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BAYESIAN ESTIMATION OF RELIABILITY FUNCTION OF RAYLEIGH DISTRIBUTION

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Abstract: Rayleigh distribution is considered. Bayesian method of estimation is employed in order to estimate the reliability function of Rayleigh distribution by using non-informative and beta priors. In this paper, the Bayes estimators of the reliability function have been obtained under squared error, precautionary and weighted loss functions.

Keywords: Rayleigh distribution, Reliability, Bayesian method, non-informative and beta priors, squared error, precautionary and weighted loss functions.

1. Introduction

Consider Rayleigh distribution with scale parameter θ. The probability density function $f(x;\theta)$ and distribution function $F(x;\theta)$ are respectively given by (Potdar and Shirke[5])

$$f(x;\theta) = \frac{2x}{\theta^2} e^{-x^2/\theta^2} \quad ; x \geq 0,\ \theta > 0. \tag{1}$$

$$F(x;\theta) = 1 - e^{-x^2/\theta^2} \quad ; x \geq 0,\ \theta > 0. \tag{2}$$

Let $R(t)$ denote the reliability function, that is, the probability that a system will survive a specified time t comes out to be

$$R(t) = e^{-t^2/\theta^2} \quad ; t > 0,\ \theta > 0. \tag{3}$$

And the instantaneous failure rate or hazard rate, h(t) is given by

$$h(t) = \frac{2t}{\theta^2}. \tag{4}$$

From equation (1) and (3), we get

$$f(x;R(t)) = \frac{2x}{t^2}\left[-\log R(t)\right]\left[R(t)\right]^{(x^2)/t^2} \quad ; 0 < R(t) \leq 1 \tag{5}$$

The joint density function or likelihood function of (5) is given by

$$f\left(\underline{x} \mid R(t)\right)=\left(\prod_{i=1}^{n} x_i\right)\left[\frac{-2\log R(t)}{t^2}\right]^n \left[R(t)\right]^{\left(\sum_{i=1}^{n} x_i^2\right)/t^2} \tag{6}$$

The log likelihood function is given by

$$log\ f\left(\underline{x} \mid R(t)\right)=log\left(\prod_{i=1}^{n} x_i\right)+n\,log\left(\frac{2}{t^2}\right)+n\,log\left[-log\,R(t)\right]+\frac{\sum_{i=1}^{n} x_i^2}{t^2}log\left[R(t)\right] \tag{7}$$

Differentiating (7) with respect to R (t) and equating to zero, we get the maximum likelihood estimator of R (t) as

$$\hat{R}(t)=exp\left(\frac{-nt^2}{\sum_{i=1}^{n} x_i^2}\right). \tag{8}$$

2. Bayesian Method of Estimation

The Bayesian estimation procedure have been developed generally under squared error loss function

$$L\left(\hat{R}(t),R(t)\right)=\left(\hat{R}(t)-R(t)\right)^2 \tag{9}$$

where $\hat{R}(t)$ is an estimate of $R(t)$. The Bayes estimator under the above loss function, say $\hat{R}(t)_S$, is the posterior mean, i.e.,

$$\hat{R}(t)_S = E\left[R(t)\right]. \tag{10}$$

The squared error loss function is often used also because it does not lead extensive numerical computation but several authors (Zellner [6], Basu & Ebrahimi [2]) have recognized the inappropriateness of using symmetric loss function. Canfield [3] points out that the use of symmetric loss function may be inappropriate in the estimation of reliability function. Norstrom [4] introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss function with quadratic loss function as a special case. A very useful and simple asymmetric precautionary loss function is

$$L\left(\hat{R}(t), R(t)\right) = \frac{\left(\hat{R}(t) - R(t)\right)^2}{\hat{R}(t)} \tag{11}$$

The Bayes estimator of $R(t)$ under precautionary loss function is denoted by $\hat{R}(t)_P$, and is obtained by solving the following equation

$$\hat{R}(t)_P = \left[E\left(R(t)\right)^2\right]^{\frac{1}{2}} \tag{12}$$

Weighted loss function (Ahamad et al. [1]) is given as

$$L\left(\hat{R}(t), R(t)\right) = \frac{\left(\hat{R}(t) - R(t)\right)^2}{R(t)}. \tag{13}$$

