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RADICAL SCREEN TRANSVERSAL SLANT LIGHTLIKE SUBMERSIONS

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Abstract: In the present article, we study radical screen transversal slant lightlike submersions from an indefinite Kaehler manifold onto a lightlike manifold with several examples. We study some properties of proper radical screen transversal slant lightlike submersions and give two characterization theorems. We also obtain integrability conditions of distributions involved in the definition of these submersions and give necessary and sufficient conditions for foliations obtained by the above distributions to be totally geodesic.

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1. Introduction

A C^∞ map between Riemannian manifolds M_1 and M_2 is called a Riemannian submersion if the derivative map φ_* is surjective and $g_1(U,V) = g_2(\varphi_*U, \varphi_*V)$, where U and V are vector fields tangent to the horizontal space $(Ker\varphi_*)^\perp$. The theory of Riemannian submersions between Riemannian manifolds was initiated by O'Neill [10] and Gray [7]. In [9], O'Neill studied Semi-Riemannian submersions between semi-Riemannian manifolds. In [17], Sahin and Gündüzalp defined lightlike submersions from semi-Riemannian manifolds onto lightlike manifolds. On the other hand, Sahin [14, 15, 16, 17] introduced the notions of slant and screen-slant lightlike submanifolds of an indefinite Hermitian manifold. In [20], Thakur et al. gave the notion of radical screen transversal slant lightlike submanifolds of indefinite Kaehler manifolds. Barros and Romero [1] discussed indefinite Kaehler manifold, Chen [2] discussed geometry of slant manifolds. Similarly Duggal [3, 4] on light like submanifolds, Falcitelli et al. [5, 6] on Riemannian and Kaehler submersions, Kumar et al. [7] on conformal hemi slant submersions, Prasad et al. [11, 12] on quasi bi-slant submersion, Park and Prasad [13] on semi slant submersion, Shukla and Omar [18] on screen Cauchy-Riemann light like submersion, Shukla et al. [19] on radial transversal screen C-R light like submersion.

This motivated us to study radical screen transversal slant lightlike submersions. The article is arranged as follows. In Section 2, we give some basic definitions and formulas related to this paper. In Section 3, we define radical screen transversal slant lightlike submersions with some non-trivial examples. In this section, we also obtain two characterization theorems and investigate integrability conditions of distributions involved in the definition of such submersions. Finally, we give necessary and sufficient conditions for foliations obtained by the above distributions to be totally geodesic.

2. Preliminaries

Let $(\mathrm{M}, g, \mathrm{J})$ be a 2m-dimensional almost complex manifold with an almost complex structure J such that $\mathrm{J}^2 = -\mathrm{I}$, where I is an identity operator and semi-Riemannian metric g of index $0 < r \le 2m$. We say that M is an indefinite almost Hermitian manifold if

$$g(\mathrm{J}\,\mathrm{U}, \mathrm{J}\,\mathrm{V}) = g(\mathrm{U}, \mathrm{V}), \quad \forall\, \mathrm{U}, \mathrm{V} \in \Gamma(T\mathrm{M}) \tag{1}$$

An indefinite almost Hermitian manifold $(\mathrm{M}, \mathrm{J}, \bar{g})$ with Levi-Civita connection ∇ is called an indefinite Kaehler manifold if

$$(\nabla_{\mathrm{U}}\mathrm{J})\mathrm{V} = 0, \quad \forall\, \mathrm{U}, \mathrm{V} \in \Gamma(T\mathrm{M}). \tag{2}$$

The Radical (or null) space $\mathrm{R}ad(\mathrm{T}_p\mathrm{M})$ of $\mathrm{T}_p\mathrm{M}$ on a smooth manifold M is given by $\mathrm{R}ad(\mathrm{T}_p\mathrm{M}) = \{\xi \in \mathrm{T}_p\mathrm{M} : g(\xi, \mathrm{U}) = 0, \forall\, \mathrm{U} \in \mathrm{T}_p\mathrm{M}\}$. If $\mathrm{R}ad(\mathrm{TM}) : p \in \mathrm{M} \to \mathrm{R}ad\mathrm{T}_p\mathrm{M}$ defines a smooth distribution of rank $r > 0$ on M such that $0 < r \le m$, then it is called the radical or null distribution and in this case the manifold M is called an r-lightlike manifold.

