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THE SAMPLE COEFFICIENT OF DETERMINATION AND ITS ADJUSTED VERSION UNDER PITMAN NEARNESS, WHEN DISTURBANCES ARE NOT NECESSARILY NORMAL

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Abstract: In this paper, an attempt has been made to compare sample coefficient of determination and adjusted version under Pitman Nearness Criterion when distribution is not necessarily normal. The detailed theoretical results have been derived based on defined model and sample estimators under Pitman Nearness Criterion. The simulation study has been conducted to strength the theoretical findings.

Keywords: Linear Models, Sample coefficient of determination (R-square), Adjusted R-Square, Pitman Nearness Criterion.

Mathematics subject classification: 62J05

1. Introduction

The reign of minimum mean squares error as a primary criterion of comparison between two competing estimators remain for a long. The fundamental reasons for its popularity are simplicity, mathematical convenience, and straightforwardness. Some anomalies of minimum mean squares criterion are discussed by Rao [19], which highlights the drawback of this criterion while comparison of estimators under consideration. The mean squares error may not always reflect the true essential properties of an estimator(s) under consideration as it places too much emphasis on the large deviation of the estimates from the parameter, which generally occur with smaller probabilities.

Rao [19] with a view to circumvent this advocated the use of criteria based on the probability of nearness of the estimator to the parameter as proposed by Pitman [18]. Afterwards, Pitman Nearness Criterion (PNC), which is based on the fundamentals of the probability of the nearness among competing estimators, is being posed as an alternative concept to mean squared error. The PNC remained in the shadow, may be due to the complexity of the mathematical expressions involved, till Rao [19] renewed the interest by citing it's advantages over traditional criteria. Since then numerous amount of extensive work has been witnessed in the literature, see; Keating and Mason [3], Keating and Mason [6], Keating et al. [5], Kourouklis [9], Chaturvedi and Shalabh [1]. The

Pitman measure of nearness has also been explored for comparing shrinkage estimators; refer, Keating and Mason [5], Van Hoa and Chaturvedi [24], Shalabh [18].

The Sample coefficient of determination is a well-known measure of goodness of the linear regression models. Moreover, its adjusted version is simultaneously used by researchers and social scientists to overcome the limitations of R^2 i.e. approaching to upper limit with increase in explanatory variables in a given model. The R^2 is adjusted by the correction measure based on the degree of freedom leads to Adjusted R^2 (generally known as adjusted R^2). The population coefficient of determination is a non-linear parametric function to which both R^2 and adjusted R^2 serves as consistent estimators as they have identical probability limits. Cramer [2] by employing the exact distribution theory has obtained bias and variances of R-Square and adjusted R-Square assuming disturbances following normal. Further, the exact moments of R^2 and its adjusted version have been obtained by Ohtani and Hasegawa [13], when disturbances follow a multivariate t-distribution. However, no analytical comparison was obtained due to extremely complicated exact results derived therein. Magnus [10], Smith [19], and Ullah [22] have derived r^{th} order of sample coefficient of determination by expressing the R^2 as a ratio of quadratic forms. Srivastava et al. [20] have compared these two popular measures by utilizing large sample asymptotic theory to draw inferences about their behaviours under the criteria of bias and mean squares error (MSE) for disturbances not necessarily normal. Kumar and Srivastava [8] have compared sample coefficient of determination & its adjusted version under Pitman nearness for normal disturbances. PNC has been widely used for comparing proposed estimators with ordinary least square estimators, see; Shalabh [18].

In many applications, the inferences under the conditions of normality lose their validity. The behaviour of most of estimators or measures seems to be entirely different when exposed to the change in assumption on disturbances.

In this paper, an attempt has been made to expose R-Square and adjusted R-Square to the Pitman nearness Criterion, when disturbance term is not necessarily following normal. The layout of the paper is given in ensuing lines; Section 2 pertains to a brief overview of Pitman nearness criteria; in section 3, the model, sample estimators & population parameters corresponding to the goodness of fit are discussed. The comparison among estimators i.e. R^2 and R_a^2 under Pitman nearness criterion have been carried out in section 4. Lastly, some important results pertaining to derivations being carried out are presented in the appendix.

2. An overview of Pitman Nearness criterion

Pitman [15] has forwarded the measure of nearness for comparing competing estimators, the details pertaining to this measure is given as follows:

The Pitman nearness of an estimator $(\hat{\beta})$ with another estimator $(\tilde{\beta})$, for estimating the unknown true population parameter (β), is the probability that $\hat{\beta}$ is closer than $\tilde{\beta}$ to the true parameter (β), Mathematically, one can interpret that $\hat{\beta}$ is closer to β than $\tilde{\beta}$ if,

$$\Pr\left(\left|\hat{\beta}-\beta\right|<\left|\tilde{\beta}-\beta\right|\right)>\frac{1}{2} \tag{1}$$

Further, the aforesaid criterion receives the significant attention of researchers only when Rao [16] has discussed the shortcoming of mean squares error criterion by pointing out the weight age given to large deviation is more than that of smaller ones. It is worthwhile to mention that the probability of occurrence of small deviation is significantly more than that of larger ones in the mean squared criterion.