The Bayes estimator of $R(t)$ under weighted loss function is denoted by $\hat{\theta}_W$ and is obtained as

$$\hat{R}(t)_W = \left[E\left(\frac{1}{R(t)}\right)\right]^{-1}. \tag{14}$$

For the situation where we have no prior information about $R(t)$, we may use non-informative prior distribution

$$h_1\left(R(t)\right) = \frac{1}{R(t)\ logR(t)};\quad 0 < R(t) \leq 1. \tag{15}$$

The most widely used prior distribution for $R(t)$ is a beta distribution with parameters $\alpha, \beta > 0$, given by

$$h_2\left(R(t)\right) = \frac{1}{B(\alpha,\beta)}\left[R(t)\right]^{\alpha-1}\left[1 - R(t)\right]^{\beta-1};\quad 0 < R(t) \leq 1. \tag{16}$$

3. Bayes Estimator of $R(t)$ under $h_1\left(R(t)\right)$

Under $h_1\left(R(t)\right)$, the posterior distribution is defined by

$$f\left(R(t)\,|\,\underline{x}\right)=\frac{f\left(\underline{x}\,|\,R(t)\right)h_1\left(R(t)\right)}{\int_0^1 f\left(\underline{x}\,|\,R(t)\right)h_1\left(R(t)\right)dR(t)} \tag{17}$$

Substituting the values of $h_1\left(R(t)\right)$ and $f\left(\underline{x}\,|\,R(t)\right)$ from equations (15) and (6) in (17), we get

$$f\left(R(t)\,|\,\underline{x}\right)=\frac{\left(\prod_{i=1}^{n}x_i\right)\left[\frac{-2\,log\,R(t)}{t^2}\right]^n\left[R(t)\right]^{\left(\sum_{i=1}^{n}x_i^2\right)/t^2}\frac{1}{R(t)\;log\,R(t)}}{\int_0^1\left(\prod_{i=1}^{n}x_i\right)\left[\frac{-2\,log\,R(t)}{t^2}\right]^n\left[R(t)\right]^{\left(\sum_{i=1}^{n}x_i^2\right)/t^2}\frac{1}{R(t)\;log\,R(t)}dR(t)}$$

$$=\frac{\left[R(t)\right]^{\frac{\left(\sum_{i=1}^{n}x_i^2\right)}{t^2}-1}\left[\text{-}log\,R(t)\right]^{n-1}}{\int_0^1\left[R(t)\right]^{\frac{\left(\sum_{i=1}^{n}x_i^2\right)}{t^2}-1}\left[\text{-}log\,R(t)\right]^{n-1}dR(t)}$$

or, $$f\left(R(t)\,|\,\underline{x}\right)=\frac{\left(\sum_{i=1}^{n}x_i^2\Big/t^2\right)^n}{\Gamma(n)}\left[R(t)\right]^{\frac{\left(\sum_{i=1}^{n}x_i^2\right)}{t^2}-1}\left[\text{-}log\,R(t)\right]^{n-1} \tag{18}$$

Theorem 1. Assuming the squared error loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_S=\left(1+\frac{t^2}{\sum_{i=1}^{n}x_i^2}\right)^{-n} \tag{19}$$

Proof. From equation (10), on using (18),

$$\hat{R}(t)_S=E\left[R(t)\right]$$

$$=\int_0^1 R(t)\,f\left(R(t)\,|\,\underline{x}\right)dR(t)$$

$$=\int_0^1 R(t)\frac{\left(\sum_{i=1}^{n}x_i^2 \Big/ t^2\right)^n}{\Gamma(n)}\left[R(t)\right]^{\frac{\left(\sum_{i=1}^{n}x_i^2\right)}{t^2}-1}\left[-log\,R(t)\right]^{n-1}dR(t)$$

$$=\frac{\left(\sum_{i=1}^{n}x_i^2 \Big/ t^2\right)^n}{\Gamma(n)}\int_0^1\left[R(t)\right]^{\frac{\left(\sum_{i=1}^{n}x_i^2\right)}{t^2}}\left[-log\,R(t)\right]^{n-1}dR(t)$$

