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SCREEN PSEUDO SLANT LIGHTLIKE SUBMANIFOLDS OF A METALLIC SEMI-RIEMANNIAN MANIFOLD

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Abstract: The geometry of screen pseudo slant lightlike submanifolds of a metallic semi-Riemannian manifold is introduced in this article. The integrability and fully geodesic foliations of $Rad(TM)$, $\dot{D}$ and $\widetilde{D}$ distributions are investigated.

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1. Introduction

Manifolds with certain differential-geometric structures have a lot of diverse geometric structures. In differential geometry, such manifolds and their relationships have been extensively investigated. Many writers have investigated virtually complex manifolds, almost contact manifolds, and almost product manifolds, as well as the relationships between these manifolds. Duggal and Bejancu [6] investigated the geometry of lightlike submanifolds of semi-Riemannian manifolds. A general notion of lightlike submanifolds of indefinite Sasakian manifolds and differential geometry of lightlike submanifolds have been introduced by Duggal and Sahin [7, 8].

One of the most researched subjects in differential geometry is Riemannian manifolds with metallic features. De Spinadel [5] developed the metallic means family in 2002 as a generalisation of the Golden mean, which includes the Silver mean, Bronze mean, Copper mean, Nickel mean, and so on. The family of metallic means is crucial in establishing a link between mathematics and architecture.

In religious art from Egypt, Turkey, India, China, and other ancient civilizations, for example, Golden mean and Silver mean can be seen in [4]. Goldberg and Petridis [9], as well as Goldberg and Yano [10] introduced polynomial structures on manifolds. Golden structure is defined by Crasmareanu and Hretcanu [3] as a special case of polynomial structure and some generalisations of Golden structure are referred to as metallic structure. Because it is an important instrument for investigating the geometry of submanifolds, metallic structure on the ambient Riemannian manifold yields crucial geometrical results on submanifolds. Invariant, anti-invariant, semi-invariant, slant, semi-slant and hemi-slant submanifolds of metallic Riemannian manifolds have been studied in [11, 12, 13]. Poyraz and Dogan [14] investigated the study of semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds.

Many researchers have recently focused on Golden Riemannian manifolds and their submanifolds. Poyraz and Yasar [15] were the first to investigate lightlike geometry in Golden semi-Riemannian manifolds. Acet [1, 2] studied lightlike hypersurfaces of a metallic semi-Riemannian manifold and the screen pseudo slant of Lightlike submanifolds of Golden semi-Riemannian manifolds. We discuss the generlization of a screen pseudo slant Golden semi-Riemannian manifold in this study. The paper is formatted as follows:

In section 2, we recall some definitions and properties of the Metallic manifold. In section 3, we give definition and result of screen pseudo-slant semi-Riemannian manifold. In last section, we establish a sufficient condition for integrability and totally geodesic foliations.

2. Preliminaries

If a submanifold $\mathcal{M}^m$ immersed in a semi-Riemannian manifold $(\bar{\mathcal{M}}^{m+n})$, admits a degenerate metric g induced by $\bar{g}$ on $\mathcal{M}$, it is called a lightlike submanifold [6]. When g degenerates on $\mathcal{M}$ tangent bundle $\mathcal{TM}$, $\mathcal{M}$ is referred to as a lightlike submanifold.

For a degenerate metric g on $\mathcal{M}$, $\mathcal{TM}^\perp$ is a degenerate n-dimensional subspace of $\mathcal{T}_x\bar{\mathcal{M}}$. Thus both $\mathcal{T}_x\mathcal{M}$ and $\mathcal{T}_x\mathcal{M}^\perp$ are degenerate orthogonal subspaces but not complementary to each other. Therefore, there exists a subspace $Rad(\mathcal{TM}) = \mathcal{T}_x\mathcal{M} \cap \mathcal{T}_x\mathcal{M}^\perp$, known as Radical subspace. If the mapping $Rad(\mathcal{TM}): \mathcal{M} \longrightarrow \mathcal{TM},$ such that $x \in \mathcal{M} \mapsto Rad(\mathcal{T}_x\mathcal{M})$, defines a smooth distribution of rank $r > 0$ on $\mathcal{M}$, then $\mathcal{M}$ is said to be an r-lightlike submanifold and the distribution $Rad(\mathcal{TM})$ is said to be radical distribution on $\mathcal{M}$. The non-degenerate complementary subbundles $S(\mathcal{TM})$ and $S(\mathcal{TM}^\perp)$ of $Rad(\mathcal{TM})$ are known as screen distribution in $\mathcal{TM}$ and screen transversal distribution in $\mathcal{TM}^\perp$ respectively, i.e.,

$$\mathcal{TM} = Rad(\mathcal{TM}) \perp S(\mathcal{TM}) \,\&\, \mathcal{TM}^\perp = Rad(\mathcal{TM}) \perp S(\mathcal{TM}^\perp). \tag{1}$$

Let $ltr(\mathcal{TM})$ (lightlike transversal bundle) and $tr(\mathcal{TM})$ (transversal bundle) be complementary but not orthogonal vector bundles to $Rad(\mathcal{TM})$ in $S(\mathcal{TM}^\perp)^\perp$ and $\mathcal{TM}$ in $\mathcal{T}\bar{\mathcal{M}}|_{\mathcal{M}}$ respectively.

