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SOME HODIERNAL PROPERTIES OF 3x3 MAGIC SQUARE

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Abstract: In this paper, we have established some new results in the area of square magic and proved some new properties, in which one is M(Aα) + M(A-α) = 2X M(A). Also, Neoteric properties of Magic Constant of Magic Square and it's derived matrix in the view of eigen values, eigen vectors have been established.

Keywords: Magic Square, Magic Constant, Matrix, Eigen Value and Eigen Vector.
MSC Classification: 97A20, 97A40, 97A80

1. Introduction

The History of the development of Magic Squares in India leads irresistibly to the conclusion that the Magic Square originated in India. The knowledge of these Squares might have gone outside India at any time between the first century and 10th century AD, but it appears to be the most probable that the west as well as China got the Magic Squares from India through the Arabs about tenth century. This would account for the simultaneous occurrence of the Magic Square in such for off places as China, Arabia and Western Europe. Many Chinese scholars visited India time to time and quenched their thirst for knowledge and carried academic materials to their own country and it is not only Chinese but so many scholars from the different part of the world enjoyed the same pattern on the line of 'Guest of Honour' of Indian sacraments. But no one became the witness of reverse.

Farrugia[5] (i) Scalar multiple of the same matrix have the same eigenvectors. So, the matrix A, 2A and 5A have same eigenvectors.

(ii) Adding any multiple of the identity matrix to the matrix A does not change any of it's eigen vectors as well, so the matrices A, A+I, A-2I and A-3I have the same set of eigen vectors. i.e, If a matrix A has an eigen vector v then any one can find the new matrix $A+vW^T$ would still have the eigen vector v, no matter what vector W can be chosen.

$(A+vW^T)v = Av + vW^Tv = \lambda v + (W^Tv)v = (\lambda+W^Tv)v$. So, $(\lambda+W^Tv)$ is an eigenvalue of $(A+vW^T)$ with associated eigen vector v.

But this is very tedious and laborious process.

Our methodology, Hodiernal Property minimize these work in square magic cases, which is Hodiernal Property of Square Magic. This is important feature of this paper. In our process, we only break our principal eigen values and get the similar result.

In this paper, we have also verified some interesting properties of the square magic of order 3X3.

2. Definition & Preliminaries

A Magic Square of order n is a square matrix having array of n2 numbers such that the sum of each row and column as well as the main diagonal and main back diagonal [1], is the same number called Magic Constant (Magic Sum or Lowest Required Sum). If A is a Magic Square and each element of B is obtained by addition, subtraction, multiplication or division to the corresponding element of A by the same number (not 0 for multiplication or division), then B will be a Magic Square.

An upper bound for the number of normal Magic Squares of order n can be given by $n^2!/8(2n+1)$. There is only one distinct third order normal Magic Square with lowest sum.

Magic Square follows the properties of matrix as sum of the Eigen values of the Magic Square is equal to trace of Magic Square and product of the Eigen values of the Magic Square is equal to the determinant of the Magic Square. The principal Eigen value of a Magic Square composed of positive elements is it's Magic Constant. If a Magic Square has some negative elements, then its Magic Constant is one of its Eigen values. Many mathematical properties of square matrix are also good factors to study in this area [2, 3, 4,6,7,8,9,10 and 11].

3. Methodology

Example 3.1: Let us consider a third-order Magic Square.

	15	12	27	
$A=$	30	18	6	
	9	24	21	
				54

The Magic Constant of the Magic Square is M (A) =54.

Taking a combination of the two numbers 30 and 24 of the Magic Constant 54 such that 30+24. Now, adding 30 to each element of the Magic Square A, we get a Magic Square A30 (Say).

	45	42	57	
A_{30} =	60	48	36	
	39	54	51	
				144

$M(A_{30}) = 144$

Similarly,

	39	36	51	
A_{24} =	54	42	30	
	33	48	45	
				126

$M(A_{24}) = 126$

	-15	-18	-3	
A_{30} =	0	-12	-24	
	-21	-6	-9	
				-36

$M(A_{-30}) = -36$

	-9	-12	3	
A_{24} =	6	-6	-18	
	-15	0	-3	
				-18

$M(A_{-24}) = -18$

Now Combining

$$M(A_{30}) + M(A_{-30}) = 144 - 36 = 2 * 54 = 2M(A)$$

$$M(A_{24}) + M(A_{-24}) = 126 - 18 = 2 * 54 = 2M(A)$$

$$M(A_{30}) + M(A_{24}) = 144 + 126 = (2+3) * 54 = (2+3) * M(A) \quad \&$$

$$M(A_{-30}) + M(A_{-24}) = -36 - 18 = (2-3) * 54 = (2-3) * M(A).$$

Example 3.2: Let us consider a third-order Magic Square.

	15	12	27	
A =	30	18	6	
	9	24	21	
				54

The Magic Constant of the Magic Square is M (A) = 54.

Taking a combination of the three numbers 10, 20 and 24 of the Magic Constant 54 such that 10+20 +24. Now, adding 10 to each element of the Magic Square A, we get a Magic Square A10 (Say).

