+1}\phi^{\tau}(a;c;-\alpha x^{-\delta})}\right.$$
$$\left. - \frac{\beta\delta a'\Gamma(c')\Gamma(b'+\tau')}{\Gamma(b')\Gamma(c'+\tau')}\,\frac{{}_2R_1^{\tau'}\left(a'+1,b'+\tau';c'+\tau';-\beta x^{\delta}\right)}{x^{1-\delta}\,{}_2R_1^{\tau'}(a',b';c';-\beta x^{\delta})}\right] \quad (19)$$

Now we shall obtain some statistical functions such as the mathematical expectation, moments, mean, moment generating function, the survivor function $S(x)$, the hazard rate function $h(x)$ and the mean residue life function $k(x)$ for the p.d.f. defined by (4.1).

The Mathematical Expectation: The mathematical expectation of any function $\phi(x)$ with respect to probability density function $f(x)$ is given by

$$E[\phi(x)] = \int_{-\infty}^{\infty} f(x)\phi(x)dx = \int_0^{\infty} f(x)\phi(x)dx \quad (20)$$

where $f(x)$ is defined by equation (16).

Moments: If we take $\phi(x) = x^k$ in equation (20), we get the following k^{th} moment about the origin of the p.d.f. defined by equation (16)

$$E\left(x^k\right) = \int_0^{\infty} x^k f(x)dx = \frac{1}{E}E_{\alpha,\beta}^{\tau,\tau'}(a;c;a',b';c';\lambda+k,\delta,p) \quad (21)$$

If we take $k = 1$ in the above equation, we get the **mean** which is defined as

$$E(x) = \int_0^{\infty} x\, f(x)dx = \frac{1}{E}E_{\alpha,\beta}^{\tau,\tau'}(a;c;a',b';c';\lambda+1,\delta,p) \quad (22)$$

Further, in equation (20) if we take $\phi(x) = x^{s-1}$, we obtain the **Mellin Transform** of the pd.f. $f(x)$ as follows.

$$E\,[x^{s-1}] = M[f(x);s] = \int_0^{\infty} x^{s-1}f(x)dx$$
$$= \frac{1}{E}E_{\alpha,\beta}^{\tau,\tau'}(a,c;a',b';c';\lambda+s-1,\delta;p) \quad (23)$$

The Moment Generating Function: The moment generating function of x, denoted by $\mathbf{M}_x(t)$ for the p.d.f. defined by (4.1) with $\delta = 1$ is given by

$$M_x(t) = E(e^{xt}) = \int_0^{\infty} e^{xt}f(x)dx = \frac{1}{E}E_{\alpha,\beta}^{\tau,\tau'}(a,c;a',b';c';\lambda,1,p-t) \quad (24)$$

Also, the **characteristic function** is given by

$$E\left(e^{itx}\right) = \int_0^{\infty} e^{itx}f(x)dx$$
$$= \frac{1}{E}E_{\alpha,\beta}^{\tau,\tau'}(a,c;a',b';c';\lambda;1,p-it) \quad (25)$$

Distribution Function: The distribution function (or cumulative distribution function) for the p.d.f. f(x) is given by

$$F(x)=\int_{-\infty}^{x} f(t)dt=\int_{0}^{x} f(t)dt$$
$$=\frac{1}{E}\int_{0}^{x} t^{\lambda-1}e^{-pt^{\delta}}\phi^{\tau}\left(a;c;-\alpha t^{-\delta}\right)\,{}_2R_1^{\tau'}\left(a',b';c';-\beta t^{\delta}\right)dt$$
$$=\frac{1}{E}\gamma_{\alpha,\beta}^{\tau,\tau'}(a,c;a',b';c';\lambda,\delta,p) \tag{26}$$

whereas its **survivor function S(x)** can be expressed as :

$$S(x)=1-F(x)=\int_{x}^{\infty} f(t)dt=\frac{\Gamma_{\alpha,\beta}^{\tau,\tau'}(a,c;a',b';c';\lambda,\delta,p;x)}{E_{\alpha,\beta}^{\tau,\tau'}(a,c;a',b';c';\lambda,\delta,p)} \tag{27}$$

The Hazard Rate Function: The hazard rate function (or failure rate) is defined as $\mathrm{h(x)}=\frac{\mathrm{f(x)}}{\mathrm{S(x)}}$ and it can be expressed using equation (16) and (27) as follows.

$$h(x)=\frac{x^{\lambda-1}e^{-px^{\delta}}\phi^{\tau}\left(a,c;-\alpha x^{-\delta}\right)\,{}_2R_1^{\tau'}\left(a',b';c';-\beta x^{\delta}\right)}{\Gamma_{\alpha,\beta}^{\tau,\tau'}(a;c;a',b';c';\lambda,\delta,p;x)} \tag{28}$$

The **Mean Residue Life Function**: For a random variable x, the mean residue life function is defined by.

$$K(x)=E[X-x\mid X\geq x]=\frac{1}{S(x)}\int_{x}^{\infty}(t-x)f(t)dt$$

which can easily be written in the following form with the help of equations (23) and (27)

$$k'(x)=\frac{\Gamma_{\alpha,\beta}^{\tau,\tau'}(a;c;a',b';c';\lambda+1,\delta,p;x)}{\Gamma_{\alpha,\beta}^{\tau,\tau'}(a;c;a',b';c';\lambda,\delta,p;x)}-x \tag{29}$$

5. Conclusion

In the present paper, we study and develop a unified probability density function which involves the product of τ-confluent hypergeometric function and τ-Gauss hypergeometric function. To achieve this, we defined a new generalization of gamma-type function along with their incomplete forms. The statistical functions associated with this distribution, including the moment, expectation, moment generating function, hazard rate function, and mean residue life function, have been derived. Our objective of introducing a mathematical model for a density function using a generalized gamma function provides a foundation for further research and applications. The emphasis on potential applications in applied statistics and reliability theory underscores the practical significance of your work. By extending and generalizing well-known probability distributions, your method offers a versatile framework that researchers in various fields can leverage for innovative applications.

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