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PASCU-TYPE HARMONIC FUNCTIONS WITH POSITIVE COEFFICIENTS INVOLVING GENERALIZED SALAGEAN OPERATOR

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Abstract: In the present paper, we introduce a new class of univalent functions and study some of their interesting properties such as coefficient bounds, distortion inequalities, extreme points, convolution and convex combination.

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1. Introduction

A continuous function $f = u + iv$ is a complex – valued harmonic function in a complex domain D if both u and v are real and harmonic in D. In any simply-connected domain $E \subset D$, we can write

$f = h + \bar{g}$, where h and g are analytic in E. We call h the analytic and g the co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and orientation preserving in E is that $|h'(z)| > |g'(z)|$ in E (see[2]).

Denote by $\mathcal{H}$ the family of functions

$$f = h + \bar{g} \tag{1}$$

which are harmonic, univalent and orientation-preserving in the open unit disc $U = \{ z : z < 1\}$ so that f is normalized by $f(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \bar{g} \in \mathcal{H}$, the functions h and g are analytic in U and can be expressed in the following forms:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k \;\; (0 \le |b_1| < 1), \tag{2}$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \;\; (0 \le |b_1| < 1). \tag{3}$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent reduces to the well-known class S of normalized univalent functions if the co-analytic part of f is identically zero; that is, $g \equiv 0$.

For functions $f \in \mathcal{H}$, Jahangiri et al. [7] defined Salagean operator on harmonic functions given by

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)}, \tag{4}$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \qquad D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k. \tag{5}$$

Al oboudy [1] define for a function $f \in A$, ($f(z) = z + \sum_{k=2}^{\infty} a_k z^k$) the following Integral operator

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z) \qquad \lambda \ge 0$$

$$D^n f(z) = D_\lambda (D^{n-1} f(z))$$

where

$$D^n f(z) = z + \sum_{k=2}^{\infty} (1 + (k-1)\lambda)^n a_k z^k$$

If $= 1$, above operator reduces as Salagean operator given by Salagean [14].

For a function $f \in \mathcal{H}$, given by (1) we define generalized Salagean derivative operator on harmonic functions given by

$$D_\lambda^n f(z) = D_\lambda^n h(z) + (-1)^n \overline{D_\lambda^n g(z)} \tag{6}$$

where

$$D_\lambda^n h(z) = z + \sum_{k=2}^{\infty} (1 + (k-1)\lambda)^n a_k z^k, D_\lambda^n g(z) = \sum_{k=1}^{\infty} (1 + (k-1)\lambda)^n b_k z^k. \tag{7}$$

In 1975, Silverman [15] introduced a new class $\mathcal{T}$ of analytic functions of the form $f(z) = z - \sum_{k=2}^{\infty}|a_k|z^k$ and opened up a new direction of studies in the theory of univalent functions as well as in harmonic functions with negative coefficients [16]. Uralegaddi et al. [17] introduced analogous subclasses of star-like, convex functions with positive coefficients and opened up a new and interesting direction of research. In fact, they considered the functions where the coefficients are positive rather than negative real numbers. Motivated by the initial work of Uralegaddi et al. [17], many researchers (see [3–6,11,12]) introduced and studied various new subclasses of analytic functions with positive coefficients but analogues results on harmonic univalent functions have not been explored in the literature. Very recently, Dixit and Porwal [6] attempted to fill this gap by introducing a new subclass of harmonic univalent functions with positive coefficients.

Denote by $V_{\mathcal{H}}$ the subfamily of $\mathcal{H}$ consisting of harmonic functions $f = h + \bar{g}$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \ (a_k \geq 0, b_k \geq 0, 0 \leq |b_1| < 1). \tag{8}$$

Motivated by the earlier works of [8–10,13] on the subject of harmonic functions, in this paper an attempt has been made to study the class of functions $f \in V_{\mathcal{H}}$ associated with generalized Salagean operator on harmonic functions. Further, we obtain a sufficient coefficient condition for functions $f \in \mathcal{H}$ given by (3) and also show that this coefficient condition is necessary for functions $f \in V_{\mathcal{H}}$, the class of harmonic functions with positive coefficients. Distortion results, extreme points, convolution properties and convex combination.

