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CERTAIN RESULTS INVOLVING MOCK THETA FUNCTIONS OF ORDER EIGHT AND TEN

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Abstract: In this paper, making use of an identity derived from Bailey's transform certain results have been established involving Mock theta functions of order 8 and 10.

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1. Introduction, Notations and Definitions

In 1920, Ramanujan's listed 17 mock theta functions of order 3,5 and 7 associated with identities delighted by them in his last letter to Hardy. This is the last contribution of Ramanujan's to mathematical world "The Mock Theta Functions" about which Ramanujan's informed to Hardy in his last letter just two months before his death. Ramanujan's quoted that I discovered very interesting functions just which I call 'Mock Theta Functions'. Ramanujan's introduced about mock theta functions of order 3, 5 and 7 respectively. In Ramanujan's collected papers of mock theta functions [10] are characterized by Hardy. Mathematicians working in the field of basic hypergeometric series (q-series) like Andrews and Hickerson [1], Gordon and McIntosh [8], Choi [6] have introduced new class of mock theta functions. Andrews and Hickerson [1] have introduced seven mock theta functions of order 6, Gordon and McIntosh [8] have introduced seven mock theta functions of order 8, Choi [6] provided four mock theta functions of order 10. Denish et al. [7], Singh and Singh [13] have established many results which provide relationship between any two mock theta functions of different orders. Similarly, various authors viz. Andrews [2] discussed a fifth and seventh order mock theta functions, Agarwal [3] on Resonance of Ramanujan's Mathematics, Chand et al. [5] on product formulas for mock theta functions, Gasper and Rahman [9] on basic hyper geometric series, on Ramanujan [10]. The Lost note book and other unpublished works, Srivastava et al. [11, 12] on certain derived WP-Bailey, pairs and transformation formulas. Singh [14], Singh and Mishra [15] on mock-theta function of order 3 and continued fraction.

For $|q|<1,$ the $q-$shifted factorial is defined by

$$(a;q)_n = \begin{cases} 1, & n=0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & for \;\; n \in N. \end{cases}$$

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n) = \lim_{n\to\infty}(a;q)_n$$

$$(a_1,a_2,\dots,a_r;q)_n = (a_1;q)_n(a_2;q)_n,\dots,(a_r;q)_n.$$

A generalized basic hypergeometric series with base q is defined as,

$${}_r\phi_s\left[\begin{matrix} a_1,a_2,\dots,a_r;q,z \\ b_1,b_2,\dots,b_s \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_r;q)_n z^n}{(q,b_1,b_2,\dots,b_s;q)_n},$$

where $|z|<1$, $|q|<1$.

Bailey's Transform:

Bailey's $[4]$ in 1947 established the following simple and extremely useful transform,

If $\beta_n = \sum_{r=0}^{n}\alpha_r u_{n-r} v_{n+r}$ (1)

and $\gamma_n = \sum_{r=0}^{\infty}\delta_{r+n} u_r v_{r+2n}$ (2)

then under suitable convergence conditions,

$$\sum_{n=0}^{\infty}\alpha_n\gamma_n = \sum_{n=0}^{\infty}\beta_n\delta_n, \quad (3)$$

where $\alpha_{r,}\ \delta_{r,} u_r$ and v_r are arbitrary functions of r alone.

Taking $u_r = v_r = 1$ in (1) &(2), it takes the form,

$$\beta_n = \sum_{r=0}^{n}\alpha_r$$

and $\gamma_n = \sum_{r=0}^{\infty}\delta_r$

then under suitable convergence conditions,

$$\sum_{n=0}^{\infty}\alpha_n\gamma_n=\sum_{n=0}^{\infty}\beta_n\delta_n. \tag{4}$$

By simple calculation (4) can be written as,

$$\left(\sum_{n=0}^{\infty}\alpha_n\right)^2+\sum_{n=0}^{\infty}(\alpha_n)^2=2\sum_{n=0}^{\infty}\alpha_n\sum_{r=0}^{n}\alpha_r. \tag{5}$$

[13,(1.6),p.158]

Mock theta functions of order 8 :