$$=\frac{\left(\sum_{i=1}^{n}x_i^2 \Big/ t^2\right)^n}{\Gamma(n)}\frac{\Gamma(n)}{\left(\left(\sum_{i=1}^{n}x_i^2 \Big/ t^2\right)+1\right)^n}$$

or, $$\hat{R}(t)_S=\left(1+\frac{t^2}{\sum_{i=1}^{n}x_i^2}\right)^{-n}.$$

Theorem 2. Assuming the precautionary loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_P=\left[1+\frac{2t^2}{\sum_{i=1}^{n}x_i^2}\right]^{-\frac{n}{2}} \tag{20}$$

Proof. From equation (12), on using (18),

$$\hat{R}(t)_P=\left[E\left(R(t)\right)^2\right]^{\frac{1}{2}}$$

$$=\left[\int_0^1\left(R(t)\right)^2 f\left(R(t\,|\,\underline{x})\right)dR(t)\right]^{\frac{1}{2}}$$

$$=\left[\int_0^1 \left(R(t)\right)^2 \frac{\left(\sum_{i=1}^n x_i^2 \big/ t^2\right)^n}{\Gamma(n)} \left[R(t)\right]^{\frac{\left(\sum_{i=1}^n x_i^2\right)}{t^2}-1} \left[-log\, R(t)\right]^{n-1} dR(t)\right]^{\frac{1}{2}}$$

$$=\left[\frac{\left(\sum_{i=1}^n x_i^2 \big/ t^2\right)^n}{\Gamma(n)} \int_0^1 \left[R(t)\right]^{\frac{\left(\sum_{i=1}^n x_i^2\right)}{t^2}+1} \left[-log\, R(t)\right]^{n-1} dR(t)\right]^{\frac{1}{2}}$$

$$=\left[\frac{\left(\sum_{i=1}^n x_i^2 \big/ t^2\right)^n}{\Gamma(n)} \frac{\Gamma(n)}{\left(\left(\sum_{i=1}^n x_i^2 \big/ t^2\right)+2\right)^n}\right]^{\frac{1}{2}}$$

$$=\left[\frac{\left(\sum_{i=1}^n x_i^2 \big/ t^2\right)^n}{\left(\left(\sum_{i=1}^n x_i^2 \big/ t^2\right)+2\right)^n}\right]^{\frac{1}{2}}$$

or, $$\hat{R}(t)_P = \left[1+\frac{2t^2}{\sum_{i=1}^n x_i^2}\right]^{-\frac{n}{2}}.$$

Theorem 3. Assuming the weighted loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_W = \left[1-\frac{t^2}{\sum_{i=1}^n x_i^2}\right]^n \tag{21}$$

Proof. From equation (14), on using (18),

$$\hat{R}(t)_W = \left[E\left(\frac{1}{R(t)}\right)\right]^{-1}$$

$$= \left[\int_0^1 \frac{1}{R(t)} f\left(R(t \mid \underline{x})\right) dR(t)\right]^{-1}$$

$$= \left[\int_0^1 \frac{1}{R(t)} \frac{\left(\sum_{i=1}^{n} x_i^2 \big/ t^2\right)^n}{\Gamma(n)} \left[R(t)\right]^{\frac{\left(\sum_{i=1}^{n} x_i^2\right)}{t^2}-1} \left[\text{-}log\, R(t)\right]^{n-1} dR(t)\right]^{-1}$$

$$= \left[\frac{\left(\sum_{i=1}^{n} x_i^2 \big/ t^2\right)^n}{\Gamma(n)} \int_0^1 \left[R(t)\right]^{\frac{\left(\sum_{i=1}^{n} x_i^2\right)}{t^2}-2} \left[\text{-}log\, R(t)\right]^{n-1} dR(t)\right]^{-1}$$

$$= \left[\frac{\left(\sum_{i=1}^{n} x_i^2 \big/ t^2\right)^n}{\Gamma(n)} \frac{\Gamma(n)}{\left(\left(\sum_{i=1}^{n} x_i^2 \big/ t^2\right)-1\right)^n}\right]^{-1}$$

