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FINSLER SPACES FROM CONFORMAL β-CHANGE

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Abstract: We have considered the conformal β-change of the Finsler metric given by $L(x,y) \to \bar{L}(x,y) = e^{\sigma(x)} f(L(x,y), \beta(x,y))$, where σ (x) is a function of x, $\beta(x,y) = b_i(x)y^i$ is a 1-form on the underlying manifold M^n, and $f(L(x,y), \beta(x,y))$ is a homogeneous function of degree one in L and β.Let F^n and $\bar{F}^n$ denote Finsler spaces with metric functions L and $\bar{L}$ respectively. It has been investigated how S_3-likeness and S_4-likeness of F^n are linked with corresponding properties of $\bar{F}^n$. Further, necessary and sufficient conditions for a Killing vector field of F^n to be a vector field of the same kind in $\bar{F}^n$ have been obtained.

Keywords: Finsler metric, conformal β-change, S_3-like Finsler space, S_4-like Finsler space, v-curvature tensor, Killing vector field.

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1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space on the differentiable manifold M^n equipped with the fundamental function L(x, y). Prasad and Kumari [11] and Shibata [12] have studied the general case of β-change, that is, $L^*(x,y) = f(L,\beta)$, where f is a positively homogeneous function of degree one in L and β, and β given by $\beta(x,y) = b_i(x)y^i$ is a one-form on M^n. The β-change of special Finsler spaces has been studied by Shukla, Pandey and Mandal [14].

The conformal theory of Finsler space was initiated by Knebelman [8] in 1929 and has been investigated in detail by many authors (Hashiguchi [3], Izumi [5, 6] and Kitayama [7]). The conformal change is defined as $L'(x,y) = e^{\sigma(x)} L(x,y)$, where $\sigma(x)$ is a function of position only and is known as conformal factor. In 2008, Abed [1, 2] introduced the change $L''(x,y) = e^{\sigma(x)} L(x,y) + \beta(x,y)$, which he called a β-conformal change, thus he generalized the conformal and Randers changes. Moreover, he studied some special Finsler spaces under this change such as C-reducible and S_3-like Finsler spaces. In 2009 and 2010, Youssef, Abed and Elgendi [18,19] introduced the transformation $L'''(x,y) = f(e^{\sigma}L, \beta)$, which is general β-change of conformally changed Finsler metric L. They have not only established the relationships between some important tensors of (M^n, L) and the corresponding tensors of (M^n, L'''), but have also studied several properties of this change.

Shukla and Mishra [13] have changed the order of combination of the above two changes, i.e., they have applied β-change first and conformal change afterwards as follows :

$$\bar{L}(x,y) = e^{\sigma(x)} f(L(x,y), \beta(x,y)), \tag{1}$$

where $\sigma(x)$ is a function of x , $\beta(x,y) = b_i(x)y^i$ is a 1-form.They have called this change as conformal β-change of Finsler metric. In this paper they have investigated the condition under which a conformal β-change of Finsler metric leads a Douglas space into a Douglas space. They have also found the necessary and sufficient conditions for this change to be a projective change.

They have studied quasi-C-reducibility, C-reducibility and semi-C-reducibility of the Finsler space with this metric in their paper [15], wherein they have also calculated the T-tensor [4] of $\bar{F}^n$. When $\sigma = 0$, it reduces to a β–change. When $\sigma = constant$, it becomes a homothetic β–change. When $f(L, \beta)$ has special forms as $L + \beta, \frac{L^2}{L-\beta}, \frac{L^2}{\beta}, \frac{L^{m+1}}{\beta^m}$ ($m \neq 0, -1$), one obtains conformal Randers change, conformal Matsumoto change, conformal Kropina change, conformal generalized Kropina change of Finsler metric respectively.

In the present paper, we investigate some other properties of conformal β-change. The Finsler space equipped with the metric $\bar{L}$ given by (1) will be denoted by $\bar{F}^n$. Throughout the paper the quantities corresponding to $\bar{F}^n$ will be denoted by putting bar on the top of them.

