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FRACTIONAL CALCULUS OF THE GENERALIZED MITTAG-LEFFLER (*p*,*s*,*k*)-FUNCTION

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Abstract: The object of this paper is to establish fractional integral and differential formulas involving the generalized Mittag-Leffler (*p*,*s*,*k*)-function. The considered fractional integral and differential operators contain the Appell's function $F_3(.)$ as a kernel and are introduced by Saigo-Maeda[15]. We obtain the images of the generalized Mittag-Leffler (*p*,*s*,*k*)-function in terms of the Wright hypergeometric function.[18] We also consider some new and known results from the derived results.

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1. Introduction and Preliminaries

In 1903, the Swedish mathematician Mittag-Leffler [12] introduced the function $E_\alpha(z)$ defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, (\alpha \in \mathbb{C}; \Re(\alpha) > 0) \quad (1)$$

A generalization of $E_\alpha(z)$ was studied by Wiman [17] and known as generalized Mittag-Leffler function or Wiman's function in the following form

$$E_{\propto,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, (\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0) \quad (2)$$

In 1971, Prabhakar [13] introduced the Mittag-Leffler function $E_{\propto,\beta}^{\gamma}(z)$ in the form

$$E_{\propto,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \quad (3)$$

where $\alpha, \beta, \gamma \in \mathbb{C}$; $\Re(\alpha) > 0, \Re(\beta) > 0$ and $(\gamma)_n$ the Pochhammer symbol given by

$$(\gamma)_n = \gamma(\gamma+1)(\gamma+2) \dots (\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$$

Another generalization of the Mittag-Leffler function called *k*-Mittag-Leffler function has been introduced by Dorrego and Cerutti [6] and defined as

$$E_{k,\propto,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\propto n+\beta)} \frac{z^n}{n!} \tag{4}$$

where $k > 0$; $\alpha, \beta, \gamma, z \in \mathbb{C}$; $\Re(\alpha) > 0, \Re(\beta) > 0$ and $(\gamma)_{n,k}$ the Pochhammer *k*-symbol given by Diaz and Pariguan [5] as

$$(\gamma)_{n,k} = \gamma(\gamma+k)(\gamma+2k)\dots(\gamma+(n-1)k) = \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)}$$

and Γ_k the *k*-Gamma function given by

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1}\, e^{-\frac{t^k}{k}}\, dt, \ (\Re(z) > 0).$$

Cerutti et al. [4] introduced the Mittag-Leffler (*p*-*k*)-function $p\boldsymbol{E}_{k,\alpha,\beta}^{\gamma}(z)$ as

$$pE_{k,\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{p(\gamma)_{n.k}}{p^{\Gamma_k(\alpha n+\beta)}} \frac{z^n}{n!} \tag{5}$$

where $\alpha, \beta, \gamma, z \in \mathbb{C}$; $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0; p, k \in \mathbb{R}^+\setminus\{0\}$ and $p^{(\gamma)_{n,k}}$ the Pochhammer (*p*-*k*)- sympol defined by Gehlot [8] as

$$p^{(\gamma)_{n,k}} = \left(\frac{\gamma p}{k}\right)\left(\frac{\gamma p}{k}+p\right)\left(\frac{\gamma p}{k}+2p\right)\dots\dots\left(\frac{\gamma p}{k}+(n-1)p\right) = \frac{p^{\Gamma_k}(\gamma+nk)}{p^{\Gamma_k}(\gamma)}$$

and $p^{\Gamma_k(z)}$ the (*p*- *k*)-Gamma function defined as

$$p^{\Gamma_k}(z) = \int_0^{\infty} t^{z-1}\, e^{-\frac{t^k}{p}}\, dt, (z \in \mathbb{C}\setminus k\mathbb{Z}^-; p, k \in \mathbb{R}^+\setminus\{0\})$$