$$= \left[\frac{\left(\sum_{i=1}^{n} x_i^2 \big/ t^2\right)^n}{\left(\left(\sum_{i=1}^{n} x_i^2 \big/ t^2\right)-1\right)^n}\right]^{-1}$$

or, $$\hat{R}(t)_W = \left[1 - \frac{t^2}{\sum_{i=1}^{n} x_i^2}\right]^n .$$

4. Bayes Estimator of $R(t)$ under $h_2(R(t))$

Under $h_2(R(t))$, the posterior distribution is defined by

$$f(R(t)|\underline{x}) = \frac{f(\underline{x}|R(t))h_2(R(t))}{\int_0^1 f(\underline{x}|R(t))h_2(R(t))dR(t)} \tag{22}$$

Substituting the values of $h_2(R(t))$ and $f(\underline{x}|R(t))$ from equations (16) and (6) in (22), we get

$$f(R(t)|\underline{x}) = \frac{\left(\prod_{i=1}^n x_i\right)\left[\frac{-2\,log\,R(t)}{t^2}\right]^n [R(t)]^{\left(\sum_{i=1}^n x_i^2\right)/t^2} \frac{1}{B(\alpha,\beta)}[R(t)]^{\alpha-1}[1-R(t)]^{\beta-1}}{\int_0^1 \left(\prod_{i=1}^n x_i\right)\left[\frac{-2\,log\,R(t)}{t^2}\right]^n [R(t)]^{\left(\sum_{i=1}^n x_i^2\right)/t^2} \frac{1}{B(\alpha,\beta)}[R(t)]^{\alpha-1}[1-R(t)]^{\beta-1} dR(t)}$$

$$= \frac{[R(t)]^{\left(\sum_{i=1}^n x_i^2/t^2\right)+\alpha-1}[\text{-}log\,R(t)]^n[1-R(t)]^{\beta-1}}{\int_0^1 [R(t)]^{\left(\sum_{i=1}^n x_i^2/t^2\right)+\alpha-1}[\text{-}log\,R(t)]^n[1-R(t)]^{\beta-1} dR(t)}$$

or,
$$f(R(t)|\underline{x}) = \frac{[R(t)]^{\left(\sum_{i=1}^n x_i^2/t^2\right)+\alpha-1}[\text{-}log\,R(t)]^n[1-R(t)]^{\beta-1}}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^n x_i^2/t^2\right)+\alpha+k\right)^{n+1}\right]} \tag{23}$$

Theorem 4. Assuming the squared error loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_S = \left[\frac{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^n x_i^2/t^2\right)+\alpha+1+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^n x_i^2/t^2\right)+\alpha+k\right)^{n+1}}\right] \tag{24}$$

Proof. From equation (10), on using (23),

$$\hat{R}(t)_S = E[R(t)]$$

$$=\int_0^1 R(t)\, f\left(R(t)\,|\,\underline{x}\right) dR(t)$$

$$=\int_0^1 R(t)\frac{\left[R(t)\right]^{\left(\sum_{i=1}^n x_i^2 / t^2\right)+\alpha-1}\left[\text{-}log\,R(t)\right]^n\left[1-R(t)\right]^{\beta-1}}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^n x_i^2 \Big/ t^2\right)+\alpha+k\right)^{n+1}\right]}dR(t)$$

$$=\frac{\int_0^1\left[R(t)\right]^{\left(\sum_{i=1}^n x_i^2 / t^2\right)+\alpha}\left[\text{-}log\,R(t)\right]^n\left[1-R(t)\right]^{\beta-1}dR(t)}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^n x_i^2 \Big/ t^2\right)+\alpha+k\right)^{n+1}\right]}$$

or, $$\hat{R}(t)_S=\left[\frac{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^n x_i^2 \Big/ t^2\right)+\alpha+1+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^n x_i^2 \Big/ t^2\right)+\alpha+k\right)^{n+1}}\right].$$

Theorem 5. Assuming the precautionary loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_P=\left[\frac{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^n x_i^2 \Big/ t^2\right)+\alpha+2+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^n x_i^2 \Big/ t^2\right)+\alpha+k\right)^{n+1}}\right]^{\frac{1}{2}} \qquad (25)$$