If $\varphi : (\mathrm{M}_1, g_1) \to (\mathrm{M}_2, g_2)$ is a C^∞ submersion from a semi-Riemannian manifold M_1 onto an r-lightlike manifold M_2, then $\mathrm{K}er\varphi_* = \{\mathrm{U} \in T_p\mathrm{M}_1 : \varphi_*\mathrm{U} = 0\}$ and $(\mathrm{K}er\varphi_*)^\perp = \{\mathrm{V} \in \mathrm{T}_p\mathrm{M}_1 : g_1(\mathrm{U}, \mathrm{V}) = 0, \forall\, \mathrm{U} \in \mathrm{K}er\varphi_*\}$. Since $\mathrm{T}_p\mathrm{M}_1$ is a semi-Riemannian vector space $(\mathrm{K}er\varphi_*)^\perp$ may not be a complementary space to $\mathrm{K}er\varphi_*$. Assume that $(\mathrm{K}er\varphi_*) \cap (\mathrm{K}er\varphi_*)^\perp = \Delta \ne \{0\}$. In this case $\Delta : p \to \Delta_p$ is a distribution on M_1 called the radical distribution. As Δ is a lightlike distribution, we have $\mathrm{K}er\varphi_* = \Delta \perp \mathrm{S}(\mathrm{K}er\varphi_*)$. Similarly $(\mathrm{K}er\varphi_*)^\perp = \Delta \perp \mathrm{S}(\mathrm{K}er\varphi_*)^\perp$. Here $\mathrm{S}(\mathrm{K}er\varphi_*)^\perp$ is the complementary distribution to Δ in $(\mathrm{K}er\varphi_*)^\perp$. Now, let $dim(\Delta) = r > 0$. Since $\Delta \subset (\mathrm{S}(\mathrm{K}er\varphi_*)^\perp)^\perp$ and $(\mathrm{S}(\mathrm{K}er\varphi_*)^\perp)^\perp$ is non-degenerate, there exists null vectors $\mathrm{N}_1, \mathrm{N}_2 ..., \mathrm{N}_r$ such that $g_1(\xi_i, \xi_j) = g_1(\mathrm{N}_i, \mathrm{N}_j) = 0$, $g_1(\xi_i, \mathrm{N}_j) = \delta_{ij}$ and $g_1(\mathrm{Z}_\alpha, \xi_j) = g_1(\mathrm{Z}_\alpha, N_j) = 0$, where $\{\mathrm{N}_i\}$, $\{\xi_i\}$ and $\{\mathrm{Z}_\alpha\}$ are smooth null vector fields of $S((Ker\varphi_*)^{\perp})^{\perp}$, lightlike basis of Δ and basis of $S(Ker\varphi_*)^{\perp}$ respectively. Assume that $ltr(Ker\varphi_*)$ denotes the distribution spanned by null vector fields $N_1, N_2..., N_r$. Then $tr(Ker\varphi_*) = ltr(Ker\varphi_*) \perp S(Ker\varphi_*)^{\perp}$. Thus, the following decomposition is clear

$$TM_1|_{Ker\varphi_*} = (\Delta \oplus ltr(Ker\varphi_*)) \perp S(Ker\varphi_*) \perp S(Ker\varphi_*)^{\perp}. \quad (3)$$

Let $\varphi : (M_1, g_1) \to (M_2, g_2)$ be a smooth submersion. We say that φ is an r-lightlike submersion if the length of horizontal vectors is preserved under the derivative map φ_* and $dim(\Delta) = dim\{(Ker\varphi_*) \cap (Ker\varphi_*)^{\perp}\} = r,\ 0 < r < min\{dim(Ker\varphi_*), dim(Ker\varphi_*)^{\perp}\}$. A lightlike submersion $\varphi : M_1 \to M_2$ determines two (1, 2) type tensors fields T and A on M_1, given by O'Neill as

$$T_U V = \hbar\nabla_{\nu U}\nu V + \nu\nabla_{\nu U}\hbar V, \quad (4)$$

$$A_U V = \nu\nabla_{\hbar U}\hbar V + \hbar\nabla_{\hbar U}\nu V, \quad (5)$$

where U and V are vector fields on $Ker\varphi_*$. Tensors T and A are called the vertical and horizontal tensors respectively. More over, for the tensor field T, we have

$$T_U V = T_V U, \quad \forall\, U, V \in \Gamma(Ker\varphi_*). \quad (6)$$

Now, let φ be a lightlike submersion from a real (m + n)-dimensional semi-Riemannian manifold (M_1, g_1) onto a lightlike manifold (M_2, g_2). Further assume that $Ker\varphi_*$ is an m-dimensional lightlike distribution on M_1 and $tr(Ker\varphi_*)$ is the complementary distribution to $Ker\varphi_*$ in M_1 with respect to the pair $\{S(Ker\varphi_*), S(Ker\varphi_*)^{\perp}\}$.

Denote by $\hat{g}_1$ the induced metric on $Ker\varphi_*$ of g_1 and by ∇ the Levi-Civita connection on M_1. Then, using (2.4), we obtain

$$\nabla_U V = \hat{\nabla}_U V + T_U V, \quad (7)$$

$$\nabla_U Z = T_U Z + \nabla^{\perp}_U Z, \quad (8)$$

where $U, V \in \Gamma(Ker\varphi_*), Z \in \Gamma(Ker\varphi_*)^{\perp}$, $\hat{\nabla}_U V = \nu\nabla_U V$ and $\nabla^{\perp}_U X = \hbar\nabla_U X$. Here $\{\hat{\nabla}_U V, T_U Z\}$ and $\{T_U V, \nabla^{\perp}_U Z\}$ belong to $\Gamma(Ker\varphi_*)$ and $\Gamma(tr(Ker\varphi_*))$, respectively.

Let $S(Ker\varphi_*)^\perp \neq \{0\}$ and denote by L and S the projections of $tr(Ker\varphi_*)$ on $ltr(Ker\varphi_*)$ and $S(Ker\varphi_*)^\perp$, respectively. Then, using (7) and (8), we have

$$\nabla_U V = \hat{\nabla}_U V + T_U^\ell V + T_U^s V, \tag{9}$$

$$\nabla_U N = T_U N + \nabla_U^{\perp\ell} N + D^{\perp s}(U, N), \tag{10}$$

$$\nabla_U Z = T_U Z + D^{\perp\ell}(U, Z) + \nabla_U^{\perp s} Z, \tag{11}$$

for any $U, V \in \Gamma(Ker\varphi_*), N \in \Gamma(ltr(Ker\varphi_*))$ and $Z \in \Gamma(S(Ker\varphi_*)^\perp)$. If φ is an r-lightlike submersion, then we put

$$\hat{\nabla}_U \xi = T_U^* \xi + \nabla_U^{*\perp} \xi, \tag{12}$$

for any $U \in \Gamma(Ker\varphi_*), \xi \in \Gamma(\Delta)$. Here $T_U^* \xi \in \Gamma(S(Ker\varphi_*))$ and $\nabla_U^{*\perp} \xi \in \Gamma(\Delta)$.