In light of above facts, Rao [16] argued to prevent the practice of employing criterion based on quadratic loss functions for judging the performance properties at all the instances. Therefore, he has recommended that the use of criterion based on the probability of concentration around the population parameter. Although, Pitman measure of nearness is the only meaningful alternative criterion available to mean squared error for comparing competing regression estimators; absence of methodology to ascertain probabilities disgusted the researchers to add meaning contributions till Mason et al. [11]. Mason et al. [11], under normality conditions, suggested a technique pertaining to determination of Pitman measure for two linear forms of a common random vector. Peddada [14] has worked out an equivalence relation between both criteria i.e. Pitman closeness and Minimum mean squared error. However, Keating and Mason [4], and Sen et al. [17] by imposing some conditions pertaining to the moments of random variables have attempted to derive the exact expression for Pitman nearness criterion along with conditions of the dominance of Stein-rule estimator(s) over maximum likelihood estimator(s). Srivastava and Srivastava [21] have simplified the exact expressions forwarded by Keating and Mason [4] & Sen et al. [17] by deriving an asymptotic approximation for Pitman nearness criterion. They have also demonstrated and matches conditions in the exact expression of the dominance of Stein-rule estimator over maximum likelihood estimator as long as the number of regression coefficients to be estimated are at-least two. In all aforesaid research articles, the conditions derived pertain to the assumption that disturbances follow the normal distribution. But practically speaking in most of the situations the assumption of normality becomes untenable and may lead to some impious consequences. Shalabh [18] employed Pitman nearness criteria for making a comparison of Stein-rule and least squares estimators when disturbances not necessarily following a normal distribution. He found that role played by skewness of the distribution is quite significant and reflected the change in the admissible range of characterizing scalar due to different assumptions about the distribution being followed by disturbances.

Kumar and Srivastava [8] & Keating and Mason [6] have also exhibited Pitman criterion to study the performance properties of a measure of goodness of fit, R-Square and its adjusted version, in linear regression models.

3. The Model Specification, the Estimators and their Properties

Let us consider the following linear regression model

$$y=\alpha e+X\beta+u \tag{2}$$

Where, y is a $n \times 1$ vector of n observations on the dependent variable; e is a $n\times1$ vector with all its elements unity, X is a $n \times p$ column rank matrix of n-observations on p explanatory non-stochastic variables; α & β are associated regression parameters of $n \times 1$ and $p\times 1$ respectively and u is a $n\times 1$ vector of disturbances with mean zero and variance-covariance matrix $\sigma^2 I_n$, where σ^2 is an unknown quantity.

If we write

$$A = \left(I_n - \frac{1}{n} ee'\right) \tag{3}$$

The sample coefficient of determination $R^2 (0 \leq R^2 \leq 1)$ is given by

$$R^2 = \frac{y'AX(X'AX)^{-1}X'Ay}{yA'y} = \frac{SSR}{SSR+SSE} \tag{4}$$

while adjusted version of R-Square is derived as under:

$$R_a^2 = R^2 - r(1-R^2) = R^2 - \frac{p}{n-p-1}(1-R^2) \tag{5}$$

Where,

$$r = \frac{p}{n-p-1} > 0 \tag{6}$$

Both the versions of sample coefficient of determination are frequently used by researchers to measure the goodness of fit in given regression model. The population Coefficient of determination forwarded by Koerts and Abrahamse [7] is as under:

$$\theta = \frac{\beta'X'AX\beta}{n\sigma^2+\beta'X'AX\beta} = \frac{1}{1+\frac{n\sigma^2}{\beta'X'AX\beta}} = \left(1 + \frac{n\sigma^2}{\beta'X'AX\beta}\right)^{-1} \tag{7}$$

4. Efficiency Properties under Pitman Nearness Criteria

As discussed in earlier sections, it is a known fact that R-Square and adjusted R-Square are the consistent estimator of the population parameter (θ). Also, Ohtani and Giles [12] have forwarded the exact expressions pertaining to the risk of R-Square and adjusted R-Square under an absolute error loss function and employed numerical evaluation method to explain that adjusted R-square performs better in case of large samples. In order to compare R^2 and R_a^2 under Pitman nearness criterion for non-normal error distribution, let us first write Pitman nearness as follows:

$$PN = P[(R_a^2 - \theta)^2 < (R^2 - \theta)^2] \tag{8}$$

With a view to have some clear inference, we employ a large sample approximation technique. Further, as per results in appendix, it is easy to re-write the expression as

$$(R^2 - \theta) = v(1-\theta)\left(\frac{f}{n^{\frac{1}{2}}} + \frac{f^*}{n}\right) + O_p(n^{-\frac{3}{2}}) \tag{9}$$

Where,

$f = \frac{1}{n^{1/2} v\sigma^2}[2(1-\theta)u'AX\beta - \theta(u'u - n\sigma^2)]$ (10)

$$f^* = \frac{1}{v\sigma^2}\left[u'M_1u - \frac{2}{n\sigma^2}(1-\theta)(1-2\theta)n\omega u'AX\beta + \frac{\theta(1-\theta)}{n\sigma^2}n^2\omega^2\right] \quad (11)$$

and

$$v = 2\theta(1-\theta) \quad (12)$$

Further,

$$M_1 = \left[AX(X'AX)^{-1}X'A + \theta\frac{ee'}{n} - 4\frac{(1-\theta)^2}{n\sigma^2}AX\beta\beta'X'A\right] \quad (13)$$

Utilizing (9), it is easy to express

$$\begin{aligned} W_1 &= \frac{n^{\frac{3}{2}}}{2vp(1-\theta)^2}[(R^2-\theta)^2 - (R_a^2-\theta)^2] \\ &= \frac{n^{\frac{3}{2}}}{2vp(1-\theta)^2}\left[(R^2-\theta)^2 - \left[R^2 - \frac{p}{n-p-1}(1-R^2) - \theta\right]^2\right] \\ &= \frac{n^{\frac{3}{2}}}{2vp(1-\theta)^2}\left[2(R^2-\theta) - \frac{p}{n}\left(1-\frac{p+1}{n}\right)^{-1}(1-R^2)\right]\left[\frac{p(1-R^2)}{n-p-1}\right] \\ &= f + \frac{1}{n^{1/2}}\left[f^* - \frac{p}{2v} - vf^2\right] + O(n^{-1}) \end{aligned} \quad (14)$$

$$W_1 = A_1 + \frac{1}{n^{1/2}}B_1 + O(n^{-1}) \quad (15)$$

Where,

$$A_1 = f \quad (16)$$

$$B_1 = f^* - \frac{p}{2v} - vf^2 \quad (17)$$

With a view to generate the probability density function (pdf) of W1 the method of cumulative generating function (CGF) is being utilized. Further, using expectations derive in appendix; the cumulants of W1 can be generated up-to-the order $O\left(n^{-\frac{1}{2}}\right)$ as