Then, the transversal vector bundle is given by [7]

$$tr(\mathcal{TM}) = ltr(\mathcal{TM}) \perp S(\mathcal{TM}^\perp). \tag{2}$$

From (1) and (2), we get

$$\mathcal{T}\bar{\mathcal{M}}|_{\mathcal{M}} = \mathcal{TM} \oplus tr(\mathcal{TM}) = (Rad(\mathcal{TM}) \oplus ltr(\mathcal{TM})) \perp S(\mathcal{TM}) \perp S(\mathcal{TM}^\perp). \tag{3}$$

Theorem 2.1 *[6] Let $(\mathcal{M}, \mathrm{g}, S(\mathcal{TM}), S(\mathcal{TM}^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{\mathcal{M}}, \bar{\mathrm{g}})$. Then there exists a complementary vector bundle $ltr(\mathcal{TM})$ of $Rad(\mathcal{TM})$ in $S(\mathcal{TM}^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(\mathcal{TN}')|u)$ consisting of a smooth section $\{\mathcal{N}_i\}$ of $S(\mathcal{TM}^{\perp})^{\perp}|_u$, where u is a coordinate neighbourhood of $\mathcal{M}$ such that*

$$\bar{\mathrm{g}}_{ij}(\mathcal{N}_i, \xi_j) = \delta_{ij}, \bar{\mathrm{g}}_{ij}(\mathcal{N}_i, \mathcal{N}_j) = 0, \tag{4}$$

for any i, j $\in \{1,2,\ldots,r\}$.

A submanifold $(\mathcal{M}, \mathrm{g}, S(\mathcal{TM}), S(\mathcal{TM}^{\perp}))$ of $\bar{\mathcal{M}}$ is said to be

(i) r-lightlike if $r < min\{m, n\}$;

(ii) coisotropic if $r = n < m, S(\mathcal{TM}^{\perp}) = \{0\}$;

(iii) isotropic if $r = m = n, S(\mathcal{TM}) = \{0\}$;

(iv) totally lightlike if $r = m = n, S(\mathcal{TM}) = \{0\} = S(\mathcal{TM}^{\perp})$.

The linear connections on $\bar{\mathcal{M}}, \mathcal{M}$, and vector bundle $tr(\mathcal{TM})$ are denoted by $\bar{\nabla}, \nabla$ and ∇^t, respectively. The Gauss and Weingarten equations are then provided.

$$\bar{\nabla}_{\mathbf{U}}\mathbf{V} = \nabla_{\mathbf{U}}\mathbf{V} + h(\mathbf{U}, \mathbf{V}), \forall\ \mathbf{U}, \mathbf{V} \in \Gamma(\mathcal{TM}), \tag{5}$$

$$\bar{\nabla}_{\mathbf{U}}N = -A_N\mathbf{U} + \nabla^t_{\mathbf{U}}N, \forall\ \mathbf{U} \in \Gamma((\mathcal{TM})), N \in \Gamma(tr(\mathcal{TM})), \tag{6}$$

where $\{\nabla_{\mathbf{U}}\mathbf{V}, A_N\mathbf{U}\}$ and $\{h(\mathbf{U}, \mathbf{V}), \nabla^t_{\mathbf{U}}N\}$ belong to $\Gamma(\mathcal{TM})$ and $\Gamma(tr(\mathcal{TM}))$ respectively, the linear connections ∇ and ∇^t are on $\mathcal{M}$ and on the vector bundle $tr(\mathcal{TM})$ respectively.

From (5) and (6), for any $\mathbf{U}, \mathbf{V} \in \Gamma(tr(\mathcal{TM}))$, $N \in \Gamma(ltr(\mathcal{TM}))$ and $\mathbf{W} \in \Gamma(S(\mathcal{TM}^{\perp}))$, we have

$$\bar{\nabla}_{\mathbf{U}}\mathbf{V} = \nabla_{\mathbf{U}}\mathbf{V} + h^l(\mathbf{U}, \mathbf{V}) + h^s(\mathbf{U}, \mathbf{V}), \tag{7}$$

$$\bar{\nabla}_{\mathbf{U}}N = -A_N\mathbf{U} + \nabla^l_{\mathbf{U}}(N) + D^s(\mathbf{U}, N), \tag{8}$$

$$\bar{\nabla}_{\mathbf{U}}W = -A_W\mathbf{U} + \nabla^s_{\mathbf{U}}(\mathbf{W}) + D^l(\mathbf{U}, \mathbf{W}), \tag{9}$$

where $D^l(\mathbf{U}, \mathbf{W}), D^s(\mathbf{U}, N)$ are the projections of ∇^t on $\Gamma(ltr(\mathcal{TM}))$ and $\Gamma(S(\mathcal{TM}^{\perp}))$ respectively, ∇^l, ∇^s are linear connections on $\Gamma(ltr(\mathcal{TM}))$ and $\Gamma(S(\mathcal{TM}^{\perp}))$, respectively and the shape operators $A_N, A_{\mathbf{W}}$ on $\mathcal{M}$ with respect to $\mathcal{N}$ and $\mathcal{W}$, respectively.