3. Radical Screen Transversal Slant Lightlike Submersions

In this section we define radical screen transversal slant lightlike submersions, giving some non-trivial examples by using the following lemmas:

Lemma 3.1 : Let $\varphi:(M_1, g_1) \to (M_2, g_2)$ be a 2r-lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2 such that $Ker\varphi_*$ is a lightlike distribution on $Ker\varphi_*$. Then the screen distribution $S(Ker\varphi_*)$ is Riemannian.

Proof. Let M_1 be a real $(m+n)$-dimensional indefinite Kaehler manifold and $Ker\varphi_*$ be a lightlike distribution of dimension m on M_1. Then there exists a local quasi orthonormal field of frames on M_1 along $Ker\varphi_*$ $\{\xi_i, N_i, U_\alpha, Z_a\}, i \in \{1, ..., 2r\}, \alpha \in \{2r+1, ..., m\}, a \in \{2r+1, ..., n\}$,

where $\{\xi_i\}, \{N_i\}$ are lightlike basis of (0.1) Δ, $ltr(Ker\varphi_*)$ and U_α, Z_a are orthonormal basis of $S(Ker\varphi_*), S(Ker\varphi_*)^\perp$, respectively. With the help of null basis $\{\xi_1, ..., \xi_{2r}, N_1, ..., N_{2r}\}$ of $\Delta \oplus ltr(Ker\varphi_*)$, we construct following orthonormal basis $\{X_1, ..., X_{4r}\}$

$$X_1 = \frac{1}{\sqrt{2}}(\xi_1 + N_1), \qquad X_2 = \frac{1}{\sqrt{2}}(\xi_1 - N_1),$$

$$X_3 = \frac{1}{\sqrt{2}}(\xi_2 + N_2), \qquad X_4 = \frac{1}{\sqrt{2}}(\xi_2 - N_2),$$

$$\ldots \qquad \ldots$$

$$\dots \qquad\qquad \dots$$

$$X_{4r-1} = \frac{1}{\sqrt{2}}(\xi_{2r} + N_{2r}), \qquad X_{4r} = \frac{1}{\sqrt{2}}(\xi_{2r} - N_{2r}).$$

Thus, $Span\{\xi_i, N_i\}$ is a non-degenerate space of index 2r, which enables us to conclude that $\Delta \oplus ltr(Ker\varphi_*)$ is non-degenerate with constant index 2r on M_1. Moreover,

$$ind(TM_1) = ind(\Delta \oplus ltr(Ker\varphi_*)) + ind(S(Ker\varphi_*) \perp (S(Ker\varphi_*))^{\perp}),$$

implies that $S(Ker\varphi_*) \perp S(Ker\varphi_*)^{\perp}$ has a constant index zero. Hence, $S(Ker\varphi_*)$ and $S(Ker\varphi_*)^{\perp}$ are Riemannian.

Lemma 3.2 : Let $\varphi:(M_1, g_1, J) \to (M_2, g_2)$ be an r-lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. If $J(\Delta)$ is a vector subbundle of $S(Ker\varphi_*)^{\perp}$, then $J(ltr(Ker\varphi_*))$ is also a vector subbundle of $S(Ker\varphi_*)^{\perp}$ and $J(\Delta) \cap J(ltr(Ker\varphi_*)) = \{0\}$.

Proof. Suppose $J(ltr(Ker\varphi_*)) = ltr(Ker\varphi_*)$. Now, since φ is an r-lightlike submersion, so for $\xi \in \Gamma(\Delta)$ and $N \in \Gamma(ltr(Ker\varphi_*))$, using (1), we have $g(J\xi, JN) = g(\xi, N) = 1.$ Now, if $JN \in \Gamma(ltr(Ker\varphi_*))$, then by hypothesis of lemma, we arrive at $g(J\xi, JN) = 0$. Thus, we get a contradiction. It follows that $JN \notin \Gamma(ltr(Ker\varphi_*))$. Further, assume that $JN \in \Gamma(S(Ker\varphi_*))$. Then, we obtain $0 = g(J\xi, JN) = g(\xi, N) = 1.$ It implies that $JN \notin \Gamma(S(Ker\varphi_*))$. Moreover, in a similar way, we can obtain that $JN \notin \Gamma(\Delta)$. Thus, we conclude that $JN \in \Gamma(S(Ker\varphi_*)^{\perp})$. Finally, suppose that $V \in \Gamma(J(\Delta) \cap J(ltr(Ker\varphi_*)))$. It follows that $V \in \Gamma(J(\Delta))$, which implies $g(V, JN) = 0$, as $V \in \Gamma(J(ltr(Ker\varphi_*)))$. But it is already known that for an r-lightlike submersion, there exists $\overline{J}V \in \Gamma(\Delta)$, such that $g(JV, N) \neq 0$. So, from (1), we have $0 \neq g(JV, N) = -g(V, JN) = 0$, which is absurd. Thus, the proof is completed.

