Journal of Rajasthan Academy of Physical Sciences
ISSN : 0972-6306; URL : http://raops.org.in
Vol.19, No.1&2, January-June, 2020, 69-80

CERTAIN CURVATURE CONDITIONS ON N(K)-CONTACT METRIC MANIFOLDS WITH RESPECT TO SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract: In the present paper, we study N(k)-contact metric manifolds with respect to semi-symmetric non-metric connection in which we study projective curvature tensor which is ξ-projectively flat and φ-projectively flat. We also study N(k)-contact metric manifolds with respect to semi-symmetric non-metric connection which satisfy $\bar{P}.\bar{S} = 0$, $\bar{P}(\xi, X).\bar{R} = 0$ and projectively pseudo-symmetric where $\bar{P}, \bar{R}$ and $\bar{S}$ denotes the projective curvature tensor, Riemannian tensor and Ricci tensor of N(k)-contact metric manifold with respect to semi-symmetric non-metric connection respectively.

Keyword: Semi-symmetric non-metric connection, projective curvature tensor, N(k)-contact metric manifold, Einstein manifold.

1. Introduction

From the point of view of differential geometry, projective curvature tensor is important tensor. Let M be (2n+1) dimensional Riemannian manifold. If there is a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if projective curvature tensor P vanishes. Here, P is given by [22]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y] \tag{1}$$

for all $X, Y, Z \in T(M)$.

Here, we define $\bar{P}$ i.e; projective curvature tensor with respect to semi-symmetric non-metric connection

$$\bar{P}(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{2n}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y] \tag{2}$$

for all $X, Y, Z \in T(M)$.

In [7] A. Barman studied N(k)-contact metric manifolds admitting a type of semi-symmetric non-metric connection. Let M be (2n+1) dimensional Riemannian manifold with Levi-Civita connection ∇. If $\overline{\nabla}$ is the semi-symmetric non-metric connection of a Riemannian manifold M, a linear connection $\overline{\nabla}$ is given by [7]

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X. \tag{3}$$

Using (3), the torsion tensor T of M with respect to $\overline{\nabla}$ is given by

$$T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] \tag{4}$$

$$= \eta(Y)X - \eta(X)Y.$$

Hence, relation (4) is called semi-symmetric connection.

From (3), it yields

$$(\overline{\nabla}_U g)(X,Y) = -\eta(X)g(Y,U) - \eta(Y)g(X,U) \neq 0 \tag{5}$$

$\overline{\nabla}$ defined by (3), satisfying (4) and (5) is a type of semi-symmetric non-metric connection.

Then, $\bar{R}$ and R are related by [7]

$$\bar{R}(X,Y)Z = R(X,Y)Z + g(X,\varphi Z)Y + g(hX,\varphi Z)Y - \eta(X)\eta(Z)Y -$$

$$g(Y,\varphi Z)X - g(hY,\varphi Z)X + \eta(Y)\eta(Z)X.$$

In the present paper, it will be organized as follows: In Section 2, we will give some preliminary results and the definitions that will be needed further. In Section 3, we study projective curvature tensor with respect to semi-symmetric non-metric connection, we have manifold is same as Riemannian connection. We also study φ-projectively flat and ξ-projectively flat N(k)-contact metric manifold with respect to semi-symmetric non-metric connection. Section 4 is devoted to study projectively pseudosymmetric N(k)-contact metric manifold with respect to semi-symmetric non-metric connection and obtain manifold is projectively flat. In Section 5, we consider N(k)-contact metric manifold with respect to semi-symmetric non-metric connection satisfying $\bar{P}(\xi,X).\bar{R} = 0$ and finds that manifold is Einstein manifold. In Section 6, we discuss N(k)-contact metric manifold with respect to semi-symmetric non-metric connection satisfying $\bar{P}.\bar{S} = 0$ and it is shown that manifold is Einstein manifold.