Gordon B. and McIntosh R.J. [8] establish 8 mock theta functions of order 8 which are given below:

$$S_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}\left(-q;q^2\right)_n}{\left(-q^2;q^2\right)_n} \tag{6}$$

$$S_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+2)}\left(-q;q^2\right)_n}{\left(-q^2;q^2\right)_n} \tag{7}$$

$$T_0(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)}\left(-q^2;q^2\right)_n}{\left(-q;q^2\right)_{n+1}} \tag{8}$$

$$T_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}\left(-q^2;q^2\right)_n}{\left(-q;q^2\right)_{n+1}} \tag{9}$$

$$U_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}\left(-q;q^2\right)_n}{\left(-q^4;q^4\right)_n} \tag{10}$$

$$U_1(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}\left(-q;q^2\right)_n}{\left(-q^2;q^4\right)_{n+1}} \tag{11}$$

$$V_0(q)=-1+2\sum_{n=0}^{\infty}\frac{q^{2n^2}\left(-q^2;q^4\right)_n}{\left(q;q^2\right)_{2n+1}} \tag{12}$$

$$V_1(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2+2n+1}\left(-q^4;q^4\right)_n}{\left(q;q^2\right)_{2n+2}} \tag{13}$$

Mock theta functions of order 10 :

Mock theta functions of tenth order defined by Choi Y.S. [6] establish in Ramanujan's Lost Notebook I, II, IV are given below:

$$\phi_{LC}(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{\left(q;q^2\right)_{n+1}} \tag{14}$$

$$\psi_{LC}(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)/2}}{\left(q;q^2\right)_{n+1}} \tag{15}$$

$$X_{LC}(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-q;q)_{2n}} \tag{16}$$

$$\chi_{LC}(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}} \tag{17}$$

2. Main Results

In this paper we shall establish results for mock theta functions of order 8 and 10 respectively.

(a) Putting $\alpha_n=\frac{q^{n^2}\left(-q;q^2\right)_n}{\left(-q^2;q^2\right)_n}$ from (6) in (5) we get

$$S_0^{\,2}(q)+\sum_{n=0}^{\infty}\frac{q^{2n^2}\left(-q;q^2\right)_n^{\,2}}{\left(-q^2;q^2\right)_n^{\,2}}=2\sum_{n=0}^{\infty}\frac{q^{n^2}\left(-q;q^2\right)_n}{\left(-q^2;q^2\right)_n}S_{0,n}(q). \tag{18}$$

(b) Taking $\alpha_n=\frac{q^{n(n+2)}\left(-q;q^2\right)_n}{\left(-q^2;q^2\right)_n}$ from (7) in (5) we have

$$S_1^{\,2}(q)+\sum_{n=0}^{\infty}\frac{q^{2n(n+2)}\left(-q;q^2\right)_n^{\,2}}{\left(-q^2;q^2\right)_n^{\,2}}=2\sum_{n=0}^{\infty}\frac{q^{n(n+2)}\left(-q;q^2\right)_n}{\left(-q^2;q^2\right)_n}S_{1,n}(q). \tag{19}$$

(c) Choosing $\alpha_n = \dfrac{q^{(n+1)(n+2)}\left(-q^2;q^2\right)_n}{\left(-q;q^2\right)_{n+1}}$ from (8) in (5) we obtain

$$T_0^{\,2}(q)+\sum_{n=0}^{\infty}\frac{q^{2(n+1)(n+2)}\left(-q^2;q^2\right)_n^{\,2}}{\left(-q;q^2\right)_{n+1}^{\,2}}=2\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)}\left(-q^2;q^2\right)_n}{\left(-q;q^2\right)_{n+1}}T_{0,n}(q). \tag{20}$$

(d) For $\alpha_n = \dfrac{q^{n(n+1)}\left(-q^2;q^2\right)_n}{\left(-q;q^2\right)_{n+1}}$ from (9) in (5) yields

$$T_1^{\,2}(q)+\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}\left(-q^2;q^2\right)_n^{\,2}}{\left(-q;q^2\right)_{n+1}^{\,2}}=2\sum_{n=0}^{\infty}\frac{q^{n(n+1)}\left(-q^2;q^2\right)_n}{\left(-q;q^2\right)_{n+1}}T_{1,n}(q). \tag{21}$$