For $0 \leq \beta \leq 1, 1 < \gamma \leq 4/3$, we let $\mathcal{P}_{\mathcal{H}}^{n}(\lambda, \beta, \gamma)$ be a new subclass of $\mathcal{H}$, consisting of all functions of the form (1.3) satisfying the condition

$$\mathcal{R}e \left\{ \frac{(1-\beta)D_{\lambda}^{n+1}f(z) + \beta D_{\lambda}^{n+2}f(z)}{(1-\beta)D_{\lambda}^{n}f(z) + \beta D_{\lambda}^{n+1}f(z)} \right\} < \gamma, \tag{9}$$

where $D_{\lambda}^{n}f(z)$ is given by (6). Also let $V_{\mathcal{H}}^{n}(\lambda, \beta, \gamma) = \mathcal{P}_{\mathcal{H}}^{n}(\lambda, \beta, \gamma) \cap V_{\mathcal{H}}$.

2. Coefficient Bounds

In our first theorem, we obtain a sufficient condition for harmonic functions in $\mathcal{P}_{\mathcal{H}}^{n}(\lambda, \beta, \gamma)$.

Theorem 2.1 Let $f = h + \bar{g}$ be given by (3) then $f \in V_{\mathcal{H}}^{n}(\lambda, \beta, \gamma)$ if and only if

$$\sum_{k=2}^{\infty} \frac{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma-1}|a_k| + \sum_{k=1}^{\infty} \frac{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda+\gamma]}{\gamma-1}|b_k| \leq 1 \tag{10}$$

where $a_1 = 1$, $1 < \gamma \leq 4/3$ and n is an odd positive integer.

Proof. Assume that $f \in \mathrm{P}_{\mathcal{H}}^{n}(\lambda, \beta, \gamma)$ Then, in view of (3) to (9), we have

$$\mathcal{R}e\left\{\frac{(1-\beta)D_{\lambda}^{n+1}f(z)+\beta D_{\lambda}^{n+2}f(z)}{(1-\beta)D_{\lambda}^{n}f(z)+\beta D_{\lambda}^{n+1}f(z)}\right\}$$

$$= \mathcal{R}e\left\{\frac{\begin{array}{c}(1-\beta)z+\sum_{k=2}^{\infty}(1+(k-1)\lambda)^{n+1}(1-\beta)a_k z^k+(-1)^{n+1}\sum_{k=1}^{\infty}(1+(k-1)\lambda)^{n+1}(1-\beta)b_k z^k\\ +\beta z+\sum_{k=2}^{\infty}(1+(k-1)\lambda)^{n+2}\beta a_k z^k+(-1)^{n+2}\sum_{k=2}^{\infty}(1+(k-1)\lambda)^{n+2}\beta a_k z^k\end{array}}{\begin{array}{c}(1-\beta)z+\sum_{k=2}^{\infty}(1+(k-1)\lambda)^{n}(1-\beta)a_k z^k+(-1)^{n}\sum_{k=1}^{\infty}(1+(k-1)\lambda)^{n}(1-\beta)b_k z^k\\ +\beta z+\sum_{k=2}^{\infty}(1+(k-1)\lambda)^{n+1}\beta a_k z^k+(-1)^{n+1}\sum_{k=2}^{\infty}(1+(k-1)\lambda)^{n+1}\beta a_k z^k\end{array}}\right\}$$

$$= \mathcal{R}e\left\{\frac{z+\sum_{k=2}^{\infty}(1+(k-1)\lambda)^{n+1}(1+\beta(k-1)\lambda)a_k z^k-(-1)^{n}\sum_{k=1}^{\infty}(1+(k-1)\lambda)^{n+1}(1-2\beta-\beta(k-1)\lambda)b_k z^k}{z+\sum_{k=2}^{\infty}(1+(k-1)\lambda)^{n}(1+\beta(k-1)\lambda)a_k z^k+(-1)^{n}\sum_{k=1}^{\infty}(1+(k-1)\lambda)^{n}(1-2\beta-\beta(k-1)\lambda)b_k z^k}\right\} < \gamma$$

If we choose z to be real and let $r \to 1^{-}$, the last inequality leads desired assertion (10) of Theorem 2.1.