A further generalization of (5) was studied by Ayub et al. [2] and known as the Mittag-Leffler (*p*,*s*,*k*)-function in the following form

$$pE_{k,\alpha,\beta}^{\gamma,s}(z) = \sum_{n=0}^{\infty} \frac{p^{(\gamma)_{n,k,s}}}{p^{\Gamma_{s,k}(\alpha n+\beta)}} \frac{z^n}{n!} \tag{6}$$

where $k, p \in \mathbb{R}, \alpha, \beta, \gamma \in \mathbb{C}, \Re e(\alpha) > 0, \Re e(\beta) > 0, \Re e(\gamma) > 0$ and $p^{(\gamma)_{n,k,s}}$ is the Pochhammer (*p*,*s*,*k*)-symbol defined by Gehlot and Nantomah [9] (see also [2]) as

$$p^{(\gamma)_{n,k,s}} = \left[\frac{\gamma p}{k}\right]_s \left[\frac{\gamma p}{k}+p\right]_s \dots \left[\frac{\gamma p}{k}+(n-1)p\right]_s = \prod_{i=0}^{n-1}\left[\frac{\gamma p}{k}+ip\right]_s$$

where $[\gamma]_s = \frac{1-s^{\gamma}}{1-s}, \ \forall\ \gamma \in \mathbb{R}, 0 < s < 1$

and $p^{\Gamma_{s,k}}$ gamma (*p*,*s*,*k*)-function defined as

$$p^{\Gamma_{s,k}(\xi)} = \frac{s}{k}\lim_{n\to\infty} \frac{n!\, p^{n+1}(snp)^{\frac{\xi}{k}-1}}{p^{(\xi)_{n,k}}}$$

By putting specific values of the parameters in (6), we get (1) - (5).

Proposition 1.1 [[2], p.3, eq. (35)]: The relation between three parameters, two parameters and the classical Pochhamer's symbol is given by

$$p^{(\gamma)_{n,k,s}} = s^n \, p^{(\gamma)_{n,k}} = \left(\frac{sp}{k}\right)^n (\gamma)_{n,k} = (sp)^n \left(\frac{\gamma}{k}\right)_n \tag{7}$$

Proposition 1.2 [[2], p.3, eq. (36)]: The relation between gamma (p,s,k)-function, gamma (p,k)-function, gamma k-function and classical gamma function is given by

$$p^{\Gamma_{s,k}(\xi)} = (s)^{\xi/k} \, p^{\Gamma_k(\xi)} = \left(\frac{sp}{k}\right)^{\xi/k} \Gamma_k(\xi) = \frac{(sp)^{\xi/k}}{k} \, \Gamma\left(\frac{\xi}{k}\right) \tag{8}$$

To study, we need the following well-known definitions and results:

(1) Relation with generalized Wright hypergeometricfunction ${}_p\Psi_q$ [z]

$$pE^{\gamma,s}_{k,\alpha,\beta}(z) = \frac{k\left(\frac{sp}{k}\right)^{-\beta/k}}{\Gamma\left(\frac{\gamma}{k}\right)} \, {}_1\Psi_1\left[\mathrm{z}(sp)^{1-\frac{\alpha}{k}} \middle| \begin{matrix} \left(\frac{\gamma}{k},1\right) \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right) \end{matrix}\right] \tag{9}$$

(2) Relation with Fox H-function

$$pE^{\gamma,s}_{k,\alpha,\beta}(z) = \frac{k\left(\frac{sp}{k}\right)^{-\beta/k}}{\Gamma\left(\frac{\gamma}{k}\right)} \, H^{1,1}_{1,2}\left[-\mathrm{z}(sp)^{1-\frac{\alpha}{k}} \middle| \begin{matrix} \left(1-\frac{\gamma}{k},1\right) \\ (0,1),\left(1-\frac{\beta}{k},\frac{\alpha}{k}\right) \end{matrix}\right] \tag{10}$$