Using (5) and (7) -(9) , we obtain

$$\bar{\mathrm{g}}(h^s(\mathbf{U}, \mathbf{V}), \mathbf{W}) + \bar{\mathrm{g}}(\mathbf{V}, D^l(\mathbf{U}, \mathbf{W})) = \mathrm{g}(A_{\mathbf{W}}\mathbf{U}, \mathbf{V}), \tag{10}$$

$$\bar{\mathrm{g}}(D^s(\mathbf{U}, N), \mathbf{W}) = \bar{\mathrm{g}}(N, A_{\mathbf{W}}\mathbf{U}). \tag{11}$$

for $\mathbf{U}, \mathbf{V} \in \Gamma(\mathcal{TM})$, $\mathbf{W} \in \Gamma(S(\mathcal{TM}^{\perp})$ and $N \in \Gamma(ltr(\mathcal{TM}))$.

Let J denote the projection of $\mathcal{TM}$ on $S(\mathcal{TM})$ and let ∇^*, ∇^{*t} stand for linear connections on $S(\mathcal{TM}\nabla$ and $Rad(\mathcal{TM})$, respectively. After that, we have the decomposition of the tangent bundle of lightlike submanifold.

$$\nabla_{\mathbf{U}} J\mathbf{V} = \nabla^*_{\mathbf{U}} J\mathbf{V} + h^*(\mathbf{U}, J\mathbf{V}), \tag{12}$$

$$\nabla_{\mathbf{U}} E = -A^*_E \mathbf{U} + \nabla^{*t}_{\mathbf{U}}(E), \tag{13}$$

for $\mathbf{U}, \mathbf{V} \in \Gamma(\mathcal{TM})$ and $E \in \Gamma(Rad\mathcal{TM})$, where $\{\nabla^*_{\mathbf{U}} J\mathbf{V}, A^*_E\mathbf{U}\}$ and $\{h^*(\mathbf{U}, J\mathbf{V}), \nabla^{*t}_{\mathbf{U}}(E)\}$ belong to $\Gamma(S(\mathcal{TM}))$ and $\Gamma Rad(\mathcal{TM})$, respectively.

Using above equations, we getting

$$\bar{g}(h^l(\mathbf{U}, J\mathbf{V}), E) = g(A^*_E \mathbf{U}, J\mathbf{V}), \tag{14}$$

$$\bar{g}(h^*(\mathbf{U}, J\mathbf{V}), N) = g(A_N \mathbf{U}, J\mathbf{V}), \tag{15}$$

$$\bar{g}(h^l(\mathbf{U}, E), E) = 0, A^*_E E = 0. \tag{16}$$

A metallic structure J on a differentiable manifold $\bar{\mathcal{M}}$ is defined as

$$J^2 = Jp + qI, \tag{17}$$

where, positive integers are p, q and the identity map is I on $\bar{\mathcal{M}}$. Also

$$\bar{g}(J\mathbf{U}, \mathbf{V}) = \bar{g}(\mathbf{U}, J\mathbf{V}), \tag{18}$$

then $(\bar{\mathcal{M}}, \bar{g}, J)$ is called metallic semi-Riemannian manifold. Also, we have

$$\mathbf{U}' J\mathbf{V} = J\mathbf{U}'\mathbf{V}, \tag{19}$$

if J is a metallic structure, then equation (18) can be written as

$$\bar{g}(J\mathbf{U}, J\mathbf{V}) = p\bar{g}(J\mathbf{U}, \mathbf{V}) + q\bar{g}(\mathbf{U}, \mathbf{V}), \tag{20}$$

for any $\mathbf{U}, \mathbf{V} \in \Gamma(\mathcal{TM})$.

3. Screen Pseudo Slant Lightlike Submanifolds of Metallic semi-Riemannian Manifold

Definition 3.1 *Let $\mathcal{M}$ be a lightlike submanifold of a metallic semi-Riemannian manifold $\bar{\mathcal{M}}$. $\mathcal{M}$ is called a screen pseudo-slant submanifold of a Metallic semi-Riemannian manifold $\bar{\mathcal{M}}$ if the following conditions are satisfied:*

- The $Rad\mathcal{TM}$ is an invariant distributions with respect to J, i.e.,

$$J(Rad\mathcal{TM}) = Rad\mathcal{TM}.$$

- The non-degenerate orthogonal distributions $\dot{D}$ and $\widetilde{D}$ on $\mathcal{M}$ are exist, such that

$$S(\mathcal{TM}) = \dot{D} \perp \widetilde{D}.$$

- The $\dot{D}$ distribution is anti-invariant, i.e.,

$$J(\dot{D}) \subset S(\mathcal{TM}^{\perp}).$$

- The distribution $\widetilde{D}$ is slant with angle $(\theta \neq \frac{\phi}{2})$, i.e., for each $x \in \mathcal{M}$ and each non-zero vector $X \in (\widetilde{D})_x$, the angle θ between JX and the vector subspace $(\widetilde{D})_x$ is a constant $(\neq \frac{\phi}{2})$, which is independent of the choice of $x \in \mathcal{M}$ and $X \in (\widetilde{D})_x$.