$$C_1 = E(W_1) = E(A_1) + \frac{1}{n^{\frac{1}{2}}}E(B_1) = \frac{1}{vn^{\frac{1}{2}}}\left[\frac{p}{2} + \theta(4\theta-5) + \theta(1-2\theta)\gamma_2\right] + O(n^{-1}) \quad (18)$$

$$\begin{aligned} C_2 = E[W_1 - E(W_1)]^2 &= E(A_1{}^2) + \frac{2}{n^{\frac{1}{2}}}[E(A_1B_1) - E(A_1)E(B_1)] \\ &= \frac{1}{vn^{\frac{1}{2}}}[\theta^2\gamma_2 - 2\theta^2 + 4\theta] + O(n^{-1}) \end{aligned} \quad (19)$$

$$C_3 = E[W_1 - E(W_1)]^3 = E\left({A_1}^3\right) + \frac{3}{n^{\frac{1}{2}}}\left[E\left({A_1}^2 B_1\right) - E\left({A_1}^2\right)E(B_1)\right]$$
$$= -\left(\frac{\theta^3}{v^3 n^{\frac{1}{2}}}\right)J_1 - \left(\frac{12\theta^2}{v^3 n^{\frac{1}{2}}}\right)[\gamma_2 + 2] + \left(\frac{3\theta}{v^3 n^{\frac{1}{2}}}\right)J_2 + O(n^{-1}) \tag{20}$$

Where,

$$J_1 = \gamma_4 + 24\,\gamma_2 + 10{\gamma_1}^2 + 32 \tag{21}$$

$$J_2 = \Big[\gamma_2(-24\theta - 32\theta^3 + 41\theta^2 + 4) + {\gamma_2}^2(3\theta^2) + 178\theta^2 - 96\theta^3 - 76\theta + 8$$
$$+ p\left(8\theta - 6 - \frac{\theta\gamma_2}{2}\right) - (4 - 2\theta + \theta\gamma_2)\{p + \theta(1-\theta)\gamma_2 - \theta(1-2\theta)\}\Big] \tag{22}$$

and so on. Further, like C3, it can be observed that all the higher order cummulants of W1 up-to-the order $O(n^{-1/2})$. Further, using aforesaid results, the cumulant generating function (CGF) of W_1 can be expressed as under:

$$CGF = C(t) = \sum_{j=1} \frac{(it)^j}{j!} C_j = \frac{(it)^1 C_1}{1!} + \frac{(it)^2 C_2}{2!} + \frac{(it)^3 C_3}{3!} + \cdots + O(n^{-1})$$
$$= \frac{it}{n^{1/2}}\left[\frac{p}{2} + \theta(4\theta - 5) + \theta(1-2\theta)\gamma_2\right]$$
$$+ \frac{i^2 t^2}{2}\left[\frac{\theta^2\gamma_2 - 2\theta^2 + 4\theta}{v^2}\right]$$
$$+ \frac{i^3 t^3}{6}\left[-\left(\frac{\theta^3}{v^3 n^{1/2}}\right)J_1 - \left(\frac{12\theta^2}{v^3 n^{1/2}}\right)(\gamma_2 + 2) + \left(\frac{3\theta}{v^3 n^{1/2}}\right)J_2\right]$$
$$+ O(n^{-1}) = -\frac{t^2}{2}a + \frac{1}{n^{1/2}}\left[itb + \frac{i^3 t^3}{6}c\right] \tag{23}$$

Where,

$$a = \frac{\theta^2\gamma_2 - 2\theta^2 + 4\theta}{v^2} \tag{24}$$

$$b = \frac{1}{v}\left(\frac{p}{2} + \theta(4\theta - 5) + \theta(1-2\theta)\gamma_2\right) \tag{25}$$

and

$$c = -\left(\frac{\theta^3}{v^3}\right)J_1 - \left(\frac{12\theta^2}{v^3}\right)(\gamma_2 + 2) + \left(\frac{3\theta}{v^3}\right)J_2 \tag{26}$$

Accordingly, the characteristic function of W_1 up-to-the $O(n^{-1})$ is obtained as

$$\phi(t) = \exp[C(t)] = \exp\left[-\frac{t^2}{2}a + \frac{1}{n^{\frac{1}{2}}}\left(itb + \frac{i^3t^3}{6}c\right)\right] + O(n^{-1})$$
$$= \exp\left[-\frac{t^2}{2}a\right] exp\left[\frac{1}{n^{\frac{1}{2}}}\left(itb + \frac{i^3t^3}{6}c\right)\right] + O(n^{-1})$$
$$= \exp\left[-\frac{t^2}{2}a\right] + \left[\frac{1}{n^{\frac{1}{2}}}\left(itb + \frac{i^3t^3}{6}c\right)\right] exp\left[-\frac{t^2}{2}a\right] + O(n^{-1}) \tag{27}$$

Now, inverting the characteristic function $\phi(t)$ to obtain probability generating function (pdf) of W_1 as

$$g(W_1) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp[-itW_1]\phi(t)dt \tag{28}$$

along with some results pertaining to integral part of expression as

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-at^2/2}e^{-itW_1}dt = f(W_1) \tag{29}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} te^{-at^2/2}e^{-itW_1}dt = -\left[\frac{iW_1}{a}\right]f(W_1) \tag{30}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} t^2e^{-at^2/2}e^{-itW_1}dt = -\left[\frac{i^2{W_1}^2}{a^2}\right]f(W_1) \tag{31}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} t^3e^{-at^2/2}e^{-itW_1}dt = -\left[\frac{3W_1}{a^2} - \frac{{W_1}^3}{a^3}\right]f(W_1) \tag{32}$$