Proof. From equation (12), on using (23),

$$\hat{R}(t)_P=\left[E\left(R(t)\right)^2\right]^{\frac{1}{2}}$$

$$=\left[\int_0^1\left(R(t)\right)^2 f\left(R(t\,|\,\underline{x})\right)dR(t)\right]^{\frac{1}{2}}$$

$$= \left[\int_0^1 \left(R(t)\right)^2 \frac{\left[R(t)\right]^{\left(\sum_{i=1}^n x_i^2 / t^2\right)+\alpha-1} \left[\text{-}log\, R(t)\right]^n \left[1-R(t)\right]^{\beta-1}}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k} \left(1 \Big/ \left(\sum_{i=1}^n x_i^2 \Big/ t^2\right)+\alpha+k\right)^{n+1}\right]} dR(t) \right]^{\frac{1}{2}}$$

$$= \left[\frac{\int_0^1 \left[R(t)\right]^{\left(\sum_{i=1}^n x_i^2 / t^2\right)+\alpha+1} \left[\text{-}log\, R(t)\right]^n \left[1-R(t)\right]^{\beta-1} dR(t)}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k} \left(1 \Big/ \left(\sum_{i=1}^n x_i^2 \Big/ t^2\right)+\alpha+k\right)^{n+1}\right]} \right]^{\frac{1}{2}}$$

or, $$\hat{R}(t)_P = \left[\frac{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k} \left(1 \Big/ \left(\sum_{i=1}^n x_i^2 \Big/ t^2\right)+\alpha+2+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k} \left(1 \Big/ \left(\sum_{i=1}^n x_i^2 \Big/ t^2\right)+\alpha+k\right)^{n+1}} \right]^{\frac{1}{2}}.$$

Theorem 6. Assuming the weighted loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_W = \left[\frac{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k} \left(1 \Big/ \left(\sum_{i=1}^n x_i^2 \Big/ t^2\right)+\alpha+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k} \left(1 \Big/ \left(\sum_{i=1}^n x_i^2 \Big/ t^2\right)+\alpha-1+k\right)^{n+1}} \right] \tag{26}$$

Proof. From equation (14), on using (23),

$$\hat{R}(t)_W = \left[E\left(\frac{1}{R(t)}\right)\right]^{-1}$$

$$= \left[\int_0^1 \frac{1}{R(t)} f\left(R(t \mid \underline{x})\right) dR(t) \right]^{-1}$$

$$=\left[\int_0^1 \frac{1}{R(t)} \frac{\left[R(t)\right]^{\left(\sum_{i=1}^{n} x_i^2 / t^2\right)+\alpha-1}\left[-log\, R(t)\right]^n\left[1-R(t)\right]^{\beta-1}}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n} x_i^2 \Big/ t^2\right)+\alpha+k\right)^{n+1}\right]} dR(t)\right]^{-1}$$

$$=\left[\frac{\int_0^1 \left[R(t)\right]^{\left(\sum_{i=1}^{n} x_i^2 / t^2\right)+\alpha-2}\left[-log\, R(t)\right]^n\left[1-R(t)\right]^{\beta-1} dR(t)}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n} x_i^2 \Big/ t^2\right)+\alpha+k\right)^{n+1}\right]}\right]^{-1}$$

$$=\left[\frac{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n} x_i^2 \Big/ t^2\right)+\alpha-1+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n} x_i^2 \Big/ t^2\right)+\alpha+k\right)^{n+1}}\right]^{-1}$$

or,

$$\hat{R}(t)_W=\left[\frac{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n} x_i^2 \Big/ t^2\right)+\alpha+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n} x_i^2 \Big/ t^2\right)+\alpha-1+k\right)^{n+1}}\right].$$

5. Conclusion

We have obtained a number of Bayes estimators of reliability function $R(t)$ of Rayleigh distribution. In equations (19), (20), and (21), we have obtained the Bayes estimators by using non-informative prior and in equations (24), (25), and (26) under beta prior. From the above said equations, it is clear that the Bayes estimators of $R(t)$ depend upon the parameters of the prior distribution. In this case, the risk function and corresponding Bayes risks do not exist.

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