2. Preliminaries

A (2n+1) dimensional smooth manifold M is said to be a contact manifold if it comes a global differentiable 1-form η which satisfies the condition $\eta^\wedge(d\eta)^\eta \neq 0$ everywhere on M. Also, a contact manifold admits an almost contact structure (φ, ξ, η), where φ is (1,1) tensor field, ξ is a characteristic vector field and η is a global 1-form s.t.

$$\varphi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \varphi\xi = 0,\ \eta o\varphi = 0. \tag{6}$$

An almost contact structure is said to be normal if the induced almost contact structure J on the product manifold M×R defined by

$$J\left(X,\lambda\frac{d}{dt}\right)=\left(\varphi X-\lambda\xi,\eta\otimes\frac{d}{dt}\right)$$

is integrable, where X is tangent to M, t is the coordinate of R and λ a smooth function on M×R. The condition of almost contact metric structure being normal is equivalent to vanishing of the torsion tensor [φ, φ] + 2d$\eta\otimes\xi$, where [φ, φ] is the Nijenhuis tensor of φ. Let g be the compatible Riemannian metric with almost contact structure (φ, ξ,η), that is

$$\left.\begin{array}{c} g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y), \\ g(X,\xi)=\eta(X),\ g(X,\varphi Y)=-g(\varphi X,Y) \end{array}\right\} \tag{7}$$

for all vector fields $X,Y\in T(M)$. A manifold M together with this almost contact metric structure is said to be almost contact metric manifold denoted by M(φ, ξ,η,g). An almost contact metric structure reduces to a contact metric structure if $g(X,\varphi Y)=d\eta(X,Y)$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds,

$$\nabla_X\xi=-\varphi X-\varphi hX. \tag{8}$$

Now, we are going to discuss about N(k)-contact metric manifold. Tanno [23], [24], introduced k-nullity distribution on a contact metric manifold. The k-nullity distribution N(k) of a Riemannian manifold is defined by

$$N(k):p\rightarrow N_p(k)=\{U\in T_p(M)|R(X,Y)U=k(g(Y,U)X-g(X,U)Y)\}, \tag{9}$$

for any $X,Y,U\in T(M)$ and k being constant, where R denotes the Riemannian curvature tensor and $T(M)$ is the tangent vector space of M^{2n+1} at any point $p\epsilon M$.

If the characteristics vector field of a contact metric manifold belongs to k-nullity distribution, then the relation

$$R(X,Y)\xi=k[\eta(Y)X-\eta(X)Y] \text{ holds.} \tag{10}$$

A contact metric manifold with $\xi\in N(k)$ is called N(k)-contact metric manifold. If (10) holds on N(k)-contact metric manifold, it becomes contact metric manifold. From (9) and (10) it follows that N(k)-contact metric manifold is Sasakian manifold if $k=1$.

There are so many authors studied about N(k)-contact metric manifolds. P. Majhi and U.C De [19] studied N(k)-contact metric manifolds satisfying certain curvature conditions on the projective curvature tensor. N(k)-contact metric manifolds have been studied by many other authors such as $\ddot{O}zg\ddot{u}r$ and Sular [20], Ghosh, De and Taleshian [16], Blair, Konfogiorgos and Papantoniou [9], Blair [8], [10], De and Gazi[13].

In an N(k)-contact metric manifold, the following relation holds [9], [10] :

$$(\nabla_X\varphi)Y=g(X+hX,Y)\xi-\eta(Y)(X+hX), \tag{11}$$

$$(\nabla_X\eta)(Y)=g(X+hX,\varphi Y), \tag{12}$$

$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X],$ (13)

$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y],$ (14)

$R(X,\xi)Y = k[\eta(Y)X - g(X,Y)\xi],$ (15)

$S(X,Y) = 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) +$

$[2nk - 2(n-1)]\eta(X)\eta(Y), n \geq 1.$ (16)

$S(\varphi X, \varphi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 4(n-1)g(hX,Y),$ (17)

$S(Y,\xi) = 2nk\eta(Y),$ (18)

$\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)],$ (19)

$(\nabla_X h)(Y) = [(1-k)g(X,\varphi Y) + g(X,h\varphi Y)]\xi + \eta(Y)[h(\varphi X + \varphi hX)],$ (20)

where R and S are the curvature tensor and Ricci tensor respectively with respect to Levi-Civita connection.

In a (2n+1) dimensional almost contact metric manifold, if $\{e_1, e_2 \dots\dots\dots. e_{2n}, \xi\}$ is a local orthonormal basis of tangent space of the manifold, then $\{\varphi e_1, \varphi e_2 \dots\dots\dots. \varphi e_{2n}, \xi\}$ is a local orthonormal basis. It is easy to verify that

$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\varphi e_i, \varphi e_i) = 2n,$ (21)

$\sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\varphi e_i, \varphi e_i) = r - 2nk,$ (22)

$\sum_{i=1}^{2n} g(e_i, Z)\, S(Y, e_i) = \sum_{i=1}^{2n} g(\varphi e_i, Z)\, S(Y, \varphi e_i) = S(Y,Z) - 2nk\eta(Y)\eta(Z),$ (23)

$\sum_{i=1}^{2n} g(R(e_i, Y)Z, e_i) = \sum_{i=1}^{2n} g(R(\varphi e_i, Y)Z, \varphi e_i)$ (24)

$=S(Y,Z) - kg(\varphi Y, \varphi Z),$

for all $Y, Z \in T(M)$.