By utilizing above results, we can obtain the approximate expression for $g(W_1)$ as

$$g(W_1) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi(t)\exp - itW_1 dt$$
$$= f(W_1) + \frac{1}{n^{\frac{1}{2}}}\left[\left\{b + \left(\frac{W_1^2}{a^2} - \frac{3}{a}\right)c\right\}\frac{W_1}{a}\right]f(W_1) + O(n^{-1}) \tag{33}$$

Where, f(W1) denotes the pdf of a normal variate with mean zero and $\sigma^2 = a > 0$. In case, all non-normality parameters are zero, ‘a' becomes 1 (refer, 25). Thus, we obtained the expression up-to-the $O(n^{-\frac{1}{2}})$ as

$$
\begin{aligned}
P[\mathrm{W}_1 > 0] &= \int_0^{+\infty} \mathrm{g}(\mathrm{W}_1)\mathrm{dW}_1 \\
&= \int_0^{+\infty} \Bigg[\mathrm{f}(\mathrm{W}_1) \\
&\quad + \frac{1}{\mathrm{n}^{\frac{1}{2}}}\left\{\left[\mathrm{b} + \left(\frac{\mathrm{W}_1^2}{\mathrm{a}^2} - 3\right)\mathrm{c}\right]\frac{\mathrm{W}_1}{\mathrm{a}}\right\}\mathrm{f}(\mathrm{W}_1)\Bigg]\mathrm{dW}_1 \\
&= \int_0^{+\infty} \mathrm{f}(\mathrm{W}_1)\,\mathrm{dW}_1 + \frac{1}{\mathrm{n}^{\frac{1}{2}}}\left\{\left[\mathrm{b} + \left(\frac{\mathrm{W}_1^2}{\mathrm{a}^2} - 3\right)\mathrm{c}\right]\frac{\mathrm{W}_1}{\mathrm{a}}\right\}\mathrm{f}(\mathrm{W}_1)\mathrm{dW}_1 \\
&= 0.5 + \frac{1}{\sqrt{2\pi}\mathrm{n}^{1/2}\mathrm{a}^{5/2}}\left[\mathrm{b} - \left(\frac{3\mathrm{a}^2 - 2}{6\mathrm{a}^3}\right)\right]\mathrm{c}
\end{aligned}
\tag{34}
$$

Thus, it is seen that adjusted R-Square is better to R-Square w.r.t aforesaid criterion of Pitman nearness up-to the order $O\left(n^{-1/2}\right)$as long as

$$\left[\mathrm{b} - \left(\frac{3\mathrm{a}^2-2}{6\mathrm{a}^3}\right)\right]\mathrm{c} > 0 \tag{35}$$

In case of normal distribution, all non-normality parameters become zero, the expression (34) reduce to

$$P[\mathrm{W}_1 > 0] = \frac{1}{2} + \frac{6\mathrm{b}_\mathrm{n} - \mathrm{c}_\mathrm{n}}{6\mathrm{n}^{1/2}\sqrt{2\pi}} \tag{36}$$

$$\mathrm{b}_\mathrm{n} = \frac{1}{\mathrm{v}}\left[\frac{\mathrm{p}}{2} + \theta(4\theta - 5)\right] \tag{37}$$

$$
\begin{aligned}
\mathrm{c}_\mathrm{n} &= -\left(\frac{32\theta^3}{\mathrm{v}^3\mathrm{n}^{1/2}}\right) - \left(\frac{24\theta^2}{\mathrm{v}^3\mathrm{n}^{1/2}}\right) \\
&\quad + \left(\frac{3\theta}{\mathrm{v}^3\mathrm{n}^{1/2}}\right)\big[178\theta^2 - 96\theta^3 - 76\theta + 8 + \mathrm{p}(8\theta - 6) \\
&\quad - (4 - 2\theta)\{(\mathrm{p} - \theta(1 - 2\theta)\}\big]
\end{aligned}
\tag{38}
$$

Further, simulation study (10,000 simulation runs) has been carried out by using (35) and tables generated based on the probability of existence of (35) for different values of n, p, γ_1 and γ_1 are tabulated as table 5.1, 5.2, 5.3, 5.4, 5.5 & 5.6. The results of simulations completely support the theoretical findings. The table 5.1 clearly depicts that with increase in value of n and p the probability of dominance of adjusted R-Square keeps increasing. Further, with increase in values of γ_2 for same combination of n, p and γ_1 the probability of dominance keep on decreasing. For given value of n = 10 and γ_2 = 3, 6, 9, 12 in table 5.5 and 5.6 the dominance probability significantly deceases for n = 10 and γ_2 = 12 with smaller value of p.

Table 5.1: The dominance probability of Adjusted R-Square over R-Square for different values of n, p and γ_1 for $\gamma_2 = 3$ & $\gamma_4 = 0$

n	p	γ_1						
		-3	**-2**	**-1**	**0**	**1**	**2**	**3**
2	1	0.6926	0.6715	0.6523	0.6476	0.6540	0.6711	0.6911
4	1	0.6937	0.6731	0.6553	0.6478	0.6546	0.6706	0.6949
4	2	0.6937	0.6741	0.6627	0.6541	0.6591	0.6757	0.6946
4	3	0.6958	0.6780	0.6659	0.6597	0.6650	0.6799	0.6958
6	1	0.6900	0.6712	0.6547	0.6452	0.6529	0.6694	0.6921
6	2	0.6940	0.6725	0.6603	0.6541	0.6616	0.6730	0.6936
6	3	0.6951	0.6786	0.6651	0.6582	0.6628	0.6771	0.6942
6	4	0.6969	0.6808	0.6668	0.6652	0.6715	0.6813	0.6986
6	5	0.6982	0.6815	0.6745	0.6692	0.6717	0.6823	0.6964
8	1	0.6912	0.6713	0.6526	0.6499	0.6543	0.6708	0.6929
8	2	0.6929	0.6759	0.6583	0.6540	0.6585	0.6745	0.6935
8	3	0.6951	0.6763	0.6655	0.6598	0.6673	0.6788	0.6974
8	4	0.6984	0.6811	0.6700	0.6655	0.6699	0.6788	0.6969
8	5	0.6968	0.6796	0.6701	0.6664	0.6692	0.6825	0.6983
8	6	0.7011	0.6855	0.6759	0.6712	0.6746	0.6859	0.7001
8	7	0.7000	0.6870	0.6785	0.6742	0.6775	0.6893	0.7019