Definition 3.1 : A 2r-lightlike submersion $\varphi:(M_1, g_1, J) \to (M_2, g_2)$ from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2 is called a radical screen transversal slant lightlike submersion if the radical distribution Δ satisfy $J\Delta \subset S(Ker\varphi_*)^{\perp}$ and the screen distribution $S(Ker\varphi_*)$ is slant with slant angle θ.

From definition (3.1), the following decomposition is clear:

$$S(Ker\varphi_*)^{\perp} = (J\,\Delta \oplus J\; ltr(Ker\varphi_*)) \perp D, \qquad (13)$$

where D is a non-degenerate orthogonal complementary distribution to $J\,\Delta \oplus J\; ltr(Ker\varphi_*)$ in $S(Ker\varphi_*)^{\perp}$. If $S(Ker\varphi_*) \neq \{0\}$, $S(Ker\varphi_*)^{\perp} \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$, then we say that φ is a proper radical screen transversal slant lightlike submersion. Moreover

(i) If $\theta = 0$, then we say that φ is a radical screen transversal lightlike submersion.

(ii) If $\theta = \frac{\pi}{2}$, then we say that φ is a screen transversal anti-invariant lightlike submersion.

Proposition 3.1 : There exists no proper radical screen transversal slant co-isotropic or isotropic or totally lightlike submersion from an indefinite Kaehler manifold onto a lightlike manifold.

Proof. If φ is a co-isotropic submersion, then $S(Ker\varphi_*)^{\perp} = \{0\}$. If φ is a isotropic submersion, then $S(Ker\varphi_*) = \{0\}$. Moreover, if φ is a totally lightlike submersion, then $S(Ker\varphi_*) = S(Ker\varphi_*)^{\perp} = \{0\}$. Thus the proof is completed.

Denote by $\mathbb{R}^n_{r,q,p}$ the space $\mathbb{R}^n$ equipped with the semi-Riemannian metric g_1, such that $g_1(e_i, e_j)_{r,q,p} = (G_{r,q,p})_{ij}$, $i \in \{1,\dots,n\}$, where e_i is the standard basis of $\mathbb{R}^n$ and $G_{r,q,p}$ is the diagonal matrix determined by g_1, i.e., G_{ij} = diagonal $(\underbrace{0,\dots,0}_{\text{r-times}}, \underbrace{-1,\dots,-1}_{\text{q-times}}, \underbrace{1,\dots,1}_{\text{p-times}})$.

Also, let us define the almost complex structure J as $J\,\{u_1, v_1, \dots, u_n, v_n\} = \{-v_1, u_1, \dots, -v_n, u_n\}$.

Example 3.1 : Let $\mathbb{R}^{12}_{0,2,10}$ and $\mathbb{R}^{8}_{2,0,6}$ be $\mathbb{R}^{12}$ and $\mathbb{R}^{8}$ endowed with the semi-Riemannian metric $g_1 = (du_1)^2 + (du_2)^2 - (du_3)^2 + (du_4)^2 - (du_5)^2 + (du_6)^2 + (du_7)^2 \backslash + (du_8)^2 + (du_9)^2 + (du_{10})^2 + (du_{11})^2 + (du_{12})^2$,

and degenerate metric $g_2 = (dv_2)^2 + (dv_3)^2 + (dv_5)^2 + (dv_6)^2 + (dv_7)^2 + (dv_8)^2$, where $u_1, \dots, u_{12}$ and $v_1, \dots, v_8$ are the canonical coordinates on $\mathbb{R}^{12}$ and $\mathbb{R}^{8}$, respectively. Consider the map $\varphi : (\mathbb{R}^{12}_{0,2,10}, g_1) \to (\mathbb{R}^{8}_{2,0,6}, g_2)$ as $(u_1, \dots, u_{12}) \mapsto \left(u_1 - u_3, u_2, u_4, u_5 - u_7, u_6, u_8, \frac{\sqrt{3}u_9 + u_{12}}{2}, u_{10}\right)$.

Then

$$Ker\varphi_* = Span\left\{\xi_1 = \partial u_1 + \partial u_3, \xi_2 = \partial u_5 + \partial u_7, S_1 = \frac{\partial u_9 - \sqrt{3}\partial u_{12}}{2}, S_2 = \partial u_{11}\right\}$$

and

$$(\mathrm{K}er\varphi_*)^{\perp} = \mathrm{S}pan\left\{\xi_1, \xi_2, \overline{\mathrm{S}}_1 = \partial u_2, \overline{\mathrm{S}}_2 = \partial u_4, \overline{\mathrm{S}}_3 = \partial u_6, \overline{\mathrm{S}}_4 = \partial u_8, \overline{\mathrm{S}}_5 = \frac{\sqrt{3}\partial u_9 + \partial u_{12}}{2}, \overline{\mathrm{S}}_6 = \partial u_{10}, \right\}.$$

Also

$$ltr(\mathrm{K}er\varphi_*) = \mathrm{S}pan\left\{\mathrm{N}_1 = \frac{\partial u_1 - \partial u_3}{2}, \mathrm{N}_2 = \frac{-\partial u_5 + \partial u_7}{2}\right\}.$$