where $p\Psi q\,[z]$ is the generalized Wright hypergeometric function for $z \in \mathbb{C}, a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}^+(\alpha_i, \beta_j \neq 0; i = 1,2,\ldots,p; j = 1,2,\ldots,q)$ introduced by Wright [18] (see also [7], [16]) and defined by

$$\begin{aligned} p\Psi q\,[z] &= p\Psi q\left[\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p) \\ (b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}; z\right] \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 n)\ldots.\Gamma(a_p+\alpha_p n)}{\Gamma(b_1+\beta_1 n)\ldots.\Gamma(b_q+\beta_q n)} \frac{z^n}{n!} \\ &= H^{1,p}_{p,q+1}\left[-z \middle| \begin{matrix}(1-a_1,\alpha_1),\ldots,(1-a_p,\alpha_p) \\ (0,1),(1-b_1,\beta_1),\ldots,(1-b_q,\beta_q)\end{matrix}\right] \end{aligned} \tag{11}$$

Let $\alpha, \alpha', \beta, \beta', \gamma, \in \mathbb{C}$ and $x > 0$, then the generalized fractional integral operators involving the Appell function F_3 are defined by Saigo and Maeda [15,p.393, eqs. (4.12) & (4.13)], the following equations:

$$\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+} f\right)(x)$$

$$= \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha,\alpha',\beta,\beta';\gamma,1-\frac{t}{x},1-\frac{x}{t}\right) f(t)dt\,, (\Re(\gamma) > 0) \tag{12}$$

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)$$

$$=\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^{\infty}(t-x)^{\gamma-1}t^{-\alpha}\,F_3\left(\alpha,\alpha',\beta,\beta';\gamma,1-\frac{x}{t},1-\frac{t}{x}\right)f(t)dt\,,(\Re(\gamma)>0) \tag{13}$$

The corresponding fractional differential operators [15] (see also [11, p.3, eqs. (12) & (13)] are defined by the following equations:

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(I_{0+}^{-\alpha',-\alpha,\beta',-\beta,-\gamma}f\right)(x) \tag{14}$$

$$=\left(\frac{d}{dx}\right)^m\left(I_{0+}^{-\alpha',-\alpha,-\beta'+m,-\beta,-\gamma+m}f\right)(x)$$

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f\right)(x) \tag{15}$$

$$=\left(-\frac{d}{dx}\right)^m\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+m,-\gamma+m}f\right)(x)$$

where $(R(\gamma)>0; m=[R(\gamma)]+1)$.

If α is replaced by $+\beta$, $\alpha'=\beta'=0$, $\beta=-\eta$ and $\gamma=\alpha$ in (12), (13), (14) and (15), we get the following relationships:

$$(I_{0+}^{\alpha+\beta,0,-\eta,0,\alpha}f)\,(x)=\left(I_{0+}^{\alpha,\beta,\eta}\,f\right)(x) \tag{16}$$

$$(I_{-}^{\alpha+\beta,0,-\eta,0,\alpha}f)(x)=\left(I_{-}^{\alpha,\beta,\eta}f\right)(x) \tag{17}$$

where the operator $I_{0+}^{\alpha,\beta,\eta}$ denotes the Saigo fractional integral operator defined by Saigo [14].

Similarly $$(D_{0+}^{\alpha+\beta,0,-\eta,0,\alpha}f)\,(x)=\left(D_{0+}^{\alpha,\beta,\eta}\,f\right)(x) \tag{18}$$

$$(D_{-}^{\alpha+\beta,0,-\eta,0,\alpha}f)(x)=\left(D_{-}^{\alpha,\beta,\eta}f\right)(x) \tag{19}$$

The following two results will be required to establish Main theorems:

Lemma 1. [[15], p.394, eq. (4.18)]: Let $\alpha,\alpha',\beta,\beta',\gamma,\rho,\in\mathbb{C}$ be such that $\Re(\gamma)>0$ and $\Re(\rho)>max.\{0,\Re(\alpha'-\beta'),\Re(\alpha+\alpha'+\beta-\gamma)\}$ then there exists the relation