Conversely, assume that (10) holds for $f(z) \in \mathrm{P}_{\mathcal{H}}^{n}(\lambda, \beta, \gamma)$, then it suffices to show thats

$$\left|\frac{\frac{(1-\beta)D_{\lambda}^{n+1}f(z)+\beta D_{\lambda}^{n+2}f(z)}{(1-\beta)D_{\lambda}^{n}f(z)+\beta D_{\lambda}^{n+1}f(z)}-1}{\frac{(1-\beta)D_{\lambda}^{n+1}f(z)+\beta D_{\lambda}^{n+2}f(z)}{(1-\beta)D_{\lambda}^{n}f(z)+\beta D_{\lambda}^{n+1}f(z)}-(2\gamma-1)}\right| < 1 \qquad (z \in U)$$

$$\left|\frac{\frac{(1-\beta)D_{\lambda}^{n+1}f(z)+\beta D_{\lambda}^{n+2}f(z)}{(1-\beta)D_{\lambda}^{n}f(z)+\beta D_{\lambda}^{n+1}f(z)}-1}{\frac{(1-\beta)D_{\lambda}^{n+1}f(z)+\beta D_{\lambda}^{n+2}f(z)}{(1-\beta)D_{\lambda}^{n}f(z)+\beta D_{\lambda}^{n+1}f(z)}-(2\gamma-1)}\right| =$$

$$\left|\frac{\frac{z+\sum_{k=2}^{\infty}(1+(k-1)\lambda)^{n+1}(1+\beta(k-1)\lambda)a_k z^k-(-1)^{n}\sum_{k=1}^{\infty}(1+(k-1)\lambda)^{n+1}(1-2\beta-\beta(k-1)\lambda)b_k z^k}{z+\sum_{k=2}^{\infty}(1+(k-1)\lambda)^{n}(1+\beta(k-1)\lambda)a_k z^k+(-1)^{n}\sum_{k=1}^{\infty}(1+(k-1)\lambda)^{n}(1-2\beta-\beta(k-1)\lambda)b_k z^k}-1}{\frac{z+\sum_{k=2}^{\infty}(1+(k-1)\lambda)^{n+1}(1+\beta(k-1)\lambda)a_k z^k-(-1)^{n}\sum_{k=1}^{\infty}(1+(k-1)\lambda)^{n+1}(1-2\beta-\beta(k-1)\lambda)b_k z^k}{z+\sum_{k=2}^{\infty}(1+(k-1)\lambda)^{n}(1+\beta(k-1)\lambda)a_k z^k+(-1)^{n}\sum_{k=1}^{\infty}(1+(k-1)\lambda)^{n}(1-2\beta-\beta(k-1)\lambda)b_k z^k}-(2\gamma-1)}\right|$$

$$= \left|\frac{\begin{array}{c}\sum_{k=2}^{\infty}(1+(k-1)\lambda)^{n}[(1+(k-1)\lambda)(1+\beta(k-1)\lambda)-(1+\beta(k-1)\lambda)]\,a_k z^{k-1}\\ -(-1)^{n}\sum_{k=1}^{\infty}(1+(k-1)\lambda)^{n}[(1+(k-1)\lambda)(1-2\beta-\beta(k-1)\lambda)+(1-2\beta-\beta(k-1)\lambda)]b_k z^{k-1}\end{array}}{\begin{array}{c}(2\gamma-2)-\sum_{k=1}^{\infty}(1+(k-1)\lambda)^{n}[(1+(k-1)\lambda)(1+\beta(k-1)\lambda)-(2\gamma-1)(1+\beta(k-1)\lambda)]a_k z^{k-1}\\ +(-1)^{n}\sum_{k=1}^{\infty}(1+(k-1)\lambda)^{n}[(1+(k-1)\lambda)(1-2\beta-\beta(k-1)\lambda)+(2\gamma-1)(1-2\beta-\beta(k-1)\lambda)]b_k z^{k-1}\end{array}}\right|$$