Table 5.2: The dominance probability of Adjusted R-Square over R-Square for different values of n, p and γ_1 for $\gamma_2 = 6$ & $\gamma_4 = 0$

n	p	γ_1						
		-3	**-2**	**-1**	**0**	**1**	**2**	**3**
2	1	0.6209	0.5870	0.5644	0.5536	0.5637	0.5878	0.6207
4	1	0.6244	0.5891	0.5619	0.5509	0.5616	0.5876	0.6195
4	2	0.6415	0.6118	0.5898	0.5858	0.5939	0.6147	0.6422
4	3	0.6562	0.6342	0.6165	0.6136	0.6182	0.6360	0.6581
6	1	0.6228	0.5890	0.5638	0.5544	0.5640	0.5908	0.6229
6	2	0.6385	0.6129	0.5950	0.5865	0.5933	0.6155	0.6416
6	3	0.6581	0.6321	0.6191	0.6149	0.6213	0.6335	0.6596
6	4	0.6713	0.6483	0.6398	0.6364	0.6364	0.6519	0.6706
6	5	0.6811	0.6665	0.6524	0.6489	0.6524	0.6660	0.6797
8	1	0.6244	0.5867	0.5634	0.5570	0.5658	0.5900	0.6213
8	2	0.6417	0.6138	0.5955	0.5882	0.5927	0.6137	0.6413
8	3	0.6552	0.6362	0.6203	0.6147	0.6185	0.6341	0.6595
8	4	0.6710	0.6525	0.6391	0.6339	0.6355	0.6510	0.6703
8	5	0.6785	0.6615	0.6543	0.6513	0.6527	0.6661	0.6821
8	6	0.6887	0.6767	0.6666	0.6611	0.6679	0.6767	0.6933
8	7	0.7014	0.6873	0.6768	0.6744	0.6799	0.6872	0.6980

Table 5.3: The dominance probability of Adjusted R-Square over R-Square for different values of n, p and γ_1 for $\gamma_2 = 9$ & $\gamma_4 = 0$

n	**p**	γ_1						
		-3	**-2**	**-1**	**0**	**1**	**2**	**3**
2	1	0.4606	0.3962	0.3538	0.3311	0.3500	0.3929	0.4634
4	1	0.4639	0.3942	0.3483	0.3353	0.3509	0.3971	0.4631
4	2	0.5234	0.4691	0.4246	0.4074	0.4258	0.4654	0.5241
4	3	0.5659	0.5253	0.4945	0.4860	0.4961	0.5249	0.5675
6	1	0.4647	0.3951	0.3480	0.3339	0.3492	0.3947	0.4626
6	2	0.5253	0.4672	0.4256	0.4078	0.4234	0.4648	0.5223
6	3	0.5710	0.5276	0.4950	0.4851	0.4968	0.5265	0.5718
6	4	0.6045	0.5764	0.5511	0.5429	0.5523	0.5733	0.6070
6	5	0.6365	0.6109	0.5968	0.5864	0.5963	0.6106	0.6340
8	1	0.4619	0.3965	0.3483	0.3341	0.3492	0.3934	0.4644
8	2	0.5240	0.4686	0.4224	0.4088	0.4248	0.4652	0.5273
8	3	0.5710	0.5287	0.4949	0.4814	0.4949	0.5271	0.5699
8	4	0.6087	0.5785	0.5544	0.5444	0.5507	0.5772	0.6074
8	5	0.6365	0.6130	0.5932	0.5869	0.5964	0.6126	0.6369
8	6	0.6579	0.6396	0.6271	0.6215	0.6285	0.6382	0.6569
8	7	0.6768	0.6605	0.6528	0.6485	0.6539	0.6595	0.6763

Table 5.4: The dominance probability of Adjusted R-Square over R-Square for different values of n, p and γ_1 for $\gamma_2 = 12$ & $\gamma_4 = 0$

n	**p**	γ_1						
		-3	**-2**	**-1**	**0**	**1**	**2**	**3**
2	1	0.2044	0.1462	0.1097	0.0992	0.1104	0.1426	0.2030
4	1	0.2046	0.1475	0.1100	0.0996	0.1102	0.1429	0.2042
4	2	0.2681	0.2053	0.1665	0.1553	0.1691	0.2042	0.2697
4	3	0.3422	0.2732	0.2357	0.2200	0.2337	0.2751	0.3424
6	1	0.2037	0.1444	0.1112	0.0976	0.1106	0.1438	0.2069
6	2	0.2679	0.2057	0.1679	0.1553	0.1690	0.2047	0.2685
6	3	0.3415	0.2740	0.2354	0.2199	0.2351	0.2742	0.3428
6	4	0.4292	0.3575	0.3104	0.2973	0.3125	0.3598	0.4269
6	5	0.5086	0.4502	0.4024	0.3870	0.4049	0.4498	0.5078
8	1	0.2031	0.1443	0.1117	0.0976	0.1107	0.1451	0.2067
8	2	0.2688	0.2059	0.1672	0.1547	0.1689	0.2065	0.2666
8	3	0.3442	0.2791	0.2330	0.2212	0.2331	0.2762	0.3449
8	4	0.4298	0.3561	0.3135	0.2937	0.3156	0.3589	0.4273
8	5	0.5087	0.4464	0.4066	0.3906	0.4033	0.4493	0.5082
8	6	0.5740	0.5318	0.5017	0.4880	0.4990	0.5332	0.5751
8	7	0.6208	0.5914	0.5729	0.5649	0.5732	0.5943	0.6190

Table 5.5: The dominance probability of Adjusted R-Square over R-Square for different values of n, p, γ_2 and γ_1 for $n = 10$ & $\gamma_4 = 0$