Therefore, we see that $\mathrm{J}\,\Delta = \mathrm{S}pan(\overline{\mathrm{S}}_1 + \overline{\mathrm{S}}_2, \overline{\mathrm{S}}_3 + \overline{\mathrm{S}}_4) \subset \Gamma(\mathrm{S}(\mathrm{K}er\varphi_*)^{\perp})$ and $\mathrm{J}\; ltr(\mathrm{K}er\varphi_*) = \mathrm{S}pan(\frac{\overline{\mathrm{S}}_1 - \overline{\mathrm{S}}_2}{2}, \frac{-\overline{\mathrm{S}}_3 + \overline{\mathrm{S}}_4}{2}) \subset \Gamma(\mathrm{S}(\mathrm{K}er\varphi_*)^{\perp})$. So $\mathrm{J}\,\Delta = ltr(Ker\varphi_*)$. Moreover $\mathrm{S}(\mathrm{K}er\varphi_*)$ is slant with slant angle $\theta = \frac{\pi}{6}$. Now, since the length of horizontal vectors is preserved under the derivative map φ_* therefore φ is a proper radical screen transversal slant lightlike submersion.

Example 3.2 : Let $\mathbb{R}^8_{0,1,7}$ and $\mathbb{R}^5_{1,0,4}$ be equipped with the semi-Riemannian metric

$$g_1 = -(du_1)^2 + (du_2)^2 + (du_3)^2 + (du_4)^2 + (du_5)^2 + (du_6)^2 + (du_7)^2 + (du_8)^2$$

and null metric

$g_2 = (dv_2)^2 + (dv_3)^2 + (dv_4)^2 + (dv_5)^2$, where $u_1,...,u_8$ and $v_1,...,v_5$ are the canonical coordinates on $\mathbb{R}^8$ and $\mathbb{R}^5$, respectively. Define the map $\varphi : (\mathbb{R}^8_{0,1,7}, g_1) \to (\mathbb{R}^5_{1,0,4}, g_2)$ as

$$(u_1,...,u_8) \to \left(u_1 + u_7, u_2, u_8, \frac{u_3 - u_5}{\sqrt{2}}, \frac{u_4 - u_6}{\sqrt{2}}\right).$$

Then

$$\mathrm{K}er\varphi_* = \mathrm{S}pan\left\{\xi = \frac{\partial u_1 - \partial u_7}{\sqrt{2}}, \mathrm{S}_1 = \partial u_3 + \partial u_5, \mathrm{S}_2 = \partial u_4 + \partial u_6\right\},$$

which gives

$$(\mathrm{K}er\varphi_*)^{\perp} = \mathrm{S}pan\left\{\xi, \overline{\mathrm{S}}_1 = \partial u_2, \overline{\mathrm{S}}_2 = \partial u_8, \overline{\mathrm{S}}_3 = \partial u_3 - \partial u_5, \overline{\mathrm{S}}_4 = \partial u_4 - \partial u_6\right\}.$$

Thus $\Delta = \mathrm{S}pan\{\xi\}$ is such that $\mathrm{J}\,\Delta = \mathrm{S}pan\{\overline{\mathrm{S}}_2 + \overline{\mathrm{S}}_3\} \subset \Gamma(\mathrm{S}((\mathrm{K}er\varphi_*)^{\perp}))$. It is also clear that $\mathrm{J}\,\mathrm{S} = \mathrm{S}_2$. Therefore, $\mathrm{S}(\mathrm{K}er\varphi_*) = \mathrm{S}pan\{\mathrm{S}_1, \mathrm{S}_2\}$ is slant with slant angle 0. Hence, φ is a radical screen transversal lightlike submersion.

Example 3.3 : Let $\mathbb{R}^6_{0,1,5}$ and $\mathbb{R}^4_{1,0,3}$ be equipped with the semi-Riemannian metric

$$g_1 = (du_1)^2 - (du_2)^2 + (du_3)^2 + (du_4)^2 + (du_5)^2 + (du_6)^2$$

and null metric $g_2 = (dv_1)^2 + (dv_2)^2 + (dv_4)^2$, where $u_1,...,u_6$ and $v_1,...,v_4$ are the canonical coordinates on $\mathbb{R}^6$ and $\mathbb{R}^4$, respectively. Assume that the map $\varphi:(\mathbb{R}^6_{0,1,5}, g_1) \to (\mathbb{R}^4_{1,0,3}, g_2)$ is defined as $(u_1,...,u_6) \to \left(u_1, u_3, u_2 + u_4, u_6\right)$. Then $\mathrm{K}er\varphi_* = \mathrm{S}pan\left\{\xi = \partial u_2 - \partial u_4, \mathrm{S} = \partial u_5\right\}$ and N are preserved under the derivative map φ_*, so φ is a screen transversal anti-invariant lightlike submersion.

Let $\varphi:(\mathrm{M}_1, g_1) \to (\mathrm{M}_2, g_2)$ be a 2r-lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. For any $\mathrm{V} \in \Gamma(\mathrm{K}er\varphi_*)$, we put

$$\mathrm{J\,V} = \alpha\mathrm{V} + \beta\mathrm{V}. \tag{14}$$

Here $\alpha\mathrm{V} \in \Gamma(\mathrm{K}er\varphi_*)$ and $\beta\mathrm{V} \in \Gamma(tr(\mathrm{K}er\varphi_*))$. Also, for any $\mathrm{Z} \in \Gamma(tr(\mathrm{K}er\varphi_*))$, we assume