(e) Putting $\alpha_n = \dfrac{q^{n^2}\left(-q;q^2\right)_n}{\left(-q^4;q^4\right)_n}$ from (10) in (5) we have

$$U_0^{\,2}(q)+\sum_{n=0}^{\infty}\frac{q^{2n^2}\left(-q;q^2\right)_n^{\,2}}{\left(-q^4;q^4\right)_n^{\,2}}=2\sum_{n=0}^{\infty}\frac{q^{n^2}\left(-q;q^2\right)_n}{\left(-q^4;q^4\right)_n}U_{0,n}(q). \tag{22}$$

(f) Taking $\alpha_n = \dfrac{q^{(n+1)^2}\left(-q;q^2\right)_n}{\left(-q^2;q^4\right)_{n+1}}$ from (11) in (5) we get

$$U_1^{\,2}(q)+\sum_{n=0}^{\infty}\frac{q^{2(n+1)^2}\left(-q;q^2\right)_n^{\,2}}{\left(-q^2;q^4\right)_{n+1}^{\,2}}=2\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}\left(-q;q^2\right)_n}{\left(-q^2;q^4\right)_{n+1}}U_{1,n}(q). \tag{23}$$

(g) Choosing $\alpha_n = -1+2\sum_{n=0}^{\infty}\dfrac{q^{2n^2}\left(-q^2;q^4\right)_n}{\left(q;q^2\right)_{2n+1}}$ from (12) in (5) we obtain

$$V_0^{\,2}(q)+\sum_{n=0}^{\infty}\left[-1+2\sum_{n=0}^{\infty}\frac{q^{2n^2}\left(-q^2;q^4\right)_n}{\left(q;q^2\right)_{2n+1}}\right]^2=2\sum_{n=0}^{\infty}\left[-1+2\sum_{n=0}^{\infty}\frac{q^{2n^2}\left(-q^2;q^4\right)_n}{\left(q;q^2\right)_{2n+1}}\right]V_{0,n}(q). \tag{24}$$

(h) For $\alpha_n = \dfrac{q^{2n^2+2n+1}\left(-q^4;q^4\right)_n}{\left(q;q^2\right)_{2n+2}}$ from (13) in (5) yields

$$V_1{}^2(q)+\sum_{n=0}^{\infty}\frac{q^{2\left(2n^2+2n+1\right)}\left(-q^4;q^4\right)_n{}^2}{\left(q;q^2\right)_{2n+2}{}^2}=2\sum_{n=0}^{\infty}\frac{q^{\left(2n^2+2n+1\right)}\left(-q^4;q^4\right)_n}{\left(q;q^2\right)_{2n+2}}V_{1,n}(q). \quad (25)$$

(i) Putting $\alpha_n = \dfrac{q^{n(n+1)/2}}{\left(q;q^2\right)_{n+1}}$ from (14) in (5) we have

$$\phi_{LC}{}^2(q)+\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{\left(q;q^2\right)_{n+1}{}^2}=2\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{\left(q;q^2\right)_{n+1}}\phi_{LC,n}(q). \quad (26)$$

(j) Taking $\alpha_n = \dfrac{q^{(n+1)(n+2)/2}}{\left(q;q^2\right)_{n+1}}$ from (15) in (5) we get

$$\psi_{LC}{}^2(q)+\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)}}{\left(q;q^2\right)_{n+1}{}^2}=2\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)/2}}{\left(q;q^2\right)_{n+1}}\psi_{LC,n}(q). \quad (27)$$

(k) Choosing $\alpha_n = \dfrac{(-1)^n q^{n^2}}{(-q;q)_{2n}}$ from (16) in (5) we obtain

$$X_{LC}{}^2(q)+\sum_{n=0}^{\infty}\frac{(-1)^{2n} q^{2n^2}}{(-q;q)_{2n}{}^2}=2\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-q;q)_{2n}}X_{LC,n}(q). \quad (28)$$

(l) Lastly, taking $\alpha_n = \dfrac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}$ from (17) in (5) we have

$$\chi_{LC}{}^2(q)+\sum_{n=0}^{\infty}\frac{(-1)^{2n} q^{2(n+1)^2}}{(-q;q)^2{}_{2n+1}}=2\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}\chi_{LC,n}(q). \quad (29)$$

Similarly, we can find another relation between mock theta functions and partial mock theta functions.

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