Now, we are going to discuss about semi-symmetric connections. The idea of semi-symmetric connection on a differentiable manifold is given by Friedmann and Schouten [15] in 1924. A linear connection $\tilde{\nabla}$ on a differentiable manifold is said to be semi-symmetric connection if torsion tensor $\tilde{T}$ of a connection $\tilde{\nabla}$ satisfying $\tilde{T}(X,Y) = \eta(Y)X - \eta(X)Y$, where η is 1-form and ρ is a vector field defined by $\eta(X) = g(X,\rho)$ for all vector fields $X \in T(M)$, $T(M)$ is the set of all differentiable vector fields on M.

The idea of semi-symmetric connection on a Riemannian manifold (M,g) by Hayden [17] in 1932. If $\tilde{\nabla} g = 0$, then semi-symmetric connection is said to be semi-symmetric metric connection.

After a long gap, Prvanovic [21] initiated the study of semi-symmetric connection $\tilde{\nabla}$ satisfying

$\tilde{\nabla} g \neq 0$ (25)

with the name pseudo metric semi-symmetric connection and was just followed by Andonie [2].

If (25) holds on semi symmetric connection $\widetilde{\nabla}$, it becomes semi-symmetric non-metric connection. Agashe and Chafle [1] in 1932 studied semi-symmetric non-metric connection $\overline{\nabla}$, whose torsion tensor $\overline{T}$ satisfies

$$\overline{T}(X,Y) = \eta(Y)X - \eta(X)Y \text{ and}$$

$$\overline{\nabla}_X g(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(X,Y).$$

They proved that the projective curvature tensor of the manifold with respect to these connections are equal to each other.

Further, semi-symmetric non-metric connection has been developed by several other authors such as Liang[18], De and Kamilya[11], Barman[4] [5], Barman and De[6], De and Biswas[12] and many others.

A. Barman[7] studied N(k)-contact metric manifold admitting a type of semi-symmetric non-metric connection and gives the following proposition (2.1)

Proposition (2.1) : For N(k)-contact metric manifold M with respect to semi-symmetric non-metric connection $\overline{\nabla}$,

The curvature tensor $\overline{R}$ is given by

$$\bar{R}(X,Y)Z = R(X,Y)Z + g(X,\varphi Z) + g(hX,\varphi Z)Y - \eta(X)\eta(Z)Y - g(Y,\varphi Z)X - g(hY,\varphi Z)X + \eta(Y)\eta(Z)X. \tag{26}$$

The Ricci tensor $\bar{S}$ is given by

$$\bar{S}(Y,Z) = S(Y,Z) - 2ng(Y,\varphi Z) - 2ng(hY,\varphi Z) + 2n\,\eta(Y)\eta(Z) \tag{27}$$

$$\bar{R}(\xi,Y)Z = kg(Y,Z)\,\xi - (k+1)\eta(Z)Y - g(Y,\varphi Z)\,\xi - g(hY,\varphi Z)\xi + \eta(Y)\eta(Z)\xi \tag{28}$$

$$\bar{R}(X,Y)Z = \bar{R}(Y,X)Z \tag{29}$$

The scalar curvature tensor $\bar{r}$ is given by

$$\bar{r} = r + 2n \tag{30}$$

The Ricci tensor $\bar{S}$ is symmetric

$$\bar{S}(Y,\xi) = 2n(k+1)\eta(Y) = \bar{S}(\xi,Y) \tag{31}$$

$$(\overline{\nabla}_U\eta)X = g(U,\varphi)X + g(hU,\varphi)X - \eta(X)\eta(U). \tag{32}$$

3. Projective curvature tensor on N(k)-contact metric manifold with respect to semi-symmetric non-metric connection

The generalized projective curvature tensor on N(k)-contact metric manifold with respect to semi-symmetric non-metric connection is defined by (2)

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{2n}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y] \tag{33}$$

Using (22) and (23) in (33), reduces to

$$\bar{P}(X,Y)Z = P(X,Y)Z \tag{34}$$

where $P(X,Y)Z$ is projective curvature tensor is defined by (1).