$$\mathrm{J\,Z} = \eta\mathrm{Z} + \tau\mathrm{Z} = \eta_1\mathrm{Z} + \eta_2\mathrm{Z} + \tau_1\mathrm{Z} + \tau_2\mathrm{Z} + \tau_3\mathrm{Z}, \tag{15}$$

where $\eta\mathrm{Z} \in \Gamma(Ker\varphi_*)$ and $\tau\mathrm{Z} \in \Gamma(tr(Ker\varphi_*))$. It is clear that $\eta_1\mathrm{Z} \in \Gamma(\Delta)$, $\eta_2\mathrm{Z} \in \Gamma(\mathrm{S}(\mathrm{K}er\varphi_*))$, $\tau_1\mathrm{Z} \in \Gamma(\mathrm{J}\,\Delta)$, $\tau_2\mathrm{Z} \in \Gamma\mathrm{J}\ (ltr(\mathrm{K}er\varphi_*))$ and $\tau_3\mathrm{Z} \in \Gamma(\mathrm{D})$. Further, denote the projections of $\mathrm{K}er\varphi_*$ on Δ and $\mathrm{S}(\mathrm{K}er\varphi_*)$ by P_1 and P_2 respectively. So, for any $\mathrm{V} \in \Gamma(\mathrm{K}er\varphi_*)$, we have

$$\mathrm{V} = \mathrm{P}_1\mathrm{V} + \mathrm{P}_2\mathrm{V}. \tag{16}$$

Equation (16), gives

$$\mathrm{J\,V} = \mathrm{J\,P}_1\mathrm{V} + \mathrm{J\,P}_2\mathrm{V} = \beta\mathrm{P}_1\mathrm{V} + \alpha\mathrm{P}_2\mathrm{V} + \beta\mathrm{P}_2\mathrm{V}, \tag{17}$$

Where

$$\alpha\mathrm{P}_1\mathrm{V} = 0, \quad \mathrm{J\,P}_1\mathrm{V} = \beta\mathrm{P}_1\mathrm{V} \in \Gamma(\mathrm{S}(\mathrm{K}er\varphi_*)^{\perp}), \tag{18}$$

$$\alpha\mathrm{P}_2\mathrm{V} \in \Gamma\mathrm{S}(\mathrm{K}er\varphi_*), \quad \beta\mathrm{P}_2\mathrm{V} \in \Gamma\mathrm{S}(\mathrm{K}er\varphi_*)^{\perp}. \tag{19}$$

Theorem 3.1 : Let $\varphi:(\mathrm{M}_1, g_1) \to (\mathrm{M}_2, g_2)$ be a 2r-lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then φ is a radical screen transversal slant lightlike submersion if and only if

(i) $J\ (ltr(Ker\varphi_*))$ is distribution on $S(Ker\varphi_*)^{\perp}$ such that $J\ (ltr(Ker\varphi_*))\subset S(Ker\varphi_*)^{\perp}$,

(ii) for any $V\in\Gamma(S(Ker\varphi_*))$ there exists a constant $\Lambda\in[-1,0]$, such that $(P_2\alpha)^2P_2V=-\Lambda P_2V,$ where $\Lambda=cos^2\theta,\theta$ is a slant angle of $S(Ker\varphi_*)$.

Proof. Let φ be a radical screen transversal slant lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then, (i) follows from Lemma 3.2. Now, since $S(Ker\varphi_*)$ is slant, so the angle between $J\,P_2V$ and $V\in\Gamma(S(Ker\varphi_*))$ is constant. Therefore, we have

$$cos(\theta)(P_2V)=\frac{\hat{g}_1(P_2V,\alpha P_2\alpha P_2V)}{|P_2V||\alpha P_2V|}. \tag{20}$$

Also

$$cos(\theta)(P_2V)=\frac{|\alpha P_2V|}{|J\,P_2V|}. \tag{21}$$

From (20) and (21), we get

$$cos^2\theta(P_2V)=-\frac{\hat{g}_1(P_2V,(P_2\alpha)^2P_2V)}{|P_2V|^2}.$$

Further, as $\theta(P_2V)$ is constant on $S(Ker\varphi_*)$, we obtain $(P_2\alpha)^2P_2V=-\Lambda P_2V,\ \Lambda\in[-1,0]$, where $\Lambda=cos^2\theta$. The reverse implication can be proved in a similar way.

Theorem 3.2 : Let $\varphi:(M_1,g_1)\to(M_2,g_2)$ be a 2r-lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then φ is a radical screen transversal slant lightlike submersion if and only if $J\ (ltr(Ker\varphi_*))$ is distribution on $S(Ker\varphi_*)^{\perp}$ such that $J\ (ltr(Ker\varphi_*))\subset S(Ker\varphi_*)^{\perp}$ and for any $V\in\Gamma(S(Ker\varphi_*))$ there exists a constant $\Lambda\in[-1,0]$, such that $\eta_1\beta P_2V=-\Lambda P_2V,$ where $\Lambda=sin^2\theta,\theta$ is a slant angle of $S(Ker\varphi_*)$ and P_2 is the projection on $S(Ker\varphi_*)$.

Proof. Let φ be a radical screen transversal slant lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then, (i) is clear using Lemma 3.2. Next, applying J to (17), using (15) and comparing the screen components, we obtain

$$-V=(P_2\alpha)^2P_2V+\beta P_2\alpha P_2V+\eta_1\beta P_2V+\eta_2\beta P_1V. \tag{22}$$

Comparing the screen components of (3.10), we have

$$-P_2V = (P_2\alpha)^2 P_2V + \eta_1\beta P_2V. \tag{23}$$

Now, using Theorem (3.1), we have $(P_2\alpha)^2 P_2V = -cos^2\theta P_2V, \Lambda \in [-1,0]$. So, using (23), we arrive $\eta_1\beta P_2V = -\Lambda P_2V,$ where $\Lambda = sin^2\theta,$ which completes the proof.