$$\leq \frac{\begin{array}{c}\sum_{k=2}^{\infty}(1+(k-1)\lambda)^{n}[(1+(k-1)\lambda)(1+\beta(k-1)\lambda)-(1+\beta(k-1)\lambda)]\,a_k|z|^{k-1}\\ +\sum_{k=1}^{\infty}(1+(k-1)\lambda)^{n}[(1+(k-1)\lambda)(1-2\beta-\beta(k-1)\lambda)+(1-2\beta-\beta(k-1)\lambda)]b_k|z|^{k-1}\end{array}}{\begin{array}{c}(2\gamma-2)-\sum_{k=1}^{\infty}(1+(k-1)\lambda)^{n}[(1+(k-1)\lambda)(1+\beta(k-1)\lambda)-(2\gamma-1)(1+\beta(k-1)\lambda)]a_k|z|^{k-1}\\ -\sum_{k=1}^{\infty}(1+(k-1)\lambda)^{n}[(1+(k-1)\lambda)(1-2\beta-\beta(k-1)\lambda)+(2\gamma-1)(1-2\beta-\beta(k-1)\lambda)]b_k|z|^{k-1}\end{array}}$$

The last expression is bounded above by 1 if

$$\sum_{k=2}^{\infty}(1+(k-1)\lambda)^n[(1+(k-1)\lambda)(1+\beta(k-1)\lambda)-(1+\beta(k-1)\lambda)]\,a_k|z|^{k-1}$$
$$+\sum_{k=1}^{\infty}(1+(k-1)\lambda)^n[(1+(k-1)\lambda)(1-2\beta-\beta(k-1)\lambda)$$
$$+(1-2\beta-\beta(k-1)\lambda)]b_k|z|^{k-1}$$
$$\leq\Bigg\{(2\gamma-2)-\sum_{k=1}^{\infty}(1+(k-1)\lambda)^n[(1+(k-1)\lambda)(1+\beta(k-1)\lambda)$$
$$-(2\gamma-1)(1+\beta(k-1)\lambda)]a_k|z|^{k-1}$$
$$-\sum_{k=1}^{\infty}(1+(k-1)\lambda)^n[(1+(k-1)\lambda)(1-2\beta-\beta(k-1)\lambda)$$
$$+(2\gamma-1)(1-2\beta-\beta(k-1)\lambda)]b_k|z|^{k-1}\Bigg\}$$

which is equivalent to

$$\sum_{k=2}^{\infty}[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]\,a_k$$
$$+\sum_{k=1}^{\infty}[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda+\gamma]\,b_k$$
$$\leq\ \gamma-1.$$

The result is sharp for

$$f(z)$$
$$=z+\sum_{k=2}^{\infty}\frac{\gamma-1}{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}x_kz^k$$
$$+\sum_{k=1}^{\infty}\frac{\gamma-1}{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}y_kz^k \qquad (11)$$

Where $0\leq\beta\leq1, 1<\gamma\leq4/3, n\in N_0$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1.$

3. Distortion Bounds and Extreme Points

By routine procedure ([6,8-10]), we can easily prove the following result:

Theorem 3.1 Let $f_k(z)\in\mathcal{P}_{\mathcal{H}}^n(\lambda,\beta,\gamma)$. Then for $|z|=r<1$, we have

$$|f_k(z)|\leq(1+|b_1|)r+\frac{1}{[1+\lambda]^n}\left[\frac{\gamma-1}{[1+\beta\lambda][1+\lambda-\gamma]}-\frac{(1-2\beta)(1+\gamma)}{[1+\beta\lambda][1+\lambda-\gamma]}|b_1|\right]r^2$$

where n is odd positive integer.