Homogeneity of f gives

$$Lf_1 + \beta f_2 = f, \tag{2}$$

where subscripts "1" and "2" denote the partial derivatives with respect to $L \, and \, \beta$ respectively. Differentiating above equation with respect to $L \, and \, \beta$ respectively, we get

$$Lf_{11} + \beta f_{21} = 0 \text{ and } Lf_{12} + \beta f_{22} = 0. \tag{3}$$

Hence we have $\frac{f_{11}}{\beta^2} = \frac{-f_{12}}{L\beta} = \frac{f_{22}}{L^2}$, (4)

which gives

$$f_{11} = \beta^2 \omega \, , f_{12} = -L\beta\omega, f_{22} = L^2\omega, \tag{5}$$

where Weierstrass function ω is positively homogeneous of degree-3 in $L \, and \, \beta$. Therefore

$$L\omega_1 + \beta\omega_2 + 3\omega = 0 \, , \tag{6}$$

where ω_1 and ω_2 are positively homogeneous of degree – 4 in $L\ and\ \beta$. Throughout the paper we frequently use the above equations without quoting them. Also we have assumed that f is not a linear function of $L\ and\ \beta$ so that $\omega \neq 0$.

Killing equations play an important role in the study of a Finsler space whose points undergo an infinitesimal transformation. In fact, they give a characterization for the transformation to preserve distances. In 1979, Singh et al. [17] have discussed Killing correspondence between Randers space $(M^n, L = \alpha + \beta)$ and the space (M^n, L_1), where $L_1^2 = L^2 + \beta^2$. In 2014, Shukla and Gupta [16] have discussed Killing correspondence between Randers space $(\mathrm{M^n}, \mathrm{L})$ and the space $(\mathrm{M^n}, \mathrm{L^*})$, where $\mathrm{L^*} = \mathrm{f}(\mathrm{L}, \beta)$. Kumbar et al. [9] have studied Killing correspondence between $(\mathrm{M^n}, \mathrm{L})$ and $(\mathrm{M^n}, \mathrm{L'''})$, where $\mathrm{L'''} = \mathrm{f}(\mathrm{e^\sigma L}, \beta)$.

The aim of this paper is to study some special Finsler spaces arising from conformal β-change of Finsler metric, viz., S_3-like and S_4-like Finsler spaces. Further, we study Killing correspondence between the Finsler spaces F^n and $\bar{F}^n$.

2. Fundamental quantities of $\bar{F}^n$

Differentiating equation (1) with respect to y^i we have

$$\bar{l}_i = e^\sigma (f_1 l_i + f_2 b_i). \tag{7}$$

Differentiating (7) with respect to y^j , we have

$$\bar{h}_{ij} = e^{2\sigma} \left(\frac{f f_1}{L} h_{ij} + f L^2 \omega\, m_i m_j \right), \tag{8}$$

where $\quad m_i = b_i - \frac{\beta}{L} L_i.$

From (7) and (8) we get the following relation between metric tensors of $F^n\ and\ \bar{F}^n$:

$$\bar{g}_{ij} = e^{2\sigma} \left[\frac{f f_1}{L} g_{ij} - \frac{p\beta}{L} l_i l_j + p(l_i b_j + l_j b_i) + (f L^2 \omega + f_2^2) b_i b_j \right], \tag{9}$$

where $\quad p = f_1 f_2 - f L \beta \omega$.

The contravariant components $\bar{g}^{ij}$ of the metric tensor of $\bar{F}^n$, obtainable from $\bar{g}^{ij} \bar{g}_{jk} = \delta^i_k$, are as follows :

$$\bar{g}^{ij} = e^{-2\sigma} [\frac{L}{f f_1} g^{ij} + \frac{p L^3}{f^3 f_1 t} \left(\frac{f\beta}{L^2} - \Delta f_2 \right) l^i l^j - \frac{\omega L^4}{f f_1 t} b^i b^j - \frac{p L^2}{f^2 f_1 t} (l^i b^j + b^i l^j)], \tag{10}$$

where $l^i = g^{ij} l_j$, $b^i = g^{ij} b_j$, $b^2 = b^j b_j$, g^{ij} is the reciprocal tensor of g_{ij} of F^n, and

$$\Delta = b^2 - \frac{\beta^2}{L^2}, t = f_1 + L^3 \omega \Delta . \tag{11}$$

Cartan's covariant C-tensor $\mathrm{C_{ijk}}$ of $\mathrm{F^n}$ is defined by

$\mathrm{C_{ijk}} = \frac{1}{4} \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^2 = \frac{1}{2} \dot{\partial}_k g_{ij}$ and Cartan's C–vector is defined as follows:

$C_i = C_{ijk}\, g^{jk}$. (12)

Under the conformal change (1) we get the following relation between Cartan's C-tensors of F^n and $\bar{F}^n$:

$$\bar{C}_{ijk} = e^{2\sigma}[\frac{ff_1}{L} C_{ijk} + \frac{p}{2L}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \frac{qL^2}{2} m_i m_j m_k], \quad (13)$$

where $= 3f_2\omega + f\omega_2$.