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}\right)(x)=\frac{\Gamma(\rho)\Gamma(\rho-\alpha'+\beta')\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)}x^{\rho-\alpha-\alpha'+\gamma-1} \tag{20}$$

Lemma 2. [[15], p.394, eq. (4.18)]: Let $\alpha,\alpha',\beta,\beta',\gamma,\rho,\in\mathbb{C}$ be such that $\Re(\gamma)>0$ and $\Re(\rho)<1+min.\{\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha+\beta'-\gamma)\}$ then there exists the relation

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}\right)(x)=\frac{\Gamma(1+\alpha+\alpha'-\gamma-\rho)\Gamma(1+\alpha+\beta'-\gamma-\rho)\Gamma(1-\beta-\rho)}{\Gamma(1-\rho)\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\Gamma(1+\alpha-\beta-\rho)}x^{\rho-\alpha-\alpha'+\gamma-1} \tag{21}$$

2. Main Results

In this section, we establish two theorems that give the images of the generalized Mittag-Leffler (*p*,*s*,*k*)-function under the Saigo-Maeda fractional integral operators in terms of the generalized Wright function.

Theorem 1. Let $\alpha,\alpha',\beta,\beta',\gamma,\rho,\sigma_i,\theta_i,\vartheta_i \in \mathbb{C}$ with $\Re(\sigma_i)>0,\Re(\theta_i)>0,\Re(,\vartheta_i)>0,\Re(\gamma)>0$, $\Re(\rho)>\max\{0,\Re(\alpha'-\beta'),\Re(\alpha+\alpha'+\beta-\gamma)\}$ and $t>0$, $p_i,k_i,s_i>0$, $\forall\, i=1,2,\ldots,r$, then left sided fractional integral formula holds

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\left\{\prod_{i=1}^{r}p_iE_{k_i,\theta_i,\vartheta_i}^{\sigma_i,s_i}(t)\right\}\right)(t)=t^{\rho+\gamma-\alpha-\alpha'-1}\prod_{i=1}^{r}\left\{\frac{k_i(s_ip_i)^{-\vartheta_i/k_i}}{\Gamma\left(\frac{\sigma_i}{k_i}\right)}\right.$$

$$\left.\times{}_{r+3}\Psi_{r+3}\left[\left.\begin{matrix}\left(\frac{\sigma_i}{k_i},1\right)_{1,r},(\rho,r),(\rho-\alpha'+\beta',r),(\rho+\gamma-\alpha-\alpha'-\beta,r)\\ \left(\frac{\vartheta_i}{k_i},\frac{\theta_i}{k_i}\right)_{1,r},(\rho+\beta',r),(\rho+\gamma-\alpha-\alpha',r),(\rho+\gamma-\alpha'-\beta,r)\end{matrix}\right|(s_ip_i)^{\left(1-\frac{\theta_i}{k_i}\right)}t^r\right]\right\} \tag{22}$$

Proof. In order to prove (22), we first express the generalized Mittag-Leffler (p,s,k)-function with the help of (6) and interchanging the order of integration and summation, we obtain (say $\mathfrak{I}_1$)

$$\mathfrak{I}_1=\prod_{i=1}^{r}\left\{\sum_{n_i=0}^{\infty}\frac{p_i{}^{(\sigma_i)_{n_i,k_i},s_i}}{p_i{}^{\Gamma_{s_i,k_i}(n_i\theta_i+\vartheta_i)}}\frac{1}{n_i!}\right\}\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{n_ir+\rho-1}\right)(t) \tag{23}$$