We can state the following theorem:

Theorem (3.1): Projective curvature tensor on N(k)-contact metric manifold with respect to semi-symmetric non-metric connection is equivalent to the projective curvature tensor with respect to the Riemannian connection.

Like the definition of ξ-conformally flat contact metric manifold [3], we define ξ -projectively flat N(k)-contact metric manifold.

Definition (3.1): N(k)-contact metric manifold M with respect to semi-symmetric non-metric connection is called ξ-projectively flat if $P(X,Y)\xi = 0$ holds on M.

Put $Z = \xi$ in (34) and using (1), we get

$$\bar{P}(X,Y)\xi = P(X,Y)\xi = 0. \tag{35}$$

We can state the following theorem:

Theorem (3.2): N(k)-contact metric manifold M with respect to semi-symmetric non-metric connection is ξ-projectively flat.

Like the definition of φ-conformally flat contact metric manifold [3], we define φ-projectively flat N(k)-contact metric manifold.

Definition (3.2): N(k)-contact metric manifold satisfying condition

$$\varphi^2 P(\varphi X, \varphi Y)\varphi Z = 0 \tag{36}$$

is called φ-projectively flat.

Let us assume that N(k)-contact metric manifold with respect to semi-symmetric non-metric connection is φ-projectively flat. This can be easily seen that

$\varphi^2 \bar{P}(\varphi X, \varphi Y)\varphi Z = 0$ holds if

$$g(\bar{P}(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0 \tag{37}$$

for $X, Y, Z, W \in T(M)$.

Using (33) and (37), φ-projectively means

$$g(\bar{R}(\varphi X, \varphi Y)\varphi Z, \varphi W) = \frac{1}{2n}[\bar{S}(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - \bar{S}(\varphi X, \varphi Z)g(\varphi Y, \varphi W)] \tag{38}$$

Let $\{e_1, e_2 \ldots\ldots\ldots . e_{2n}, \xi\}$ be a local orthonormal basis of the vector fields in manifold and using the fact that $\{\varphi e_1, \varphi e_2 \ldots\ldots\ldots . \varphi e_{2n}, \xi\}$ is also a local orthonormal basis, putting $X = W = e_i$ in (38) and summing up with respect to $i = 1,2,3,\ldots\ldots .2n$, we have $\sum_{i=1}^{i=2n} g(\bar{R}(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) =$

$$\frac{1}{2n}\sum_{i=1}^{i=2n}[\bar{S}(\varphi Y,\varphi Z)g(\varphi e_i,\varphi e_i) - \bar{S}(\varphi e_i,\varphi Z)g(\varphi Y,\varphi e_i)] \tag{39}$$

$$= \bar{S}(\varphi Y,\varphi Z) - \frac{1}{2n}\sum_{i=1}^{i=2n}\bar{S}(\varphi e_i,\varphi Z)g(\varphi Y,\varphi e_i) \tag{40}$$

Using (21),(23), (27) and (40) , equation (39) becomes

$$S(Y,Z) = 4(n-1)g(hY,Z) + 2nkg(Y,Z) \tag{41}$$

In case of Riemannian connection from definition (3.2) an almost contact metric manifold M is said to be φ-projectively flat if

$$g(P(\varphi X,\varphi Y)\varphi Z,\varphi W) = 0$$

for $X,Y,Z,W \in T(M)$.

Hence, from (1), we get

$$g(P(\varphi X,\varphi Y)\varphi Z,\varphi W) = g(R(\varphi X,\varphi Y)\varphi Z,\varphi W) - \frac{1}{2n}[S(\varphi Y,\varphi Z)g(\varphi X,\varphi W) - S(\varphi X,\varphi Z)g(\varphi Y,\varphi W)] \tag{42}$$

for $X,Y,Z,W \in T(M)$.

For the local orthonormal basis $\{e_1, e_2 \dots\dots\dots. e_{2n}, \xi\}$ of vector fields in M, putting $X = W = e_i$ in (42), we get

$$\sum_{i=1}^{i=2n} g(R(\varphi e_i,\varphi Y)\varphi Z,\varphi e_i) =$$

$$\frac{1}{2n}\sum_{i=1}^{i=2n}[S(\varphi Y,\varphi Z)g(\varphi e_i,\varphi e_i) - S(\varphi e_i,\varphi Z)g(\varphi Y,\varphi e_i)] \tag{43}$$

for $Y,Z \in T(M)$.