We have

$$\text{(a) } m_i l^i = 0, (b)\, m_i b^i = b^2 - \frac{\beta^2}{L^2} = \Delta = b_i m^i, (c) g_{ij}m^i = h_{ij}m^i = m_j . \quad (14)$$

From (7), (9), (10) and (13), we get

$$\bar{C}^i_{jk} = C^i_{jk} + \frac{p}{2ff_1}\left(h_{jk}m^i + h^i_j m_k + h^i_k m_j\right) - \frac{pL\Delta}{2f^2 f_1 t} h_{jk}n^i - \frac{(2pL + qL^4\Delta)}{2f^2 f_1 t} m_j m_k n^i$$
$$- \frac{L}{ft} C_{.jk} n^i + \frac{qL^3}{2ff_1} m_j m_k n^i , \quad (15)$$

where $n^h = fL^2\omega b^h + pl^h$ and $h^i_j = g^{il}h_{lj}$, $C_{.jk} = C_{ijk}b^i$, $C_{..i} = C_{ijk}b^j b^k$ and so on.

Proposition 2.1. The normalized supporting element $\bar{l}_i$, angular metric tensor $\bar{h}_{ij}$, fundamental metric tensor $\bar{g}_{ij}$ and (h)hv-torsion tensor $\bar{C}_{ijk}$ of $\bar{F}^n$ are given by (7),(8), (9) and (13) respectively.

From (10), (12), (13) and (14) we get the following relations between the

C-vectors of F^n and $\bar{F}^n$:

$$\bar{C}_i = C_i - L^2\omega C_{i..} + \mu m_i , \quad (16)$$

where $= \frac{p(n+1) - 3pL^3\omega\Delta + qL^3\Delta(1 - L^3\Delta\omega)}{2ff_1}$.

3. Expression for v-curvature tensors of $\bar{F}^n$

The v-curvature tensor of Finsler space with fundamental function L is given by

$$S_{hijk} = C_{ijr}C^r_{hk} - C_{ikr}C^r_{hj} .$$

Therefore the v-curvature tensor of conformally β-changed Finsler space $\bar{F}^n$ is given by

$$\bar{S}_{hijk} = \bar{C}_{ijr}\bar{C}^r_{hk} - \bar{C}_{ikr}\bar{C}^r_{hj} . \quad (17)$$

From equations (13) and (15), we have

$$\bar{C}_{ijr}\bar{C}^r_{hk} = e^{2\sigma}[\frac{ff_1}{L} C_{ijr}C^r_{hk} + \frac{p}{2L}\left(C_{ijk}m_h + C_{ijh}m_k + C_{ihk}m_j + C_{hjk}m_i\right)$$
$$+ \frac{pf_1}{2Lt}\left(C_{.ij}h_{hk} + C_{.hk}h_{ij}\right) - \frac{ff_1 L^2\omega}{t} C_{.ij}C_{.hk} + \frac{p^2\Delta}{4fLt} h_{hk}h_{ij}$$

$$+\frac{L^2(qf_1-2p\omega)}{2t}\left(C_{.ij}m_k m_h + C_{.hk}m_i m_j\right) + \frac{p(p+L^3q\Delta)}{4Lft}\left(h_{ij}m_k m_h + h_{hk}m_i m_j\right) + \frac{p^2}{4Lff_1}\left(h_{ij}m_h m_k + h_{hk}m_i m_j + h_{hj}m_i m_k + h_{jk}m_i m_h + h_{hi}m_j m_k + h_{ik}m_h m_j\right) + \frac{L^2\{2pqt+(qf_1-2\omega p)(2p+L^3q\Delta)\}}{4tff_1}\, m_h m_k m_i m_j]. \quad (18)$$