This constant angle θ is called the slant angle of distribution $\widetilde{D}$. A screen pseudo slant lightlike submanifold is said to be proper if $\dot{D} \neq \{0\}, \widetilde{D} \neq \{0\}$ and $\theta \neq 0$.

From above definition, we can write

$$\mathcal{TM} = Rad\mathcal{TM} \perp \dot{D} \perp \widetilde{D}. \tag{21}$$

Example 3.1 *Let* $(\mathbb{R}_5^{10}, \tilde{g})$ *be a semi-Riemannian manifold with signature* $(-,\dots,-,+,+,\dots,+)$ *and* $(y_1, y_2, \dots, y_{10})$ *b e cooridinate system of* $\mathbb{R}_5^{10}$.

Suppose that

$$J(y_1, y_2, \dots, y_{10}) = (\sigma y_1, (p-\sigma)y_2, (p-\sigma)y_3, (p-\sigma)y_4, (p-\sigma)y_5, (p-\sigma)y_6, (p-\sigma)y_7, (p-\sigma)y_8, (p-\sigma)y_9, (p-\sigma)y_{10})$$

then J is metallic structure on $\mathbb{R}^{10}$. Assume that M is a submanifold of $\mathbb{R}^{10}$given by

$$y_1 = y_7 = \omega^1$$

$$y_2 = \sigma cos\alpha\omega^5 + cos\alpha\omega^6$$

$$y_3 = \sigma\omega^2$$

$$y_4 = \sigma\omega^3$$

$$y_5 = \sigma\omega^4$$

$$y_6 = \sigma sin\alpha\omega^5 + sin\alpha\omega^6$$

$$y_7 = 0$$

$$y_8 = (p-\sigma)\omega^2$$

$$y_9 = (p-\sigma)\omega^3$$

$$y_{10} = (p-\sigma)\omega^4$$

Then, $\mathcal{TM} = Sp\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}\}$,

where

$$Z_1 = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_7}$$

$$Z_2 = \sigma\frac{\partial}{\partial y_3} + (p-\sigma)\frac{\partial}{\partial y_8}$$

$$Z_3 = \sigma\frac{\partial}{\partial y_4} + (p-\sigma)\frac{\partial}{\partial y_9}$$

$$Z_4 = \sigma\frac{\partial}{\partial y_5} + (p-\sigma)\frac{\partial}{\partial y_{10}}$$

$$Z_5 = \sigma cos\alpha \frac{\partial}{\partial y_2} + \sigma sin\alpha \frac{\partial}{\partial y_6}$$

$$Z_6 = cos\alpha \frac{\partial}{\partial y_2} + sin\alpha \frac{\partial}{\partial y_6}$$

Thus $Rad\mathcal{TM} = Sp\{Z_1\}, S(\mathcal{TM}) = Sp\{Z_2, Z_3, Z_4, Z_5, Z_6\}$ and $ltr(\mathcal{TM})$ is spanned by

$$\mathcal{N} = -\frac{1}{2}(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_7})$$

Now, $S(\mathcal{TM}^{\perp})$ is spanned by

$$W_1 = -\frac{\partial}{\partial y_3} + (p-\sigma)^2 \frac{\partial}{\partial y_8}$$

$$W_2 = -\frac{\partial}{\partial y_4} + (p-\sigma)^2 \frac{\partial}{\partial y_9}$$

$$W_3 = -\frac{\partial}{\partial y_5} + (p-\sigma)^2 \frac{\partial}{\partial y_{10}}.$$

It follows that

$$JZ_1 = \sigma Z_1$$

which implies that $Rad\mathcal{TM}$ is invariant. Also, we can state that

$$\dot{D} = \{Z_2, Z_3, Z_4\}$$

such that

$$JZ_2 = \omega_1, JZ_3 = \omega_2, JZ_4 = \omega_3$$

which gives that $\dot{D}$ is anti -invariant and $\widetilde{D} = \{Z_5, Z_6\}$ is slant distribution with slant angle 2α. Therefore, $\mathcal{M}$ is a screen pseudo-slant lightlike submanifold of $\mathbb{R}_5^{10}$.

For any vector field $\mathbf{U} \in \Gamma(\mathcal{TM})$ tangent to $\mathcal{M}$, we take

$$J\mathbf{U} = R\mathbf{U} + T\mathbf{U}, \tag{22}$$

where $R\mathbf{U}$ and $T\mathbf{U}$ are the tangential and transversal part of $J\mathbf{U}$, respectively. We denote the projections on $Rad\mathcal{TM}$, $\dot{D}$ and $\widetilde{D}$ in $\mathcal{TM}$ by R_1, R_2 and R_3, respectively.