$$|f_k(z)| \geq (1-|b_1|)r - \frac{1}{[1+\lambda]^n}\left[\frac{\gamma-1}{[1+\beta\lambda][1+\lambda-\gamma]} - \frac{(1-2\beta)(1+\gamma)}{[1+\beta\lambda][1+\lambda-\gamma]}|b_1|\right]r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_k(z) \in \mathcal{P}_{\mathcal{H}}^n(\lambda,\beta,\gamma)$ taking the absolute value of f_k, we obtain

$$|f_k(z)| \leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2$$

$$\leq (1+|b_1|)r + \frac{\gamma-1}{[1+\lambda]^n[1+\beta\lambda][1+\lambda-\gamma]}\sum_{k=2}^{\infty}\frac{[1+\lambda]^n[1+\beta\lambda][1+\lambda-\gamma]}{\gamma-1}(|a_k| + |b_k|)r^2$$

$$\leq (1+|b_1|)r + \frac{\gamma-1}{[1+\lambda]^n[1+\beta\lambda][1+\lambda-\gamma]}\sum_{k=2}^{\infty}\left[\frac{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma-1}|a_k| + \frac{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda+\gamma]}{\gamma-1}|b_k|\right]r^2$$

$$\leq (1+|b_1|)r + \frac{\gamma-1}{[1+\lambda]^n[1+\beta\lambda][1+\lambda-\gamma]}\left[1 - \frac{(1-2\beta)(1+\gamma)}{\gamma-1}|b_1|\right]r^2$$

$$\leq (1+|b_1|)r + \frac{1}{[1+\lambda]^n}\left[\frac{\gamma-1}{[1+\beta\lambda][1+\lambda-\gamma]} - \frac{(1-2\beta)(1+\gamma)}{[1+\beta\lambda][1+\lambda-\gamma]}|b_1|\right]r^2$$

Remark : If we put $\lambda = 1$ we get the following result.

$$(1-|b_1|)r - \frac{1}{2^n[1+\beta]}\left[\frac{\gamma-1}{[2-\gamma]} - \frac{(1-2\beta)(1+\gamma)}{[2-\gamma]}|b_1|\right]r^2 \leq |f_k(z)| \leq (1+|b_1|)r + \frac{1}{2^n[1+\beta]}\left[\frac{\gamma-1}{[2-\gamma]} - \frac{(1-2\beta)(1+\gamma)}{[2-\gamma]}|b_1|\right]r^2.$$

It should be noted that the **result given by K. Vijaya etc. [18] is not correct** while the correct result mentioned above.

The following covering result follows from the left hand inequality in theorem.

Corollary 3.2 If $f(z) \in \mathcal{P}_{\mathcal{H}}^n(\lambda,\beta,\gamma)$, then

$$\left\{w: |w| < \frac{[1+\lambda]^n[1+\beta\lambda][1+\lambda-\gamma]-(\gamma-1)}{[1+\lambda]^n[1+\beta\lambda][1+\lambda-\gamma]} - \frac{[1+\lambda]^n[1+\beta\lambda][1+\lambda-\gamma]-(1-2\beta)(\gamma-1)}{[1+\lambda]^n[1+\beta\lambda][1+\lambda-\gamma]}|b_1|\right\} \subset f(U).$$

Next we state that the extreme points of closed convex hulls of $\mathrm{P}_{\mathcal{H}}^{n}(\lambda,\beta,\gamma)$ denoted by $clco\mathrm{P}_{\mathcal{H}}^{n}(\lambda,\beta,\gamma)$.

Theorem 3.3 A function $f_m = h + \overline{g_m}$ be given by (1.3). Then $f_m(z) \in clco\ \mathrm{P}_{\mathcal{H}}^{n}(\lambda,\beta,\gamma)$ if and only if

$$f_m(z) = \sum_{k=1}^{\infty}[x_k h_k(z) + y_k g_{mk}(z)] \tag{12}$$

where $h_1(z) = z$,

$$h_k(z) = z + \frac{\gamma - 1}{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]} z^k (k = 2,3, \dots \dots \dots \dots)$$

$$g_{mk}(z) = z + \frac{\gamma - 1}{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]} z^k\ (k = 1,2, \dots)$$

$$\sum_{k=1}^{\infty}(x_k + y_k) = 1\ , x_k \geq 0\ \ and\ \ y_k \geq 0.$$

In particular, the extreme points of $\mathrm{P}_{\mathcal{H}}^{n}(\lambda,\beta,\gamma)$ are $\{h_k\}$ and $\{g_{nk}\}$.