As an immediate consequence of the above theorems (3.1) and (3.2), we have

Corollary 3.1 : Let $\varphi:(M_1, g_1, J\ ) \to (M_2, g_2)$ be a radical screen transversal slant lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2 with slant angle θ of $S(Ker\varphi_*)$. Then $\forall\ U, V \in \Gamma(Ker\varphi_*)$, we have

$$\hat{g}_1(\alpha P_2U, \alpha P_2V) = cos^2\theta \hat{g}_1(P_2U, P_2V),$$

and

$$\hat{g}_1(\beta P_2U, \beta P_2V) = sin^2\theta \hat{g}_1(P_2U, P_2V).$$

Using (2), (9), (11), and (13)-(19), we obtain

$$T_U\beta P_1V + D^{\perp l}(U, \beta P_1V) + \nabla^{\perp s}_U \beta P_1V + \hat{\nabla}_U \alpha P_2V + TU^l \alpha P_2V + T^s_U \alpha P_2V + T_U\beta P_2V + D^{\perp l}(U, \beta P_2V) + \nabla^{\perp s}_U \beta P_2V = \beta P_1\hat{\nabla}_U V + \alpha P_2\hat{\nabla}_U V + \beta P_2\hat{\nabla}_U V + \eta T^l_U V + \tau T^l_U V + \eta T^s_U V + \tau T^s_U V.$$

Identifying the tangential, screen transversal, and lightlike transversal components, we get

$$(\hat{\nabla}_U\alpha)P_2V + T_U\beta P_1V + T_U\beta P_2V = \eta TU^s V, \tag{24}$$

$$(\hat{\nabla}_U\beta)P_1V + (\hat{\nabla}_U\beta)P_2V + T_U\alpha P_2V = \tau T^l_U V + \tau T^s_U V, \tag{25}$$

$$D^{\perp l}(U, \beta P_1V) + D^{\perp l}(U, \beta P_2V) = -TU^l \alpha P_2V. \tag{26}$$

Theorem 3.3 : Let $\varphi:(M_1, g_1) \to (M_2, g_2)$ be a radical screen transversal slant lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then, the radical distribution Δ is integrable if and only if for any $U, V \in \Gamma(\Delta)$, we have

(i) $T_U\beta P_1V = T_V\beta P_1U$,

(ii) $\nabla^{\perp s}_U \beta P_1V - \nabla^{\perp s}_V \beta P_1U \in \beta\Delta \subset S(Ker\varphi_*)^{\perp}$.

Proof. Let $U, V \in \Gamma(\Delta)$. Using (21), we have $\alpha P_2\hat{\nabla}_U V = +\eta T^s_U V - T_U\beta P_1V$. In view of (6) above equation gives

$$T_U\beta P_1V - T_V\beta P_1U = \alpha P_2[U,V]. \quad (27)$$

Further using (25), we get $\nabla^{\perp s}_U\beta P_1V - \beta P_1\hat{\nabla}_UV - \beta P_2\hat{\nabla}_UV = \tau TU^lV + \tau TU^sV$. It follows that

$$\nabla U^{\perp s}\beta P_1V - \nabla V^{\perp s}\beta P_1U = \beta P_1[U,V] + \beta P_2[U,V]. \quad (28)$$

Thus, the proof follows using (28) and (29).

Theorem 3.4. Let $\varphi:(M_1, g_1, J\ ) \to (M_2, g_2)$ be a radical screen transversal slant lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then, the the non-degenerate distribution $S(Ser\varphi_*)$ is integrable if and only if $\nabla^{\perp s}_U\beta P_2V - \nabla^{\perp s}_V\beta P_2U + T^s_U\alpha P_2V - T^s_V\alpha P_2U \in \beta S(Ker\varphi_*)$, for any $U,V \in \Gamma(S(Ker\varphi_*))$.

Proof. Let $U,V \in \Gamma(S(Ker\varphi_*))$. Using (21), we arrive at

$$-\beta P_1\hat{\nabla}UV + \nabla^{\perp s}_U\beta P_2V - \beta P_2\hat{\nabla}UV + T^s_U\alpha P_2V = \tau T^l_UV + TU^sV \in \beta S(Ker\varphi_*),$$

which gives

$$\beta P_1[U,V] + \beta P_2[U,V] = \nabla^{\perp s}_U\beta P_2V - \nabla^{\perp s}_V\beta P_2U + T^s_U\alpha P_2V - T^s_V\alpha P_2U.$$

Thus, the proof follows.

Theorem 3.5 : Let $\varphi:(M_1, g_1) \to (M_2, g_2)$ be a radical screen transversal slant lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then, the induced connection $\hat{\nabla}$ on $Ker\varphi_*$ is a Levi-Civita connection if and only if $\eta\nabla^{\perp s}_U\beta P_1\xi = -\alpha P_2T_U\beta P_1\xi,$ for any $U \in \Gamma(Ker\varphi_*)$ and $\xi \in \Gamma(\Delta)$.