Similarly, we show that the projections of $tr(\mathcal{TM})$ on $ltr(\mathcal{TM})$, $J(\dot{D})$ and $\widetilde{D}$ by Q_1, Q_2 and Q_3, respectively, where $\widetilde{D}$ is a non-degenerate orthogonal complementary subbundle of $J(\widetilde{D})$ in $S(\mathcal{TM}^{\perp})$. So, for any $\mathbf{U} \in \Gamma(\mathcal{TM})$, we get

$$\mathbf{U} = R_1\mathbf{U} + R_2\mathbf{U} + R_3\mathbf{U}. \tag{23}$$

Applying J to (23), we have

$$J\mathbf{U} = JR_1\mathbf{U} + JR_2\mathbf{U} + JR_3\mathbf{U},$$

which yields

$$J\mathbf{U} = JR_1\mathbf{U} + JR_2\mathbf{U} + WR_3\mathbf{U} + TR_3\mathbf{U}, \tag{24}$$

where $\mathbf{W}R_3\mathbf{U}$ and $TR_3\mathbf{U}$ denote the tangential and the transversal components of $JR_3\mathbf{U}$. So, we arrive at $JR_1\mathbf{U} \in \Gamma(Rad\mathcal{TM}), JR_2\mathbf{U} \in \Gamma\left(J(\dot{D})\right) \subset S(\mathcal{TM}^{\perp})$, $\mathbf{W}R_3U \in \Gamma(\widetilde{D})$, $TR_3\mathbf{U} \in (\widetilde{D})$.

Also, for any $w \in \Gamma(tr(\mathcal{TM}))$, we get

$$\mathbf{W} = f_1\mathbf{W} + f_2\mathbf{W} + f_3\mathbf{W}. \tag{25}$$

Applying J to (25), we get

$$J\mathbf{W} = Jf_1\mathbf{W} + Jf_2\mathbf{W} + Jf_3\mathbf{W},$$

which yields

$$J\mathbf{W} = Jf_1\mathbf{W} + Jf_2\mathbf{W} + Bf_3\mathbf{W} + Cf_3\mathbf{W}, \tag{26}$$

where $Bf_3\mathbf{W}$ and $Cf_3\mathbf{W}$ denote the tangential and the transversal components of $Jf_3\mathbf{W}$.

Thus, we get

$$Jf_1\mathbf{W} \in \Gamma(ltr(\mathcal{TM})), \qquad Jf_2\mathbf{W} \in \Gamma(\widetilde{D}), \qquad Bf_2\mathbf{W} \in \Gamma(\widetilde{D}), \qquad Cf_3\mathbf{W} \in \Gamma(\dot{D}).$$

Now, using (24) and (26) with (7)-(9), we obtain the following;

$$\nabla^{*t}_{\mathbf{U}}JR_1\mathbf{V} + R_1(\nabla_{\mathbf{U}}\mathbf{W}R_3\mathbf{V}) = R_1(A_{TR_3\mathbf{V}}\mathbf{U}) + R_1(A_{JR_2\mathbf{V}}\mathbf{U}) + JR_1 \square_{\mathbf{U}} \mathbf{V}, \tag{27}$$

$$R_2(A^*_{JR_1\mathbf{V}}\mathbf{U}) + R_2(A^*_{JR_2\mathbf{V}}\mathbf{U}) + R_2(A^*_{TR_3\mathbf{V}}\mathbf{U}) = R_2(\nabla_{\mathbf{U}}\mathbf{W}R_3\mathbf{V}) - Jf_1h^s(\mathbf{U},\mathbf{V}), \tag{28}$$

$$R_3(A^*_{JR_1\mathbf{V}}\mathbf{U}) + R_3(A^*_{JR_2\mathbf{V}}\mathbf{U}) + R_3(A^*_{TR_3\mathbf{V}}\mathbf{U}) = R_3(\nabla_{\mathbf{U}}\mathbf{W}R_3\mathbf{V}) - \mathbf{W}R_3(\nabla_{\mathbf{U}}\mathbf{V}) - Bf_3h^s(\mathbf{U},\mathbf{V}), \tag{29}$$

$$h^l(\mathbf{U}, JR_1\mathbf{V}) + D^l(\mathbf{U}, JR_2\mathbf{V}) + h^l(\mathbf{U}, \mathbf{W}R_3\mathbf{V}) + D^l(\mathbf{U}, TR_3\mathbf{V}) = Jh^l(\mathbf{U},\mathbf{V}), \tag{30}$$

$$f_2\nabla^s_{\mathbf{U}}JR_2\mathbf{V} + f_2\nabla^s_{\mathbf{U}}TR_3\mathbf{V} = JR_2\nabla_{\mathbf{U}}\mathbf{V} - f_2h^s(\mathbf{U}, JR_1\mathbf{V}) - f_2h^s(\mathbf{U}, \mathbf{W}R_3\mathbf{V}), \tag{31}$$

$$f_3\nabla^s_{\mathbf{U}}JR_2\mathbf{V} + f_3\nabla^s_{\mathbf{U}}TR_3\mathbf{V} - TR_3\nabla_{\mathbf{U}}\mathbf{V} = Cf_3h^s(\mathbf{U},\mathbf{V}) - f_3h^s(\mathbf{U}, \mathbf{W}R_3\mathbf{V}) - f_3h^s(\mathbf{U}, JR_1\mathbf{V}). \tag{32}$$

4. Main Theorems

Theorem 4.1 *Let $\mathcal{M}$ be a screen pseudo-slant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{\mathcal{M}}$. Then $Rad\mathcal{TM}$ is integrable if and only if*

- $f_2h^s(E_2, JR_1E_1) = f_2h^s(E_1, JR_1E_2)$,
- $f_3h^s(E_2, JR_1E_1) = f_3h^s(E_1, JR_1E_2)$,
- $R_3A^*_{JR_1E_1}E_2 = R_3A^*_{JR_1E_2}E_1$.