Using equation (29), (30), (31) and (32) in (42), yields

$$\sum_{i=1}^{i=2n} g(R(\varphi e_i,\varphi Y)\varphi Z,\varphi e_i) = (2n-1)\left[kg(\varphi Y,\varphi Z) - \frac{1}{2n}S(\varphi Y,\varphi Z)\right]$$

and

$$\sum_{i=1}^{i=2n} g(R(\varphi e_i,\varphi Y)\varphi Z,\varphi e_i) = \frac{2n-1}{2n}S(\varphi Y,\varphi Z)$$

Hence, from (43), it becomes

$$S(\varphi Y,\varphi Z) = nk\, g(\varphi Y,\varphi Z) \tag{44}$$

Replace $Y \to \varphi Y$ and $Z \to \varphi Z$ in (44), it becomes

$$S(\varphi^2 Y,\varphi^2 Z) = nk\, g(\varphi^2 Y,\varphi^2 Z) \tag{45}$$

Using (6) and (7) in (45)

$$S(Y,Z) = nk\, g(Y,Z) + nk\, \eta(Y)\eta(Z)$$

Hence, we can state the following theorem:

Theorem (3.3): If N(k)-contact metric manifold M with respect to semi-symmetric non-metric connection is φ-projectively flat then $S(Y,Z) = 4(n-1)g(hY,Z) + 2nkg(Y,Z)$.

4. Projectively pseudosymmetric N(k)-contact metric manifold with respect to semi-symmetric non-metric connection

A Riemannian manifold is said to be pseudosymmetric [14] if at every point of the manifold the following relation holds :

$$(\bar{R}(X,Y).\bar{R})(U,V)W = L_{\bar{R}}((X^{\wedge}Y).\bar{R}(U,V)W) \tag{46}$$

for all $X,Y,Z,U,V,W \in T(M)$,where $L_{\bar{R}}$ is some function on M. The endomorphism $(X^{\wedge}Y)$ is defined by

$$(X^{\wedge}Y)Z = g(Y,Z)X - g(X,Z)Y \tag{47}$$

A Riemannian manifold is said to be projectively pseudosymmetric if it satisfies the condition

$$(\bar{R}(X,Y).\bar{P})(U,V)W = L_{\bar{P}}((X^{\wedge}Y).\bar{P}(U,V)W) \tag{48}$$

where $L_{\bar{P}} \neq k+1$ is some function on M.

Let us suppose that N(k)-contact metric manifold satisfies the condition

$$(\bar{R}(X,Y).\bar{P})(U,V)W = L_{\bar{P}}((X^{\wedge}Y).\bar{P}(U,V)W) \tag{49}$$

From (49), Put $Y = W = \xi$, we obtain

$$(\bar{R}(X,\xi).\bar{P})(U,V)\xi = L_{\bar{P}}((X^{\wedge}\xi).\bar{P}(U,V)\xi) \tag{50}$$

Using (2), (6), (26), (28), in (50), we have

$$-(k+1)\bar{P}(U,V)X = L_{\bar{P}}((X^{\wedge}Y).\bar{P}(U,V)W) \tag{51}$$

Now,

$$L_{\bar{P}}((X^{\wedge}\xi).\bar{P}(U,V)\xi) = L_{\bar{P}}[(X^{\wedge}\xi).\bar{P}(U,V)\xi - \bar{P}\big((X^{\wedge}\xi)U,V\big)\xi$$

$$-\bar{P}(U,(X^{\wedge}\xi),V)\xi - \bar{P}(U,V)(X^{\wedge}\xi)\xi] \tag{52}$$

with the help of

$$(X^{\wedge}\xi).\bar{P}(U,V)\xi = 0 \tag{53}$$

$$\bar{P}\big((X^{\wedge}\xi)U,V\big)\xi = 0 \tag{54}$$

$$\bar{P}(U,(X^{\wedge}\xi),V)\xi = 0 \tag{55}$$

$$\bar{P}(U,V)(X^{\wedge}\xi)\xi = \bar{P}(U,V)X \tag{56}$$

In view of (53), (54), (55) and (56) from (51), we obtain

$$-(k+1)\bar{P}(U,V)X = -L_{\bar{P}}[\bar{P}(U,V)X] \tag{57}$$

The above equation yields,

$[L_{\bar{P}} - (k+1)]\bar{P}(U,V)X = 0$

By assumption, $L_{\bar{P}} \neq (k+1)$ and hence

$\bar{P}(U,V)X = 0$ for any vector fields U, V.