γ_2	p	γ_1						
		-3	**-2**	**-1**	**0**	**1**	**2**	**3**
3	1	0.6952	0.6719	0.6535	0.6495	0.6540	0.6718	0.6935
3	2	0.6951	0.6770	0.6647	0.6547	0.6600	0.6727	0.6944
3	3	0.6962	0.6798	0.6628	0.6620	0.6641	0.6772	0.6971
3	4	0.6961	0.6805	0.6700	0.6650	0.6696	0.6801	0.6985
3	5	0.6988	0.6821	0.6740	0.6686	0.6738	0.6830	0.7009
3	6	0.7003	0.6849	0.6771	0.6722	0.6734	0.6853	0.7025
3	7	0.7030	0.6867	0.6804	0.6746	0.6744	0.6885	0.7006
3	8	0.7008	0.6863	0.6792	0.6747	0.6798	0.6872	0.7026
3	9	0.7039	0.6879	0.6812	0.6777	0.6839	0.6933	0.7046
6	1	0.6218	0.5896	0.5647	0.5520	0.5632	0.5872	0.6223
6	2	0.6418	0.6135	0.5952	0.5867	0.5931	0.6132	0.6405
6	3	0.6536	0.6356	0.6176	0.6140	0.6190	0.6360	0.6594
6	4	0.6700	0.6485	0.6365	0.6333	0.6365	0.6530	0.6695
6	5	0.6831	0.6673	0.6546	0.6520	0.6556	0.6642	0.6781
6	6	0.6882	0.6778	0.6670	0.6629	0.6689	0.6779	0.6925
6	7	0.6993	0.6862	0.6785	0.6767	0.6781	0.6829	0.6967
6	8	0.7044	0.6935	0.6862	0.6833	0.6883	0.6955	0.7028
6	9	0.7095	0.7015	0.6966	0.6949	0.6924	0.7016	0.7114

Table 5.6: The dominance probability of Adjusted R-Square over R-Square for different values of n, p, γ_2 and γ_1 for $n = 10$ & $\gamma_4 = 0$

γ_2	p	γ_1						
		-3	**-2**	**-1**	**0**	**1**	**2**	**3**
9	1	0.4638	0.3963	0.3481	0.3323	0.3501	0.3941	0.4653
9	2	0.5213	0.4658	0.4248	0.4085	0.4266	0.4676	0.5219
9	3	0.5693	0.5287	0.4936	0.4829	0.4952	0.5277	0.5671
9	4	0.6095	0.5751	0.5533	0.5451	0.5516	0.5781	0.6025
9	5	0.6358	0.6090	0.5922	0.5882	0.5941	0.6153	0.6374
9	6	0.6569	0.6389	0.6289	0.6223	0.6293	0.6419	0.6584
9	7	0.6755	0.6629	0.6516	0.6487	0.6531	0.6637	0.6727
9	8	0.6888	0.6801	0.6714	0.6683	0.6711	0.6770	0.6897
9	9	0.7034	0.6928	0.6880	0.6852	0.6844	0.6947	0.7009
12	1	0.2039	0.1453	0.1103	0.0995	0.1092	0.1457	0.2049
12	2	0.2675	0.2052	0.1663	0.1556	0.1673	0.2054	0.2685
12	3	0.3445	0.2764	0.2345	0.2194	0.2351	0.2755	0.3425
12	4	0.4281	0.3586	0.3115	0.3003	0.3136	0.3604	0.4285
12	5	0.5081	0.4497	0.4050	0.3887	0.4073	0.4482	0.5099
12	6	0.5731	0.5323	0.4980	0.4900	0.5004	0.5327	0.5724
12	7	0.6208	0.5949	0.5715	0.5672	0.5740	0.5945	0.6180
12	8	0.6514	0.6325	0.6226	0.6144	0.6231	0.6335	0.6529
12	9	0.6790	0.6649	0.6522	0.6498	0.6538	0.6619	0.6780

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Appendix: Derivations of results

Let us re-write the model (2) as

$$y = X\beta + \sigma v \; (u = \sigma v) \tag{39}$$

Where, elements of v are i.i.d and $\gamma_1, (3+\gamma_2), (\gamma_3 + 10\,\gamma_1{}^2)$ and $(\gamma_4 + 10\,\gamma_1{}^2 + 15\gamma_2 + 15)$ denote $3^{rd}, 4^{th}, 5^{th}$ and 6^{th}moments of the distribution of disturbance term respectively, we have

$$E\big(v'Cv\big) = tr(C) \tag{40}$$

$$E\big(v'Cvv\big) = \gamma_1 (I * C)e \tag{41}$$

$$E\big(v'C_1vv'C_2v\big) = \gamma_2 (I * C_1)C_2 + trC_1 trC_2 + 2trC_1C_2 \tag{42}$$

$$E\big(v'vv'Cvv'Cv\big) = \gamma_4 (I * C)C + \gamma_2 2[(n+8)tr(I * C)C + 2(trC)^2 + 4trC^2] - 2\gamma_1{}^2\left[2tr(I * C)C + 3e'(I * C)^2 e\right] + (n+4)[(trC)^2 + 2trC^2] \tag{43}$$