Interchanging j and k in (18) and subtracting the equation thus obtained from (18) and using (17), we get

$$\bar{S}_{hijk} = e^{2\sigma}[\frac{ff_1}{L}S_{hijk} + \Theta_{jk}\left(d_{hk}h_{ij} + d_{ij}h_{hk} + E_{hk}C_{.ij} + E_{ij}C_{.hk}\right)], \quad (19)$$

where $d_{ij} = QC_{.ij} + Rh_{ij} + Pm_i m_j,$ (20)

$E_{ij} = I\, m_i m_j + T\, C_{.ij}\,,$ (21)

$P = \frac{p\{p(f_1 - L^3\omega\Delta) + L^3 q\omega\}}{4tfL}\,,\ Q = \frac{f_1 p}{2Lt}\,,\ R = \frac{p^2\Delta}{8Lft}\,,$

$I = \frac{L^2(f_1 q - 2p\omega)}{2t}\,,\ T = \frac{f_1 L^2\omega f}{2t}\,,$

and Θ_{jk} denotes interchange of j and k and subtraction .

Proposition 3.1 The relation between v-curvature tensors of F^n and $\bar{F}^n$ is given by (19).

We get the following expressions for the vertical Ricci tensor $\bar{S}_{ik}$ and the vertical scalar curvature $\bar{S}$ associated with the transformed space $\bar{F}^n$:

$$\bar{S}_{ik} = S_{ik} + Kh_{ik} + \left\{\frac{L^4\omega\Delta}{ff_1t} - \frac{(n-3)L}{ff_1}\right\}d_{ik} + \varphi_{ik}\,, \quad (22)$$

where $K = \left[\frac{L^4\omega}{ff_1t}d_{..} - \frac{L}{ff_1}\{P\Delta + QC_{.} + (n-1)R\}\right],$

$$\varphi_{ik} = \frac{L}{ff_1}\left\{E_{rk}C^r_{i.} + E_{ri}C^r_{k.} - \left(I\Delta - \frac{f_1L^2\omega f}{2t}\right)C_{ik.}\right\} - \frac{L^4\omega}{ff_1t}\Big(d_{.k}m_i + d_{.i}m_k + E_{k.}C_{i..} + E_{i.}C_{k..} - E_{..}C_{ik.} - E_{ik}C_{...} + \frac{ff_1}{L}S_{hijk}b^hb^j\Big),$$

$d_{..} = d_{hj}b^hb^j\,, E_{..} = E_{hj}b^hb^j$

and

$$\bar{S} = e^{-2\sigma}[\frac{L}{ff_1}\,S - \frac{2LK}{ff_1}\left\{\frac{L^3\omega}{t} - (n-2)\right\} - \frac{L^3\omega}{ff_1t}\,\varphi_{..} + \frac{L}{ff_1t}\varphi - \frac{L^3\omega}{ff_1t}S_{hijk}b^hb^j], \quad (23)$$

where $\varphi_{..} = \varphi_{hj}b^hb^j, \varphi = \varphi_{hj}g^{hj}$

and φ_{ij} is symmetric and indicatory .

4. The S_3-likeness and S_4-likeness

In this section, following Matsumoto [10], we shall investigate special cases of $\bar{F}^n$.

Definition 4.1. A Finsler space (M^n,L) with dimension n $\geq$ 3 is called S_3-like if the v-curvature tensor S_{hijk} satisfies

$$S_{hijk} = \frac{S}{(n-1)(n-2)}\left(h_{ik}h_{hj} - h_{hk}h_{ij}\right),$$

where scalar S is vertical scalar curvature .

Define the tensor

$$K_{hijk} = S_{hijk} - \frac{S}{(n-1)(n-2)}\left(h_{ik}h_{hj} - h_{hk}h_{ij}\right).$$

It is clear that the tensor K_{hijk} vanishes iff F^n is S_3-like.