Now, using the result (20) in (23), we have

$$\mathfrak{I}_1=\prod_{i=1}^{r}\left\{\sum_{n_i=0}^{\infty}\frac{p_i{}^{(\sigma_i)_{n_i,k_i},s_i}}{p_i{}^{\Gamma_{s_i,k_i}(n_i\theta_i+\vartheta_i)}}\frac{1}{n_i!}\,\frac{\Gamma(\rho+n_ir)\Gamma(\rho+n_ir-\alpha'+\beta')}{\Gamma(\rho+n_ir+\beta')\Gamma(\rho+n_ir+\gamma-\alpha-\alpha')}\right.$$

$$\left.\times\frac{\Gamma(\rho+n_ir+\gamma-\alpha-\alpha'-\beta)}{\Gamma(\rho+n_ir+\gamma-\alpha'-\beta)}\,t^{\rho+n_ir+\gamma-\alpha-\alpha'-1}\right\} \tag{24}$$

Using the results (7) and (8) in (24), we get

$$\mathfrak{I}_1=t^{\rho+\gamma-\alpha-\alpha'-1}\prod_{i=1}^{r}\left\{\frac{k_i(s_ip_i)^{-\vartheta_i/k_i}}{\Gamma\left(\frac{\sigma_i}{k_i}\right)}\sum_{n_i=0}^{\infty}\frac{1}{n_i!}\,\frac{\Gamma\left(\frac{\sigma_i}{k_i}+n_i\right)}{\Gamma((n_i\theta_i+\vartheta_i)/k_i)}\,\frac{\Gamma(\rho+n_ir)}{\Gamma(\rho+n_ir+\beta')}\right.$$

$$\left.\times\frac{\Gamma(\rho+n_ir-\alpha'+\beta')\Gamma(\rho+n_ir+\gamma-\alpha-\alpha'-\beta)}{\Gamma(\rho+n_ir+\gamma-\alpha-\alpha')\Gamma(\rho+n_ir+\gamma-\alpha'-\beta)}\left((s_ip_i)^{\left(1-\frac{\theta_i}{k_i}\right)}t\right)^{n_i}\right\}$$

Finally, re-interpreting the above series thus obtained in terms of generalized Wright hypergeometric function defined by (11), we arrive at the right hand side of result (22).

Also, we can easily obtain the result in terms of H-function by using the relation (10).

Theorem 2. Let $\alpha,\alpha',\beta,\ \beta',\gamma,\rho,\sigma_i,\theta_i,\ \vartheta_i\in\mathbb{C}$ with $\Re(\sigma_i)>0,\Re(\theta_i)>0,\Re(,\vartheta_i)>0,\Re(\gamma)>0,\Re(\rho)<1+\min\{\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha+\beta'-\gamma)\}$ and >0, $p_i,k_i,s_i\ 0,\forall\, i=1,2,\ldots,r$, then the right sided fractional integral formula holds

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\left\{\prod_{i=1}^{r}p_iE_{k_i,\theta_i,\vartheta_i}^{\sigma_i,s_i}\left(\frac{1}{t}\right)\right\}\right)(t)=t^{\rho-\alpha-\alpha'+\gamma-1}\prod_{i=1}^{r}\left\{\frac{k_i(s_ip_i)^{\frac{-\vartheta_i}{k_i}}}{\Gamma\left(\frac{\sigma_i}{k_i}\right)}\right.$$

$$\left.\times{}_{r+3}\Psi_{r+3}\left[\begin{matrix}\left(\frac{\sigma_i}{k_i},1\right)_{1,r},(1-\rho-\beta,r),(1-\rho+\alpha+\alpha'-\gamma,r),(1-\rho+\alpha+\beta'-\gamma,r)\\\left(\frac{\vartheta_i}{k_i},\frac{\theta_i}{k_i}\right)_{1,r},(1-\rho,r),(1-\rho+\alpha+\alpha'+\beta'-\gamma,r),(1-\rho+\alpha-\beta,r)\end{matrix}\middle|\frac{(s_ip_i)^{\left(1-\frac{\theta_i}{k_i}\right)}}{t^r}\right]\right\}\tag{25}$$