Proof. For the functions $f_m\ (z)$ of the form of theorem (3.1) , we have

$$\begin{aligned} f_m(z) &= \sum_{k=1}^{\infty}[x_k h_k(z) + y_k g_{mk}(z)] \\ &= \sum_{k=1}^{\infty}(x_k + y_k)\, z \\ &+ \sum_{k=2}^{\infty}\frac{\gamma - 1}{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]} x_k z^k \\ &+ \sum_{k=1}^{\infty}\frac{\gamma - 1}{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]} y_k \bar{z}^k. \end{aligned}$$

Then

$$\sum_{k=2}^{\infty}\frac{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma-1}\left(\frac{\gamma - 1}{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}x_k\right)$$
$$+\sum_{k=1}^{\infty}\frac{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma-1}\left(\frac{\gamma - 1}{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}y_k\right)$$
$$= \sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k = 1 - x_1 \leq 1,$$

and so $f_m(z) \in\ \mathrm{P}_{\mathcal{H}}^{n}(\lambda,\beta,\gamma)$, conversely, if $f_m(z) \in clco\ \mathrm{P}_{\mathcal{H}}^{n}(\lambda,\beta,\gamma)$, then

$$a_k \leq \frac{\gamma - 1}{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}$$

and

$$b_k \leq \frac{\gamma - 1}{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}$$

set

$$x_k = \frac{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma-1} a_k \ (k = 2,3,\dots\dots)$$

$$y_k = \frac{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma-1} b_k, (k = 1,2,\dots\dots)$$

then by Theorem 2.1 $0 \leq x_k \leq 1$ $(k = 2,3,\dots.)$ and $0 \leq y_k \leq 1$ $(k = 1,2,3,\dots.)$. We define $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$ and note that by Theorem 2.1 $x_1 \geq 0$ consequently we obtain

$$f_m(z) = \sum_{k=1}^{\infty}[x_k h_k(z) + y_k g_{mk}(z)]$$

as required.

4. Convolution properties

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f_n(z) = z + \sum_{k=2}^{\infty}|a_k|z^k + \sum_{k=1}^{\infty}|b_k|\bar{z}^k$$

and

$$F_n(z) = z + \sum_{k=2}^{\infty}|A_k|z^k + \sum_{k=1}^{\infty}|B_k|\bar{z}^k.$$

We define the convolution of two harmonic functions $f(z)$ and $F(z)$ as

$$f_n(z) * F_n(z) = z + \sum_{k=2}^{\infty}|a_k||A_k|z^k + \sum_{k=1}^{\infty}|b_k||B_k|\bar{z}^k. \tag{13}$$

Using this definition, we show that the class $\mathrm{P}_{\mathcal{H}}^n(\lambda,\beta,\gamma)$ is closed under convolution.

Theorem 4.1 For $1 < \gamma_2 \leq \gamma_1 \leq \frac{4}{3}$, Let $f_n(z) \in \mathrm{P}_{\mathcal{H}}^n(\lambda,\beta,\gamma_2)$ and $F_n(z) \in \mathrm{P}_{\mathcal{H}}^n(\lambda,\beta,\gamma_1)$ then

$$f_n * F_n \in \mathrm{P}_{\mathcal{H}}^n(\lambda,\beta,\gamma_2) \subset \mathrm{P}_{\mathcal{H}}^n(\lambda,\beta,\gamma_1).$$

Proof. Let $f_n(z) = z + \sum_{k=2}^{\infty}|a_k|z^k + (-1)^n \sum_{k=1}^{\infty}|b_k|\bar{z}^k$ be in $\mathrm{P}_{\mathcal{H}}^{n}(\lambda, \beta, \gamma_2)$ and $F_n(z) = z + \sum_{k=2}^{\infty}|A_k|z^k + (-1)^n \sum_{k=1}^{\infty}|B_k|\bar{z}^k$ be in $\mathrm{P}_{\mathcal{H}}^{n}(\lambda, \beta, \gamma_1)$.