Proposition 4.1. Under the conformal β-change(1), the tensor $\bar{K}_{hijk}$

associated with the space $\bar{F}^n$ has the form

$$\bar{K}_{hijk} = e^{2\sigma}\frac{ff_1}{L}K_{hijk} + U_{hijk}\ , \tag{24}$$

where $$U_{hijk} = e^{2\sigma}\ \Theta_{jk}\Big[d_{hk}h_{ij} + d_{ij}h_{hk} + E_{hk}C_{.ij} + E_{ij}C_{.hk} - \frac{\rho f^2 f_1^2}{L^2(n-1)(n-2)}h_{ik}h_{hj} - \frac{fL^2\omega}{(n-1)(n-2)}\left(S + \frac{\rho f f_1}{L}\right)\left(h_{ik}m_h m_j + h_{hj}m_i m_k\right)\Big], \tag{25}$$

$$\rho = [\frac{L}{ff_1}\varphi - \frac{L^3\omega}{ff_1 t}\left(\varphi_{..} + S_{ik}b^i b^k\right) - \frac{2LK}{ff_1}\{\frac{\Delta L^3\omega}{t} - (n-2)\}].$$

From (24) we have the following theorem :

Theorem 4.1. Conformally β-changed Finsler space $\bar{F}^n$ is S_3-like iff F^n is S_3-like and the tensor U_{hijk} given by (25) vanishes identically.

Definition 4.2. A Finsler space (M^n,L) with dimension n $\geq$ 4 is called S_4-like if the v-curvature tensor S_{hijk} satisfies

$$S_{hijk} = \Theta_{jk}(h_{hj}K_{ik} + h_{ik}K_{hj}),$$

where $$K_{ik} = \frac{1}{(n-3)}\left\{S_{ik} - \frac{S}{2(n-2)}h_{ik}\right\}.$$

Define the tensor

$$H_{hijk} = S_{hijk} - \Theta_{jk}(h_{hj}K_{ik} + h_{ik}K_{hj}).$$

Then F^n is S_4-like iff H_{hijk} vanishes.

Proposition 4.2. Under the conformal β-change(1), the tensor $\bar{H}_{hijk}$

associated with the space $\bar{F}^n$ has the form

$$\bar{H}_{hijk} = e^{2\sigma}\frac{ff_1}{L}H_{hijk} + V_{hijk}\,, \tag{26}$$

where

$$V_{hijk} = \Xi_{hijk}[E_{hk}C_{ij.} + fL^2\omega K_{hk}m_im_j + \frac{ff_1}{L(n-3)}\left(\frac{\Delta L^3\omega}{ff_1t}d_{hk}h_{ij} + Kh_{ij}h_{hk} + \varphi_{hk}h_{ij}\right) - \frac{fL^2\omega S}{2(n-2)(n-3)}m_hm_kh_{ij} + \frac{\rho ff_1h_{ij}}{2L(n-2)(n-3)}\left(\frac{ff_1}{L}h_{hk} + fL^2\omega m_hm_k\right) + \frac{fL^2\omega h_{hk}m_im_j}{(n-3)} + \frac{L^2\omega}{f_1(n-3)}\left\{\varphi_{hk} + \left(\frac{\Delta L^2\omega}{t} - (n-3)\right)d_{hk} - \frac{\rho f^2f_1^2}{2L^2(n-2)}h_{hk}\right\}m_im_j] \tag{27}$$

and Ξ_{hijk} $(X_{hk}Y_{ij})$ denotes $X_{hk}Y_{ij} + X_{ij}Y_{hk} - X_{hj}Y_{ik} - X_{ik}Y_{hj}$.

From (26) we have the following theorem :

Theorem 4.2. Conformally β-changed Finsler space $\bar{F}^n$ is S_4-like iff F^n is S_4-like and the tensor V_{hijk} given by (27) vanishes identically.

5. Killing correspondence of F^n and $\bar{F}^n$

Let us consider an infinitesimal transformation

$$x^{i'} = x^i + \in v^i(x), \tag{28}$$

where ∈ is an infinitesimal constant and $v^i(x)$ is a contravariant vector field. This vector field $v^i(x)$ is said to be a Killing vector field in F^n if the metric tensor of the Finsler space with respect to the infinitesimal transformation (28) is Lie invariant, i.e. if

$$£_v g_{ij} = 0, \tag{29}$$

where $£_v$ is the operator of Lie differentiation. The condition (29) is equivalent to

$$v_{i|j} + v_{j|i} + 2C_{ij}^h v_{h|0} = 0, \tag{30}$$

where $v_i = g_{il}v^l$ and the symbol | denotes h-covariant differentiation with respect to the Cartan's connection CΓ.

In this section we investigate a necessary and sufficient condition for Killing vector field in F^n to be a Killing vector field in $\bar{F}^n$. For the aforesaid investigation some preliminaries are required, which are given below.