Proof. In order to prove (25), we first express the generalized Mittag-Leffler (p,s,k)-function with the help of (6) and interchanging the order of integration and summation, we obtain (say $\mathfrak{J}_2$)

$$\mathfrak{J}_2=\prod_{i=1}^{r}\left\{\sum_{n_i=0}^{\infty}\frac{p_i{}^{(\sigma_i)}{}_{n_i,k_i,s_i}}{p_i{}^{\Gamma_{s_i,k_i}}(n_i\theta_i+\vartheta_i)}\frac{1}{n_i!}\right\}\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-n_ir-1}\right)(t)\tag{26}$$

Now, using the result (21) in (26), we obtain

$$\mathfrak{J}_2=\prod_{i=1}^{r}\left\{\sum_{n_i=0}^{\infty}\frac{p_i{}^{(\sigma_i)}{}_{n_i,k_i,s_i}}{p_i{}^{\Gamma_{s_i,k_i}}(n_i\theta_i+\vartheta_i)}\frac{1}{n_i!}\right\}\frac{\Gamma(1-\rho-\beta+n_ir)\Gamma(1-\rho+n_ir-\gamma+\alpha+\beta')}{\Gamma(1-\rho+n_ir)\Gamma(1-\rho+n_ir-\gamma+\alpha+\alpha')}$$

$$\times\frac{\Gamma(1-\rho+n_ir-\gamma+\alpha+\alpha')}{\Gamma(1-\rho+n_ir+\alpha-\beta)}t^{\rho+n_ir-\alpha-\alpha'+\gamma-1}\tag{27}$$

Using the results (7) and (8) in (27), we get

$$\mathfrak{J}_2=t^{\rho+\gamma-\alpha-\alpha'-1}\prod_{i=1}^{r}\left\{\frac{k_i(s_ip_i)^{-\vartheta_i/k_i}}{\Gamma\left(\frac{\sigma_i}{k_i}\right)}\sum_{n_i=0}^{\infty}\frac{1}{n_i!}\frac{\Gamma\left(\frac{\sigma_i}{k_i}+n_i\right)}{\Gamma((n_i\theta_i+\vartheta_i)/k_i)}\frac{\Gamma(1-\rho-\beta+n_ir)}{\Gamma(1-\rho+n_ir)}\right.$$

$$\left.\times\frac{\Gamma(1-\rho+n_ir-\gamma+\alpha+\beta')\Gamma(1-\rho+n_ir-\gamma+\alpha+\alpha')}{\Gamma(1-\rho+n_ir-\gamma+\alpha+\alpha')\Gamma(1-\rho+n_ir+\alpha-\beta)}\left(\frac{(s_ip_i)^{\left(1-\frac{\theta_i}{k_i}\right)}}{t^r}\right)^{n_i}\right\}$$

Finally, re-interpreting the above series thus obtained in terms of generalized Wright hypergeometric function defined by (11), we arrive at the right hand side of result (25).

3. Special Cases:

We obtain some new and known results involving Saigo-Meada fractional integral and differential operators, Saigo operators, Riemann-Liouville, Erdélyi-Kober operators etc. of derived results.

Corollary 1. Let the conditions of Theorem 1 be satisfied with $r=1$, $p_1=p,\sigma_1=\sigma,s_1=s,k_1=k,\theta_1=\theta$ and $\vartheta_1=\vartheta$, then there holds the formula:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\left\{pE_{k,\theta,\vartheta}^{\sigma,s}(t)\right\}\right)(t)=t^{\rho+\gamma-\alpha-\alpha'-1}\left\{\frac{k(sp)^{-\frac{\vartheta}{k}}}{\Gamma\left(\frac{\sigma}{k}\right)}\right.$$

$$\times {}_4\Psi_4\left[\begin{matrix}\left(\frac{\sigma}{k},1\right),(\rho,1),(\rho-\alpha'+\beta',1),(\rho+\gamma-\alpha-\alpha'-\beta,1)\\ \left(\frac{\vartheta}{k},\frac{\theta}{k}\right),(\rho+\beta',1),(\rho+\gamma-\alpha-\alpha',1),(\rho+\gamma-\alpha'-\beta,1)\end{matrix}\middle|(sp)^{1-\frac{\vartheta}{k}}\ t\right]\Bigg\}.$$

If we put $p = s = k = 1$ in above Corollary 1, we get known result given by Chouhan et al. [3].