Then the convolution is given by (13) we wish to show that the coefficient of $f_n * F_n$ satisfies the required condition given in Theorem 2.1.

For $F_n(z) \in \mathrm{P}_{\mathcal{H}}^{n}(\lambda, \beta, \gamma_1)$, we noted that $|A_k| < 1$ and $|B_k| < 1$. Now, for the convolution function $f_n * F_n$, we obtain

$$\sum_{k=2}^{\infty} \frac{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma_1 - 1}|a_k||A_k|$$
$$+ \sum_{k=1}^{\infty} \frac{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma_1 - 1}|b_k||B_k|$$
$$\leq \sum_{k=2}^{\infty} \frac{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma_1 - 1}|a_k|$$
$$+ \sum_{k=1}^{\infty} \frac{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma_1 - 1}|b_k|$$
$$\leq \sum_{k=2}^{\infty} \frac{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma_2 - 1}|a_k|$$
$$+ \sum_{k=1}^{\infty} \frac{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma_2 - 1}|b_k|$$
$$\leq 1$$

since $1 < \gamma_2 \leq \gamma_1 \leq \frac{4}{3}$ and $f_n(z) \in \mathrm{P}_{\mathcal{H}}^{n}(\lambda, \beta, \gamma_2)$ therefore $f_n * F_n \in \mathrm{P}_{\mathcal{H}}^{n}(\lambda, \beta, \gamma_2) \subset \mathrm{P}_{\mathcal{H}}^{n}(\lambda, \beta, \gamma_1)$.

Now we show that $\mathrm{P}_{\mathcal{H}}^{n}(\lambda, \beta, \gamma)$ is closed under convex combination of its members.

Theorem 4.2 The class $\mathrm{P}_{\mathcal{H}}^{n}(\lambda, \beta, \gamma)$ is closed under convex combination.

Proof. For $i = 1,2, \ldots \ldots ..$ let $f_{n_i} \in \mathrm{P}_{\mathcal{H}}^{n}(\lambda, \beta, \gamma)$ where f_{n_i} is given by

$$f_{n_i} = z + \sum_{k=2}^{\infty}|a_{k_i}|z^k + \sum_{k=1}^{\infty}|b_{k_i}|\bar{z}^k,$$

then by (2.1)

$$\sum_{k=2}^{\infty}\frac{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma-1}\left|a_{k_i}\right| + \sum_{k=1}^{\infty}\frac{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma-1}\left|b_{k_i}\right| \leq 1 \quad (14)$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$ the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i\, f_{n_i} = z + \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i\, a_{k_i}\right) z^k + \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i\, b_{k_i}\right) \bar{z}^k,$$

Then by (4.2),

$$\sum_{k=2}^{\infty}\frac{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma-1}\left(\sum_{i=1}^{\infty} t_i\, a_{k_i}\right) + \sum_{k=1}^{\infty}\frac{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma-1}\left(\sum_{i=1}^{\infty} t_i\, a_{k_i}\right) \leq 1$$

$$\sum_{i=1}^{\infty} t_i \left\{\sum_{k=2}^{\infty}\frac{[1+(k-1)\lambda]^n[1+\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma-1}\, a_{k_i} + \sum_{k=1}^{\infty}\frac{[1+(k-1)\lambda]^n[1-2\beta-\beta(k-1)\lambda][1+(k-1)\lambda-\gamma]}{\gamma-1}\, b_{k_i}\right\} \leq \sum_{i=1}^{\infty} t_i = 1$$

This is the condition required by (10) and so $\sum_{i=1}^{\infty} t_i\, f_{n_i} \in \mathcal{P}_{\mathcal{H}}^{n}(\lambda,\beta,\gamma)$.

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