We put $\quad 2\,r_{ij} = b_{i|j} + b_{j|i}, \qquad 2\,s_{ij} = b_{i|j} - b_{j|i}$.

The transformed Christoffel symbols of the Finsler space $\bar{F}^n$ are given by

$\bar{\gamma}^i_{jk} = \frac{1}{2}\bar{g}^{ir}\left(\partial_j\bar{g}_{kr} + \partial_k\bar{g}_{jr} - \partial_r\bar{g}_{jk}\right).$ (31)

Now we deal with the well-known functions $G^i(x, y)$ which are (2)p-homogeneous in y^i and are given by

$G^i = \frac{1}{2}\ \gamma^i_{jk}y^jy^k.$ (32)

Using (9), (10), (31) and (32) ,we have

$\bar{G}^i = \frac{1}{2}\ \bar{\gamma}^i_{jk}y^jy^k = G^i + D^i$, (33)

where the vector D^i is given by

$D^i = \frac{f_2L}{f_1}s^i_0 - \frac{L}{ff_1t}(f_1r_{00} - 2Lf_2s_{r0}b^r)\left(py^i - L^2\omega fb^i\right) + \sigma_0y^i - \frac{1}{2}f^2\sigma^i,$ (34)

in which $s^i_0 = g^{ir}s_{rj}y^j$.

Let $C\bar{\Gamma} = \left(\bar{F}^i_{jk}, \bar{N}^i_j, \bar{C}^i_{jk}\right)$ be the Cartan's connection on the space $\bar{F}^n$.

For coefficient $\bar{N}^i_j = \dot{\partial}_j\bar{G}^i$ to the non-linear connection, we differentiate (33) with respect to y^j and get

$\bar{N}^i_j = N^i_j + D^i_j$, (35)

where the tensor $D^i_j = \dot{\partial}_jD^i$ is given by

$D^i_j = \frac{Le^{2\sigma}}{f\,f_1}A^i_j - Q^iA_{rj}b^r + \frac{pLf_2}{f^2f_1^2t}b_{0|j}\left\{-Lf_1b^i + (f\beta - \Delta L^2f_2)y^i\right\} + \sigma_jy^i$
$-f\,\sigma^i(f_1l_j + f_2b_j)$, (36)

in which

$A_{ij} = \frac{1}{2}r_{00}B_{ij} + e^{2\sigma}ff_2s_{ij} + s_{i0}Q_j - (\frac{e^{2\sigma}ff_1}{L}C_{imj} + V_{ijm})D^m$,

$A^i_j = g^{ir}A_{rj}$, $V_{ijm} = g_{sj}V^s_{im}$, $Q_i = e^{2\sigma}(py_i + fL^2\omega y_i + f_2^2b_i)$,

$B_{jk} = \frac{1}{2}e^{2\sigma}\left(ph_{jk} + qL^2m_jm_k\right), \dot{\partial}_kQ_j = \frac{1}{2}B_{jk}$,

$$V_{ijk} = \frac{e^{2\sigma}}{(n+1)}\pi_{(ijk)}\left\{(n+1)\left(\alpha_1h_{ij} + \alpha_2m_im_j\right)m_k + \omega L^2m_im_jC_k + \omega L^2\left(f\,f_1h_{ij} + L^3\omega m_im_j\right)C_{k..}\right\},$$

$$\alpha_1 = \frac{p}{2L} - \frac{\mu ff_1}{L(n+1)}, \qquad \alpha_2 = \frac{qL^2}{6} - \frac{\mu\omega L^2}{(n+1)}$$

and $\pi_{(ijk)}$ represents cyclic permutation and sum over the indices I, j and k .

Let $B\bar{\Gamma} = \left(\bar{G}^i_{jk}, \bar{N}^i_j, 0\right)$ be the Berwald connection on the space $\bar{F}^n$. Differentiating (35) with respect to y^k, we have connection coefficients $\bar{G}^i_{jk} = \dot{\partial}_k \bar{N}^i_j$ of $B\bar{\Gamma}$, which are given by

$$\bar{G}^i_{jk} = G^i_{jk} + B^i_{jk},\ B^i_{jk} = \dot{\partial}_k D^i_j,$$

where G^i_{jk} are connection coefficients of $B\Gamma\ on\ F^n$. Substituting from (31), (13), (10), (35) and (15) in