Where, C, C1, C2 are non-stochastic symmetric matrices of order $n \times 4$ (see, Ullah et al. [32] and Shalabh [23]). Some useful results for further derivations in this section using equations (40-43) are given by

$$E(u'u) = \sigma^2 [n] \tag{44}$$

$$E(u'uu')AX\beta = \sigma^3 [\gamma_1 e'(I * I)AX\beta = 0 \tag{45}$$

$$E\big(u'uu'u\big) = \sigma^4 [\gamma_2 n + n^2 + 2n] \tag{46}$$

$$E\big(u'uu'uu'u\big) = \sigma^6 [\gamma_4 n + \gamma_2 (3n^2 + 12n) + 10\gamma_1{}^2 n + n^3 + 6n^2 + 8n] \tag{47}$$

$$E\big(u'uu'uu'uu'u\big) = \sigma^8 [n^4 + 12n^3 + 44n^2 + 48n + \gamma_2 (6n^3 + 60n^2 + 144n) + \gamma_4 (4n^2 + 24n) + \gamma_6 n + \gamma_1{}^2 (40n^2 + 240n) + \gamma_2{}^2 (3n^2 + 32n) + 56\gamma_1\gamma_2 n] \tag{48}$$

$$E\big(u'uu'uu'uu'll'u\big) = \sigma^{10}\left(\frac{\theta}{1-\theta}\right)[n^4 + 12n^2 + 44n^2 + 48 + \gamma_4 (4n^2 + 24n) + \gamma_2 (6n^3 + 60n^2 + 144n) + \gamma_6 n + \gamma_1{}^2 (40n^2 + 240n) + \gamma_2{}^2 (3n^2 + 32n) + 56\gamma_1\gamma_3 n] \tag{49}$$

Also, rewriting

$$\theta(n\sigma^2 + \beta'X'AX\beta) = \beta'X'AX\beta \Rightarrow n\sigma^2\left(\frac{\theta}{1-\theta}\right) = \beta'X'AX\beta \tag{50}$$

Now, using the model (2) and equations (4), (7), we get

$$R^2-\theta=\frac{(\alpha e'+\beta'X'+u')AX\left(X'A'X\right)^{-1}X'A(\alpha e+X\beta+u)}{(\alpha e'+\beta'X'+u')A(\alpha e+X\beta+u)}$$
$$=\frac{\beta'X'A'X\beta+u'AX\beta+u'AX\left(X'A'X\right)^{-1}X'Au+\beta'X'Au}{\beta'X'A'X\beta+\beta'X'Au+u'AX\beta+u'Au)}$$
$$=\left(\frac{1-\theta}{n\sigma}\right)\Big[2(1-\theta)u'AX\beta+u'AX\left(X'A'X\right)^{-1}X'Au+n^{-1}\theta(u'e)^2$$
$$-\theta n\omega\Big]\left[1+\frac{1-\theta}{n\sigma}n\omega-(1-\theta)2u'AX\beta-\frac{1-\theta}{n^2\sigma^2}(u'e)^2\right]^{-1}$$
$$=(1-\theta)\left[\frac{f}{n^{1/2}}-\frac{f^*}{n}\right] \tag{51}$$

Where

$$f=\frac{1}{n^{1/2}v\sigma^2}(2(1-\theta)u'AX\beta-\theta n\omega) \tag{52}$$

$$f^*=\frac{1}{v\sigma^2}\left[u'Mu-\frac{2}{n\sigma^2}(1-\theta)(1-2\theta)n\omega u'AX\beta+\frac{\theta(1-\theta)}{n\sigma^2}n^2\omega^2\right] \tag{53}$$

$$M_1=\left[AX\left(X'AX\right)^{-1}X'A-\frac{\theta}{n}ee'-\frac{4(1-\theta)^2}{n\sigma^2}AX\beta\beta'X'A\right] \tag{54}$$

$$v^2=2\theta(2-\theta) \tag{55}$$

and

$$\omega=\left(\frac{u'u}{n}-\sigma^2\right) \tag{56}$$

Let us allow to write following quantities:

$$C_1=\frac{1}{n^{1/2}v\sigma^2}2(1-\theta)u'l \tag{57}$$

$$C_2=-\frac{1}{n^{1/2}v\sigma^2}\theta\left(u'u-n\sigma^2\right) \tag{58}$$

$$D_1=\frac{1}{v\sigma^2}u'AX(X'AX)^{-1}X'Au \tag{59}$$

$$D_2=\frac{\theta}{nv\sigma^2}u'ee'u \tag{60}$$

$$D_3=-\frac{4(1-\theta)^2}{nv\sigma^4}u'll'u \tag{61}$$

$$D_4=-\frac{2(1-\theta)(1-2\theta)}{nv\sigma^4}\left(u'u-n\sigma^2\right)u'l \tag{62}$$

$$D_5=\frac{\theta(1-\theta)}{nv\sigma^4}(u'u-n\sigma^2)^2 \tag{63}$$

The expectations of aforesaid quantities are as follows

$$E(C_1) = \frac{1}{n^{\frac{1}{2}} v\sigma^2} 2(1-\theta)E(u')l = 0$$

$$E(C_2) = -\frac{1}{n^{1/2} v\sigma^2} \theta\left(E(u'u) - n\sigma^2\right) = 0$$

$$E(D_1) = \frac{1}{v\sigma^2} E\left(u'AX(X'AX)^{-1}X'Au\right) = \frac{p}{v}$$

$$E(D_2) = \frac{\theta}{nv\sigma^2} E(u'ee'u) = \frac{\theta}{v}$$

$$E(D_3) = -\frac{4(1-\theta)^2}{nv\sigma^4} E(u'll'u) == -\frac{4(\theta)(1-\theta)}{v}$$

$$E(D_4) = -\frac{2(1-\theta)(1-2\theta)}{nv\sigma^4} \left(E(u'uu'l) - n\sigma^2 E(u')l\right) = 0$$

$$E(D_5) = \frac{\theta(1-\theta)}{nv\sigma^4} \left(\left(E(u'uu'u\right) + n^2\sigma^4 - 2n\sigma^2 E(u'u)\right) = \frac{(\theta)(1-\theta)}{v} (\gamma_2 + 2)$$

(64-70)

Similarly,

$$E(C_1 D_1) = 0$$

$$E(C_1 D_2) = 0$$

$$E(C_1 D_3) = 0 + O(n^{-1})$$

$$E(C_1 D_4) = \frac{4(1-\theta)^2(1-2\theta)}{n^{\frac{3}{2}} v^2 \sigma^6} E\left(u'uu'll'u - n\sigma^2 u'll'u\right)$$

$$= \frac{4(1-\theta)^2(1-2\theta)}{n^{\frac{3}{2}} v^2 \sigma^6} \sigma^4 \left[\gamma_2 n\sigma^2 \left(\frac{\theta}{1-\theta}\right) + 2n\sigma^2 \left(\frac{\theta}{1-\theta}\right)\right]$$

$$= \frac{4(1-\theta)^2(1-2\theta)}{n^{\frac{1}{2}} v^2} (\gamma_2 + 2)$$

$$E(C_1 D_5) = 0$$

(71 –75)