Corollary 2. Let the conditions of Theorem 1 be satisfied, α is replaced by $\alpha+\beta$, $\alpha'=\beta'=0$, $\beta=-\eta$ and $\gamma=\alpha$ in Theorem 1, we get the following new result concerning Saigo-fractional integral operator:

$$\left(I_{0+}^{\alpha,\beta,\eta}t^{\rho-1}\left\{\prod_{i=1}^{r}p_iE_{k_i,\theta_i,\vartheta_i}^{\sigma_i,s_i}(t)\right\}\right)(t)$$

$$=t^{\rho-\beta-1}\prod_{i=1}^{r}\left\{\frac{k_i(s_ip_i)^{\frac{-\vartheta_i}{k_i}}}{\Gamma\left(\frac{\sigma_i}{k_i}\right)}\ {}_{r+2}\Psi_{r+2}\left[\begin{matrix}\left(\frac{\sigma_i}{k_i},1\right)_{1,r},(\rho,r),(\rho+\eta-\beta,r)\\ \left(\frac{\vartheta_i}{k_i},\frac{\theta_i}{k_i}\right)_{1,r},(\rho-\beta,r),(\rho+\alpha+\eta,r)\end{matrix}\middle|(s_ip_i)^{\left(1-\frac{\theta_i}{k_i}\right)}t^r\right]\right\}$$

If we put $r=1, p_1=p, \sigma_1=\sigma, s_1=s, k_1=k, \theta_1=\theta$ and $\vartheta_1=\vartheta$, in the above Corollary 2, then we coincide with known result of Kabra and Nagar [[10], p.15, Cor. 3.1.2].

Also, If we put $r=1,\ p_1=p=1, s_1=s=1, k_1=k=1, \sigma_1=\sigma, \theta_1=\theta$ and $\vartheta_1=\vartheta$, in the above Corollary 2, then we coincide with known result given by Ahmed [1].

We can obtain result concerning Riemann-Liouville fractional integral operators by putting $\beta=-\alpha$ in Corollary 2.

Again, we can also obtain result concerning Erdélyi-Kober fractional integral operators by putting $\beta=0$ in Corollary 2.

Corollary 3. Let $\alpha,\alpha',\beta,\beta',\gamma,\rho,\ \sigma_i,\theta_i,\ \vartheta_i\in\mathbb{C}$ with $\Re(\sigma_i)>0, \Re(\theta_i)>0, \Re(,\vartheta_i)>0, \Re(\gamma)>0, \Re(\rho)>\max\{0,\Re(\beta-\alpha),\Re(\gamma-\alpha-\alpha'-\beta')\}$ and $t>0$, $p_i,k_i,s_i>0, \forall\ i=1,2,\ldots,r$ then left sided fractional differential formula holds

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\left\{\prod_{i=1}^{r}p_iE_{k_i,\theta_i,\vartheta_i}^{\sigma_i,s_i}(t)\right\}\right)(t)=t^{\rho+\alpha+\alpha'-\gamma-1}\prod_{i=1}^{r}\left\{\frac{k_i(s_ip_i)^{-\frac{\vartheta_i}{k_i}}}{\Gamma\left(\frac{\sigma_i}{k_i}\right)}\right.$$

$$\left.\times\ {}_{r+3}\Psi_{r+3}\left[\begin{matrix}\left(\frac{\sigma_i}{k_i},1\right)_{1,r},(\rho,r)\ (\rho+\alpha-\beta,r)\ (\rho+\alpha+\alpha'+\beta'-\gamma,r)\\ \left(\frac{\vartheta_i}{k_i},\frac{\theta_i}{k_i}\right)_{1,r},(\rho-\beta,r)(\rho+\alpha+\alpha'-\gamma,r)(\rho+\alpha+\beta'-\gamma,r)\end{matrix}\middle|(s_ip_i)^{\left(1-\frac{\theta_i}{k_i}\right)}t^r\right]\right\}\tag{28}$$