$\forall\ E_1, E_2 \in \Gamma(Rad\mathcal{TM})$.

Proof. Assume that $\mathcal{M}$ is a screen pseudo-slant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{\mathcal{M}}$ and $E_1, E_2 \in \Gamma(Rad\mathcal{T}\mathcal{M})$.

In view of (31), we get

$$JR_2\nabla_{E_1}E_2 = f_2h^s(E_1, JR_1E_2). \tag{33}$$

Interchanging E_1 and E_2, we obtain

$$JR_2\nabla_{E_2}E_1 = f_2h^s(E_2, JR_1E_1). \tag{34}$$

From (33) and (34), we have

$$JR_2[E_1, E_2] = f_2(h^s(E_1, JR_1E_2)) - h^s(E_2, JR_1E_1).$$

Similarly, by use of (32), we get

$$f_3h^s(E_1, JR_1E_2) = Cf_3h^s(E_1E_2) + TR_3\nabla_{E_1}E_2. \tag{35}$$

Now, we obtain

$$f_3h^s(E_2, JR_1E_1) = Cf_3h^s(E_2, E_1) + TR_3\nabla_{E_2}E_1. \tag{36}$$

From (35) and (36), we have

$$TR_2[E_1, E_2] = f_3(h^s(E_1, JR_1E_2) - h^s(E_2, JR_1E_1)).$$

Furthermore, from (29), we obtain

$$R_3(A^*_{JR_1E_1}E_2) = -\mathbf{W}R_3\nabla_{E_1}E_2 - Bf_3h^s(E_1, E_2), \tag{37}$$

which implies

$$R_3(A^*_{JR_1E_2}E_1) = -\mathbf{W}R_3\nabla_{E_2}E_1 - Bf_3h^s(E_2, E_1). \tag{38}$$

From (37) and (38), we obtain

$$\mathbf{W}R_3[E_1, E_2] = R_3(A^*_{JR_1E_1}E_2) - R_3(A^*_{JR_1E_2}E_1).$$

Therefore, we arrive at the required equations.

Theorem 4.2 *Let $\mathcal{M}$ be a screen pseudo-slant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{\mathcal{M}}$. Then $\dot{D}$ is integrable if and only if*

- $R_1(A_{JR_2\mathbf{U}}\mathbf{V}) = R_1(A_{JR_2\mathbf{V}}\mathbf{U})$ and $R_3(A_{JR_2\mathbf{U}}\mathbf{V}) = R_3(A_{JR_2\mathbf{V}}\mathbf{U})$,
- $f_3(\nabla^s_{\mathbf{V}}JR_2\mathbf{U}) = f_3(\nabla^s_{\mathbf{U}}JR_2\mathbf{V})$

$\forall\ \mathbf{U}, \mathbf{V} \in \Gamma(\dot{D})$.

Proof. Assume that $\mathcal{M}$ is a screen pseudo-slant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{\mathcal{M}}$ and $\mathbf{U}, \mathbf{V} \in \Gamma(\dot{D})$.

By use of (27), we get

$$R_1(A_{JR_2\mathbf{U}}\mathbf{V}) = -JR_1\nabla_{\mathbf{U}}\mathbf{V}. \tag{39}$$

Interchanging $\mathbf{U}$ and $\mathbf{V}$, we have

$$R_1(A_{JR_2\mathbf{V}}\mathbf{U}) = -JR_1\nabla_{\mathbf{V}}\mathbf{U}. \tag{40}$$

From (39) and (40), we get

$$R_1(A_{JR_2\mathbf{U}}\mathbf{V}) - R_1(A_{JR_2\mathbf{V}}\mathbf{U}) = JR_1[\mathbf{U},\mathbf{V}].$$

In view of (29), we get

$$R_3(A_{JR_2\mathbf{U}}\mathbf{V}) + Bf_3h^s(\mathbf{U},\mathbf{V}) = -WR_3(\nabla_{\mathbf{U}}\mathbf{V}), \tag{41}$$

which implies

$$R_3(A_{JR_2\mathbf{U}}\mathbf{V}) - R_3(A_{JR_2\mathbf{V}}\mathbf{U}) = WR_2[\mathbf{U},\mathbf{V}].$$

Moreover, using (32), we obtain

$$f_3\nabla^s_{\mathbf{U}}JR_2\mathbf{V} + Cf_3h^s(\mathbf{U},\mathbf{V}) = TR_3\nabla_{\mathbf{U}}\mathbf{V},$$

from which, we arrive at

$$f_3\nabla^s_{\mathbf{U}}JR_2\mathbf{V} - f_3\nabla^s_{\mathbf{V}}JR_2U = TR_3[\mathbf{U},\mathbf{V}].$$

Thus, we obtain the desired results.