Further, the expression pertaining to the combinations of various expectations are as under:

$$E(C_2 D_1) = \frac{\theta}{n^{\frac{1}{2}} v^2} [(\gamma_2 + 2)p]$$

$$E(C_2 D_2) = \frac{\theta^2}{n^{\frac{3}{2}} v^2} (\gamma_2 + 2)$$

$$E(C_2D_3) = \frac{4\theta^2(1-\theta)}{n^{\frac{1}{2}}v^2}(\gamma_2+2)$$
$$E(C_2D_4) = 0$$
$$E(C_2D_5) = \frac{\theta^2(1-\theta)}{n^{\frac{1}{2}}v^2}(\gamma_4+10\gamma_1{}^2+12\gamma_2+8)$$
$$E(C_1^2D_1) = \frac{4\theta(1-\theta)}{v^3}(\gamma_2+p+2)$$
$$E(C_1^2D_2) = \frac{4\theta^2(1-\theta)}{v^3}+O(n^{-1})$$
$$E(C_1^2D_3) = \frac{16\theta^2(1-\theta)^2}{v^3}(\gamma_2+3)+O(n^{-1})$$
$$E(C_1^2D_4) = 0+O(n^{-1})$$
$$E(C_1^2D_5) = \frac{4\theta^2(1-\theta)^2}{v^3}(\gamma_2+2)+O(n^{-1})$$
$$E(C_2^2D_1) = \frac{\theta^2}{v^3}[(\gamma_2+2)p]$$
$$E(C_2^2D_2) = \frac{\theta^3}{v^3}(\gamma_2+2)+O(n^{-1})$$
$$E(C_2^2D_3) = -\frac{(1-\theta)\theta^3}{v^3}(\gamma_2+2)+O(n^{-1})$$
$$E(C_2^2D_4) = 0+O(n^{-1})$$
$$E(C_2^2D_5) = -\frac{(1-\theta)\theta^3}{v^3}(12\gamma_2+3\gamma_2^2+12)+O(n^{-1})$$
$$E(2C_1C_2D_1) = E(2C_1C_2D_2) = E(2C_1C_2D_3) = E(2C_1C_2D_5) = 0$$
$$E(2C_1C_2D_4) = \frac{8(1-\theta)^2(1-2\theta)\theta}{n^2v^2\sigma^8}[\gamma_2+2]$$

(76– 92)

Further, using equations above, it is easy to obtain following expressions pertaining to expectations

$$E(A_1) = 0$$
$$E\left(A_1{}^2\right) = \frac{4\theta(1-\theta)}{v^2}+\frac{\theta^2(\gamma_2+2)}{v^2}+O(n^{-1})$$

$$E\left(A_1{}^3\right) = -\frac{\theta^3}{v^3 n^{\frac{1}{2}}}(\gamma_4 + 24\gamma_2 + 10\gamma_1^2 + 32) - \frac{12\theta^2(\gamma_2 + 2)}{v^3 n^{\frac{1}{2}}} + O(n^{-1})$$

$$E\left(A_1{}^4\right) = \frac{16(1-\theta^2)\theta^2}{v^4}(\gamma_2 + 3) + \frac{\theta^4(12 + 12\gamma_2 + 3\gamma_2{}^2)}{v^4} + \frac{24\theta^3(1-\theta)}{v^4}(\gamma_2 + 2) + O(n^{-1})$$

$$E(f^*) = \frac{1}{v}[p + \theta - 4(1-\theta)\theta + \theta(1-\theta)(\gamma_2 + 2)]$$

(93-97)

Using $E(f^*)$ and $E\left(A_1{}^2\right)$, we get

$$E(B_1) = \frac{1}{v}\left[\frac{p}{2} + (4\theta - 5)\theta + \theta(1-2\theta)\gamma_2\right] + O(n^{-1})$$

(98)

$$E(A_1 B_1) = \frac{1}{n^{1/2}v^2}[(\gamma_2 + 2)\{4\theta(1-\theta)(3\theta - 1) - \theta^2 - p\theta + 12\theta^2\}] - \frac{\theta^2}{n^{1/2}v^2}[\gamma_4 + 12\gamma_2 + 10\gamma_1{}^2 + 8] + \frac{\theta^3}{n^{1/2}v^2}[2\gamma_4 + 36\gamma_2 + 10\gamma_1{}^2 + 40] + O(n^{-1})$$

(99)

Where, using expectations of various combinations of C1, C2, D1, D2, D3, D4 and D5 is given by

$$E(A_1 f^*) = \frac{1}{n^{1/2}v^2}(\gamma_2 + 2)[4\theta(1-\theta)(3\theta - 1) - \theta^2 - p\theta]$$

(100)

Also,

$$E\left(A_1{}^2 B_1\right) = \frac{1}{v^3}\gamma_2(-24\theta^2 - 32\theta^4 + 41\theta^3 + 4\theta) + \gamma_2{}^2(3\theta^3) + p\left(8\theta^2 - 6\theta - \theta^2\frac{\gamma_2}{2}\right) + O(n^{-1})$$

(101)

Where, using expectations of various combinations of C_1, C_2, D_1, D_2, D_3, D_4 and D_5 is given by

$$E\left(A_1{}^2 f^*\right) = \frac{1}{v^3}[\gamma_2(-8\theta^2 - 4\theta^4 + 9\theta^3 + 4\theta + \gamma_2{}^2 3\theta^3 - 3\theta^4) - 28\theta^2 + 34\theta^3 - 12\theta^4 + 8\theta + 4p(1-\theta)\theta] + O(n^{-1})$$

(102)