$$\bar{F}^i_{jk} = \bar{\gamma}^i_{jk} + \bar{C}_{jkr}\bar{N}^r_m \bar{g}^{im} - \bar{C}^i_{kr}\bar{N}^r_j - \bar{C}^i_{rj}\bar{N}^r_k \tag{37}$$

we obtain connection coefficients $\bar{F}^i_{jk}$ of Cartan's connection $C\bar{\Gamma}\ on\ \bar{F}^n$ as

$$\bar{F}^i_{jk} = F^i_{jk} + D^i_{jk}, \tag{38}$$

where

$$D^i_{jk} = \left[\frac{e^{-2\sigma}L}{ff_1} g^{is} - Q^i b^s + y^s \frac{e^{-2\sigma}pL}{f^3 f_1 t}\left\{-Lfb^i + (f\beta - \Delta L^2 f_2)y^i\right\}\right]$$

$$(B_{sj}b_{0|k} + B_{sk}b_{0|j} - B_{kj}b_{0|s} + s_{sj}Q_k + s_{sk}Q_j + r_{kj}Q_s + \frac{e^{2\sigma}ff_1}{L}C_{jkr}D^r_s$$

$$+V_{jkr}D^r_s - \frac{e^{2\sigma}ff_1}{L}C_{skm}D^m_j - V_{sjm}D^m_k - \frac{e^{2\sigma}ff_1}{L}C_{sjm}D^m_k - V_{skm}D^m_j)$$

$$-e^{-2\sigma}\sigma^i\ \ \bar{g}_{jk}\,. \tag{39}$$

The tensor D^i_{jk}, called the difference tensor, has the following properties:

$$(a)\ D^i_{j0} = B^i_{j0} = D^i_j\ ,\ (b)\ D^i_{00} = 2\ D^i. \tag{40}$$

Theorem 5.1. A Killing vector field $v^i(x)$ in F^n is Killing vector field in $\bar{F}^n$ if and only if

$$W^h_{ij}\ v_{h|0} - v_r D^r_{ij} - \bar{C}^h_{ij} v_r D^r_h = 0, \tag{41}$$

where $\bar{C}^h_{ij}$ is the associate Cartan tensor of $\bar{F}^n$ and $W^h_{ij} = \bar{C}^h_{ij} - C^h_{ij}$.

Proof. Assume that $v^i(x)$ is Killing vector field in F^n. Then condition (30) is satisfied. The h-covariant derivatives of v_i(x) with respect to CΓ and $C\bar{\Gamma}$ are respectively given as

$$\text{(a)}\ v_{i|j} = \partial_j v^i - v_r F^r_{ij}\ ,\quad \text{(b)}\ v_{i\|j} = \partial_j v^i - v_r \bar{F}^r_{ij}\ , \tag{42}$$

where $\partial_j \equiv \frac{\partial}{\partial x^j}$.

By virtue of (42)(a) and (38), the equation (42)(b) takes the form

$$v_{i\|j} = v_{i|j} - v_r D^r_{ij}\ . \tag{43}$$

From (43) and (40)(a) we have

$$v_{i\|j} + v_{j\|i} + 2\bar{C}^h_{ij} v_{h\|0} = v_{i|j} + v_{j|i} - 2v_r D^r_{ij} + 2\bar{C}^h_{ij}\, v_{h|0} - 2\bar{C}^h_{ij} v_r D^r_h\ . \tag{44}$$

Using condition (30) in (44) and putting $W^h_{ij} = \bar{C}^h_{ij} - C^h_{ij}$, we get

$$v_{i\|j} + v_{j\|i} + 2\bar{C}^h_{ij} v_{h\|0} = 2W^h_{ij}\, v_{h|0} - 2v_r D^r_{ij} - 2\bar{C}^h_{ij} v_r D^r_h\ . \tag{45}$$

From (45) it follows that $v^i(x)$ is Killing vector field in $\bar{F}^n$ iff (41) holds.

Transvecting (41) by y^i and y^j and noting equation 40(b) and the fact that $W^h_{ij} y^i y^j = 0 = \bar{C}^h_{ij} y^i\, y^j$, we get

$v_r D^r = 0.$

Thus we have the following corollary:

Corollary 5.1. If $v^i(x)$ is Killing vector field in F^n and $\bar{F}^n$ both, then it is orthogonal to the vector $D^i(x, y)$.

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