Proof. On using result (14), we obtain (say $\mathfrak{I}_3$)

$$\mathfrak{I}_3=\left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}t^{\rho-1}\left\{\prod_{i=1}^{r}p_iE_{k_i,\theta_i,\vartheta_i}^{\sigma_i,s_i}(t)\right\}\right)(t)\tag{29}$$

We get the required result as given in (28) after applying Theorem 1 in (29).

Corollary 4. Let $\alpha, \alpha', \beta, \beta', \gamma, \rho, \sigma_i, \theta_i, \vartheta_i \in \mathbb{C}$ with $\Re(\sigma_i) > 0, \Re(\theta_i) > 0, \Re(\vartheta_i) > 0, \Re(\gamma) > 0, \Re(\rho) < 1 + \min\{\Re(\beta'), \Re(\gamma - \alpha' - \beta), \Re(\gamma - \alpha - \alpha')\}$ and $t > 0$, $p_i, k_i, s_i > 0, \forall\, i = 1,2, \ldots, r$, then right sided fractional differential formula holds

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\left\{\prod_{i=1}^{r} p_i E_{k_i,\theta_i,\vartheta_i}^{\sigma_i,s_i}\left(\frac{1}{t}\right)\right\}\right)(t) = t^{\rho+\alpha+\alpha'-\gamma-1}\prod_{i=1}^{r}\left\{\frac{k_i(s_i p_i)^{-\frac{\vartheta_i}{k_i}}}{\Gamma\left(\frac{\sigma_i}{k_i}\right)}\right.$$

$$\left.\times\ {}_{r+3}\Psi_{r+3}\left[\begin{matrix}\left(\frac{\sigma_i}{k_i},1\right)_{1,r},(1-\rho,r),(1-\rho-\alpha-\alpha'-\beta+\gamma,r),(1-\rho-\alpha'+\beta',r)\\ \left(\frac{\vartheta_i}{k_i},\frac{\theta_i}{k_i}\right)_{1,r},(1-\rho-\alpha-\alpha'+\gamma,r),(1-\rho-\alpha'-\beta+\gamma,r),(1-\rho+\beta',r)\end{matrix}\middle|\frac{(s_i p_i)^{\left(1-\frac{\theta_i}{k_i}\right)}}{t^r}\right]\right\} \tag{30}$$

Proof. On using result (15), we obtain (say $\mathfrak{J}_4$)

$$\mathfrak{J}_4 = \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} t^{\rho-1}\left\{\prod_{i=1}^{r} p_i E_{k_i,\theta_i,\vartheta_i}^{\sigma_i,s_i}\left(\frac{1}{t}\right)\right\}\right)(t) \tag{31}$$

We get the required result as given in (30) by using Theorem 2 in (31).

A number of several other new and known results can also be obtained by considering specific values.

4. Conclusion

In the present paper, we have investigated two theorems of generalized fractional integral operators given by Saigo-Maeda involving the generalized Mittag-Leffler (*p,s,k*)-function, which are expressed in terms of generalized Wright hypergeometric function. Also, we obtained the left and right sided Saigo-Maeda fractional differential operators for the generalized Mittag-Leffler (*p,s,k*)-function by using Theorem 1 and Theorem 2 respectively. The results derived in this paper also correspond to Saigo fractional calculus operators as corollaries. It can be easily seen that, if we put $\beta = -\alpha$ and $\beta = 0$ in Saigo fractional calculus operators (16), (17), (18) and (19) then we can obtain results concerning Riemann-Liouville and Erdelyi-Kober fractional integral and differential operators.

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