Theorem 4.3 *Let $\mathcal{M}$ be a screen pseudo-slant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{\mathcal{M}}$. Then $\widetilde{D}$ is integrable if and only if*

- $R_1(\nabla_{\mathbf{U}}WR_3\mathbf{V} - \nabla_{\mathbf{V}}WR_3\mathbf{U}) = R_1(A_{TR_3\mathbf{V}}\mathbf{U} - A_{TR_3\mathbf{U}}\mathbf{V})$,
- $f_2(\nabla^s_{\mathbf{U}}TR_3\mathbf{V} - \nabla^s_{\mathbf{V}}TR_3\mathbf{U}) = f_2(h^s(\mathbf{V},WR_3\mathbf{U}) - h^s(\mathbf{U},WR_3\mathbf{V}))$

$\forall\ \mathbf{U},\mathbf{V} \in \Gamma(\widetilde{D})$.

Proof. Assume that $\mathcal{M}$ is a screen pseudo-slant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{\mathcal{M}}$ and $\mathbf{U},\mathbf{V} \in \Gamma\widetilde{D}$.

If we consider (27), we get

$$R_1(\nabla_{\mathbf{U}}WR_3\mathbf{V}) = R_1(A_{TR_3\mathbf{V}}\mathbf{U}) + JR_1\nabla_{\mathbf{U}}\mathbf{V}. \tag{42}$$

Interchanging $\mathbf{U}$ and V, we have

$$R_1(\nabla_{\mathbf{V}}WR_3\mathbf{U}) = R_1(A_{TR_3\mathbf{U}}\mathbf{V}) + JR_1\nabla_{\mathbf{V}}\mathbf{U}. \tag{43}$$

In view of (42) with (43), we find

$$R_1(\nabla_{\mathbf{U}}WR_3\mathbf{V} - \nabla_{\mathbf{V}}WR_3\mathbf{U}) - R_1(A_{TR_3\mathbf{V}}\mathbf{U} - A_{TR_3\mathbf{U}}\mathbf{V}) = JR_1[\mathbf{U},\mathbf{V}].$$

Also, using (31), we have

$$f_2\nabla^s_{\mathbf{U}}TR_3\mathbf{V} + f_2h^s(\mathbf{U},WR_3\mathbf{V}) = JR_2\nabla_{\mathbf{U}}\mathbf{V}, \tag{44}$$

and

$$f_2\nabla^s_{\mathbf{V}}TR_3\mathbf{U} + f_2h^s(\mathbf{V}, WR_3\mathbf{U}) = JR_2\nabla_{\mathbf{V}}\mathbf{U}. \quad (45)$$

From last two equations, we arrive at

$$f_2(\nabla^s_{\mathbf{U}}TR_3\mathbf{V} - \nabla^s_{\mathbf{U}}TR_3\mathbf{U}) + f_2(h^s(\mathbf{U}, WR_3\mathbf{V}) - h^s(\mathbf{V}, WR_3\mathbf{U})) = JR_2(\mathbf{U}, \mathbf{V}).$$

So, we obtain the required results.

Now, we discover that some foliations specified by distributions are completely geodesic.

Theorem 4.4 *Let $\mathcal{M}$ be a screen pseudo-slant lightlike submanifold of a metallic semi-Riemannian manifold $\overline{\mathcal{M}}$. Then $Rad\mathcal{TM}$ defines a totally geodesic foliation if and only if*

$$\frac{1}{q}(\bar{g}(D^l(E_1, R_2Z) + D^l(E_1, TR_3Z), JE_2)) = -\frac{1}{q}\bar{g}(h^l(E_1, WR_3Z), JE_2), \forall E_1, E_2 \in \Gamma(Rad\mathcal{TM}) \text{ and } Z \in \Gamma(S(\mathcal{TM})).$$

Proof. We know that $Rad\mathcal{TM}$ defines a totally geodesic foliation if and only if

$\nabla_{E_1}E_2 \in \Gamma(Rad\mathcal{TM}) \forall E_1, E_2 \in \Gamma(Rad\mathcal{TM})$.

Because of $\overline{\nabla}$ is a metric connection, by use of (7), (20) and (24), we have

$$\begin{aligned}
q\bar{g}(\nabla_{E_1}E_2, E) = {} & (\bar{g}(J\overline{\nabla}_{E_1}E_2, JZ) - p\bar{g}(J\overline{\nabla}_{E_1}E_2, Z)) \\
= {} & \bar{g}(\overline{\nabla}_{E_1}JR_2Z, JE_2) + \bar{g}(\overline{\nabla}_{E_1}WR_3Z, JE_2) + \bar{g}(\overline{\nabla}_{E_1}TR_3Z, JE_2) \\
- {} & p\bar{g}(\overline{\nabla}_{E_1}JR_2Z, E_2) - p\bar{g}(\overline{\nabla}_{E_1}WR_3Z, E_2) - p\bar{g}(\overline{\nabla}_{E_1}TR_3Z, E_2) \\
= {} & \bar{g}(D^l(E_1, JR_2Z), JE_2) + \bar{g}(h^l(E_1, WR_3Z), JE_2) + \bar{g}(D^l(E_1, TR_3Z), JE_2) \\
- {} & \bar{g}(D^l(E_1, JR_2Z), E_2) - \bar{g}(h^l(E_1, WR_3Z), E_2) - \bar{g}(D^l(E_1, TR_3Z), E_2) \\
= {} & \frac{1}{q}(\bar{g}(D^l(E_1, JR_2Z), JE_2) + \bar{g}(h^l(E_1, WR_3Z), JE_2) + \bar{g}(D^l(E_1, TR_3Z), JE_2)).
\end{aligned}$$

Hence proved.