Proof. The induced connection $\hat{\nabla}$ on $Ker\varphi_*$ is a Levi-Civita connection if and only if Δ is a parallel distribution with respect to $\hat{\nabla}$ [3]. Using (2) and (15), we obtain

$$\nabla_U\xi = -J\ \nabla_UJ\ \xi = -\nabla_U\beta P_1\xi. \quad (29)$$

Further, using (9), (11), (15) and (17) in (29), we have

$$\hat{\nabla}_U\xi + T^l_U\xi + T^s_U\xi = -J\ \ (T_U\beta P_1\xi + D^{\perp l}(U,\beta P_1\xi) + \nabla^{\perp s}_U\beta P_1\xi) = -$$
$$\beta P_1T_U\beta P_1\xi - \alpha P_2T_U\beta P_1\xi - \beta P_2T_U\beta P_1\xi - \eta D^{\perp l}(U,\beta P_1\xi) - \tau D^{\perp l}(U,\beta P_1\xi) - \eta\nabla^{\perp s}_U\beta P_1\xi - \tau\nabla^{\perp s}_U\beta P_1\xi.$$

Comparing the tangential components, we get

$$\hat{\nabla}_U\xi = -\alpha P_2T_U\beta P_1\xi - \eta\nabla^{\perp s}_U\beta P_1\xi.$$

Thus, the proof follows:

Theorem 3.6 : Let $\varphi:(\mathrm{M}_1, g_1)\to(\mathrm{M}_2, g_2)$ be a radical screen transversal slant lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then, the lightlike distribution Δ defines a totally geodesic foliation on $\mathrm{K}er\varphi_*$ if and only if $\mathrm{T}\xi^s\alpha\mathrm{P}_2\mathrm{V}+\nabla\xi^{\perp s}\beta\mathrm{P}_2\mathrm{V}$ has no component in $\mathrm{J}\,\Delta$ for any $\xi\in\Gamma(\Delta)$ and $\mathrm{V}\in\Gamma(\mathrm{S}(\mathrm{K}er\varphi_*))$.

Proof. Let $\xi,\xi'\in\Gamma(\Delta)$ and $\mathrm{V}\in\Gamma(S(Ker\varphi_*))$. From (1), (2) (9), (11), (17) and taking into account that ∇ is a metric connection, we obtain

$$
\begin{aligned}
g_1(\hat{\nabla}_\xi\xi',\mathrm{V}) &= g_1(\mathrm{J}\,\xi',\nabla_\xi\mathrm{J}\,\mathrm{V})\\
&= -g_1(\mathrm{J}\,\xi',\nabla_\xi\alpha\mathrm{P}_2\mathrm{V}+\nabla_\xi\beta\mathrm{P}_2\mathrm{V})\\
&= -g_1(\mathrm{J}\,\xi',\mathrm{T}^s_\xi\alpha\mathrm{P}_2\mathrm{V}+\nabla^{\perp s}_\xi\beta\mathrm{P}_2\mathrm{V}).
\end{aligned}
$$

Thus, the proof is completed.

Theorem 3.7 : Let $\varphi:(\mathrm{M}_1, g_1,\mathrm{J})\to(\mathrm{M}_2, g_2)$ be a radical screen transversal slant lightlike submersion from an indefinite Kaehler manifold M_1 onto a lightlike manifold M_2. Then, the screen distribution $\mathrm{S}(\mathrm{K}er\varphi_*)$ defines a totally geodesic foliation on $\mathrm{K}er\varphi_*$ if and only if $\mathrm{T}^s_\mathrm{U}\alpha\mathrm{P}_2\mathrm{V}+\nabla^{\perp s}_\mathrm{U}\beta\mathrm{P}_2\mathrm{V}$ has no component in $\mathrm{J}\,ltr(\mathrm{K}er\varphi_*)$ for any $\mathrm{U},\mathrm{V}\in\Gamma(\mathrm{S}(\mathrm{K}er\varphi_*))$ and $\mathrm{N}\in\Gamma(ltr(\mathrm{K}er\varphi_*))$.

Proof. Let $\mathrm{U},\mathrm{V}\in\Gamma(\mathrm{S}(\mathrm{K}er\varphi_*))$ and $\mathrm{N}\in\Gamma(ltr(\mathrm{K}er\varphi_*))$. Then, using (1) (2), (9), (11) and (17), we obtain

$$
\begin{aligned}
g_1(\hat{\nabla}_\mathrm{U}\mathrm{V},\mathrm{N}) &= g_1(\nabla_\mathrm{U}\mathrm{J}\,\mathrm{V},\mathrm{J}\,\mathrm{N})\\
&= g_1(\nabla_\mathrm{U}\alpha\mathrm{P}_2\mathrm{V}+\nabla_\mathrm{U}\beta\mathrm{P}_2\mathrm{V},\mathrm{J}\,\mathrm{N})\\
&= g_1(\mathrm{T}^s_\mathrm{U}\alpha\mathrm{P}_2\mathrm{V}+\nabla^{\perp s}_\mathrm{U}\beta\mathrm{P}_2\mathrm{V},\mathrm{J}\,\mathrm{N}),
\end{aligned}
$$

which completes the proof.

Conclusion

In this paper, we study radical screen transversal slant lightlike submersions from an indefinite Kaehler manifold onto a lightlike manifold, which is an umbrella of radical screen transversal and screen transversal anti-invariant lightlike submersions. Such submersions can also be studied in the future if the total manifold is indefinite Sasakian or indefinite Kenmotsu.

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