Conversely, if $\bar{P} = 0$, then equation (48) holds insignificantly.

From the view of above result, we can state the following theorem.

Theoerem (4.1): A (2n+1) dimensional N(k)-contact metric manifold with respect to semi-symmetric non-metric connection is projectively pseudosymmetric if and only if it is projectively flat.

Also, projectively flat implies projectively semi-symmetric. Therefore, we can state the following theorem-

Theorem(4.2): A (2n+1) dimensional projectively pseudosymmetric N(k)-contact metric manifold with respect to semi symmetric non-metric connection is projectively semi-symmetric provided $L_{\bar{P}} \neq (k+1)$.

5. N(k)-contact metric manifold with respect to semi-symmetric non-metric connection satisfying $\bar{P}(\xi, X).\bar{R} = 0$.

Definition(5.1): N(k)-contact metric manifold is said to be η-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form

$$S(X,Y) = a\, g(X,Y) + b\, \eta(X)\eta(Y)$$

where a and b are smooth function on the manifold.

If $b = 0$, then manifold is said to be Einstein manifold.

In this section, we consider projective curvature tensor which is semi-symmetric with respect to semi-symmetric non-metric connection M^{2n+1} i.e;

$$(\bar{P}(\xi,X).\bar{R})(U,V)W = 0$$

$$\bar{P}(\xi,X)\bar{R}(U,V)W - \bar{R}(\bar{P}(\xi,X)U,V)W - \bar{R}(U,\bar{P}(\xi,X)V)W - \bar{R}(U,V)\bar{P}(\xi,X)W = 0 \tag{58}$$

Put $U = \xi$ in (58), we get

$$\bar{P}(\xi,X)\bar{R}(\xi,V)W - \bar{R}(\bar{P}(\xi,X)\xi,V)W - \bar{R}(\xi,\bar{P}(\xi,X)V)W - \bar{R}(\xi,V)\bar{P}(\xi,X)W = 0 \tag{59}$$

By using (2), (27), (28), (31) in (59), we conclude that

$$S(X,V) = 2nkg(X,V). \tag{60}$$

Therefore, $S(Y,Z) = a\,g(Y,Z) + \,b\,\eta(Y)\eta(Z)$,

where $a = 2nk$ and $b = 0$.

which implies that the manifold is Einstein manifold.

In view of above relation, we can state the following theorem:

Theorem (5.1): If N(k)-contact metric manifold is projectively semi-symmetric with respect to semi-symmetric non-metric connection, then the manifold is an Einstein manifold.

6. N(k)-contact metric manifold with respect to semi-symmetric non-metric connection satisfying $\bar{P}.\bar{S} = 0$.

Definition (6.1): N(k)-contact metric manifold is said to be Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form $S(X,Y) = a\,g(X,Y)$, where a is constant.

In this section, we consider Ricci semi-symmetric N(k)-contact metric manifold with respect to semi-symmetric non-metric connection i.e;

$$(\bar{P}(X,Y).\bar{S})(U,V) = 0$$

Then, we have

$$\bar{S}(\bar{P}(X,Y)U,V) + \bar{S}(U,\bar{P}(X,Y)V) = 0 \tag{61}$$

Put $X = \xi$ in (61), it follows that

$$\bar{S}(\bar{P}(\xi,Y)U,V) + \bar{S}(U,\bar{P}(\xi,Y)V) = 0 \tag{62}$$

Using (2), (28), (31) in (62), we obtain

$$2nkg(Y,U)\eta(V) + 2nkg(Y,V)\eta(U) - S(Y,U)\eta(V) - S(Y,V)\eta(U) = 0 \tag{63}$$

Again, putting $U = \xi$ in (63), and using (6), we get

$$S(Y,V) = 2nkg(Y,V) \tag{64}$$

Therefore, $S(Y,Z) = ag(Y,Z)$, where $a = 2nk$. From which it follows that the manifold is an Einstein manifold.

Now, we can state the following theorem:

Theorem(6.1): If (2n+1) dimensional N(k)-contact metric manifold with respect to semi-symmetric non-metric connection satisfying the curvature condition $\bar{P}.\bar{S} = 0$, then the manifold M is an Einstein manifold.

Acknowledgement: The authors are thankful to the referees for their valuable suggestions in the improvement of the paper.

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