Theorem 4.5 *Let $\mathcal{M}$ be a screen pseudo-slant lightlike submanifold of a metallic semi-Riemannian manifold $\overline{\mathcal{M}}$. Then $\dot{D}$ defines a totally geodesic foliation if and only if*

- $\bar{g}(h^s(\mathbf{U}, WZ), P\mathbf{V}) = -\frac{p}{q}(\nabla^s_{\mathbf{U}}TZ, P\mathbf{V}), \ \forall h^s(\mathbf{U}, Z) \in \Gamma(\dot{D})$.
- $D^s(\mathbf{U}, JN)$ has no component in $J(\dot{D})$ and $D^s(\mathbf{U}, N), \forall \mathbf{U}, \mathbf{V} \in \Gamma(\dot{D})$, $Z \in \Gamma(\widetilde{D})$ and $N \in \Gamma(ltr(\mathcal{TM}))$.

Proof. Let $\mathcal{M}$ be a screen pseudo-slant lightlike submanifold of a metallic semi-Riemannian manifold $\overline{\mathcal{M}}$. We know that $\dot{D}$ defines a totally geodesic foliation if and only if $\nabla_U\mathbf{V} \in \Gamma(\dot{D}), \mathbf{U}, \mathbf{V} \in \Gamma(\dot{D})$.

Using (7) with (20), we have

$$\begin{aligned}
\bar{g}(\nabla_{\mathbf{U}}\mathbf{V}, Z) = {} & -p(\bar{g}(J\nabla_{\mathbf{U}}Z, J\mathbf{V})) + q\bar{g}(\nabla_{\mathbf{U}}Z, J\mathbf{V}), \\
= {} & -p\bar{g}(\overline{\nabla}_{\mathbf{U}}JZ, J\mathbf{V}) + p\bar{g}(\nabla^s_{\mathbf{U}}J\mathbf{V}) + q\bar{g}(h^s(\mathbf{U}, Z), J\mathbf{V}),
\end{aligned}$$

which gives (i).

Also, we use (7) and (20), we get

$$q\bar{g}(\nabla_{\mathbf{U}}\mathbf{V},Z) = \bar{g}(\overline{\nabla}_{\mathbf{U}}JN,J\mathbf{V}) + p\bar{g}(\overline{\nabla}_{\mathbf{U}}N,JN),$$
$$= -\bar{g}(D^s(\mathbf{U},JN),J\mathbf{V}) + p\bar{g}(D^s(\mathbf{U},N)J\mathbf{V}),$$

which implies (ii).

Theorem 4.6 *Let $\mathcal{M}$ be a metallic semi-Riemannian manifold $\bar{\mathcal{M}}$ with a screen pseudo-slant lightlike submanifold. Then $\widetilde{D}$ defines a totally geodesic foliation if and only if*

- $A_{JZ}\mathbf{U}$ has no component in $\widetilde{D}$ and $\nabla^s_{\mathbf{U}}JZ \in \Gamma(\dot{D})$.
- $A_{JN}\mathbf{U}$ has no component in $J(\widetilde{D})$ and $D^s(\mathbf{U},JN) \in \Gamma(\dot{D}), \forall \mathbf{U},\mathbf{V} \in \Gamma(\widetilde{D})$ and $Z \in \Gamma(\dot{D})$ and $N \in \Gamma(ltr(\mathcal{TM}))$.

Proof. Let $\mathcal{M}$ be a screen pseudo-slant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{\mathcal{M}}$. We know that $\widetilde{D}$ defines a totally geodesic foliation if and only if $\nabla_{\mathbf{U}}\mathbf{V} \in \Gamma(\widetilde{D}), \mathbf{U}, V \in \Gamma(\widetilde{D})$.

In view of (7) with (20), we have

$$q\bar{g}(\nabla_{\mathbf{U}}\mathbf{V},Z) = -(\bar{g}(\overline{\nabla}_{\mathbf{U}}JZ,J\mathbf{V})) + p\bar{g}(\overline{\nabla}_{\mathbf{U}}Z,J\mathbf{V}),$$
$$= \bar{g}(A_{JZ}\mathbf{U},WV) + \bar{g}(\nabla^s_{\mathbf{U}}JZ,T\mathbf{V}) + p\bar{g}(A_{JZ}\mathbf{U},J\mathbf{V}),$$

which yields (i).

From (7) and (20), we get

$$q\bar{g}(\nabla_{\mathbf{U}}\mathbf{V},Z) = -\bar{g}(\overline{\nabla}_{\mathbf{U}}JN,J\mathbf{V}) + p\bar{g}(\overline{\nabla}_{\mathbf{U}}N,JN),$$
$$= -\bar{g}(A_{JN}\mathbf{U},W\mathbf{V}) + \bar{g}(D^s(\mathbf{U},JN)T\mathbf{V}) - p\bar{g}(A_{JN}\mathbf{U},\mathbf{V}),$$

which implies (ii).

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