	25	22	37	
A_{10} =	40	28	16	
	19	34	31	
				84

$M(A_{10}) = –84$

Similarly,

	35	32	47	
A_{20} =	50	38	26	
	29	44	41	
				114

Similarly,

	39	36	51	
A_{24} =	54	42	30	
	33	48	45	
				126

$M(A_{24}) = 126$

	5	2	17	
A_{-10} =	20	8	-4	
	-1	14	11	
				24

$M(A_{-10}) = 24$

	-5	-8	7	
A_{-20} =	10	-2	-14	
	-11	4	1	
				-6

$M(A_{-20}) = -6$

	-9	-12	3	
A_{-24} =	6	-6	-18	
	-15	0	-3	
				-18

$M(A_{-24}) = -18$

Now Combining

$$M(A_{10}) + M(A_{-10}) = 84 + 24 = 2*54 = 2M(A)$$

$$M(A_{20}) + M(A_{-20}) = 114 - 6 = 2*54 = 2M(A)$$

$$M(A_{24}) + M(A_{-24}) = 126 - 18 = 2*54 = 2M(A)$$

$$M(A_{10}) + M(A_{20}) + M(A_{24}) = 84 + 114 + 126 = (3+3)*54 = (3+3)*M(A) \ \&$$

$$M(A_{-10}) + M(A_{-20}) + M(A_{-24}) = 24 - 6 - 18 = (3-3)*54 = (3-3)*M(A).$$

4. Results and Discussion

(i) If M(Aα) = Magic Constant of Magic Square Aα

M(A-α) = Magic Constant of Magic Square A-α

M(A(M-α)) = Magic Constant of Magic Square A(M-α)

M(A-(M-α)) = Magic Constant of Magic Square A-(M-α)

M(A) = Magic Constant of Magic Square A

n = Order of Magic Square A

m = Number of partitions of M(A).

Then, the properties of Magic Square are:

M(Aα) + M(A-α) = 2X M(A)

M(A(M-α)) + M(A-(M-α)) = 2X M(A)

M(Aα) + M(A(M-α)) = (m+n)X M(A)

M(A-α) + M(A-(M-α)) = (m-n)XM(A).

We can choose randomly any number of combinations for the Magic Constant.

(ii) A square matrix A has an Eigen vector v, then ($A + \nu W^T$) is a new matrix, still have the Eigen vector ν, where W is any compatible vector of choice and Eigen vector v is associated with the Eigen value λ of A. After Eigen decomposition process, the Eigen values and Eigen vectors of these obtained matrices are also linearly related together. Eigen vector of one matrix are the basis of an invariant subspace within the range of the corresponding linear map.

Hence, it is possible to find a large number of essentially different matrices. But it is a difficult and time consuming task.

It is easy to assume both matrices have corresponding Eigen vector. Then, when two matrices are under test added together (in case of addition) on the line of our methodology (those overlapping Eigen vectors) and only overlapping Eigen vectors will be produced linearly as new Eigen value. Since the outputs from both maps will be along the same line.

However, any Eigen vector that is not showed by both matrices (notice that Eigen values does not matter here) will not be preserved.

In short, adding matrices together will seriously change the geometry of their corresponding linear maps and linear subspaces. The places where they overlap will stay the same but everywhere they differ gets thrown out.

But in our methodology they always overlap.

Discussion on Example 3.1

M(A)=54

$\lambda_1 = 54$, $\lambda_2 = 14.6969i$, $\lambda_3 = -14.6969i$

$X_3 = 1$,

$v_1 = (1,1,1)$, $v_2 = (-0.2-0.979796i, -0.8+0.979796i,1)$ $v_3 = (-0.2+ 0.979796i, -0.8-0.979796i,1)$

M(A30)=144

$\lambda_1 = 144$, $\lambda_2 = 14.6969i$, $\lambda_3 = -14.6969i$

$X_3 = 1$,

$v_1 = (1,1,1)$, $v_2 = (-0.2-0.979796i, -0.8+0.979796i,1)$ $v_3 = (-0.2+ 0.979796i, -0.8-0.979796i,1)$

M(A24) = 126

$\lambda_1 = 126$, $\lambda_2 = 14.6969i$, $\lambda_3 = -14.6969i$

X_3 = 1,

v_1 = (1,1,1), v_2 = (-0.2-0.979796i, -0.8+0.979796i,1) v_3 = (-0.2+ 0.979796i, -0.8-0.979796i,1)

M(A-30) = -36

λ_1 = -36, λ_2 = 14.6969i, λ_3 = -14.6969i

X_3 = 1,

v_1 = (1,1,1), v_2 = (-0.2-0.979796i, -0.8+0.979796i,1) v_3 = (-0.2+ 0.979796i, -0.8-0.979796i,1)

M(A-24) = -18

λ_1 = -18, λ_2 = 14.6969i, λ_3 = -14.6969i

X_3 = 1,

v_1 = (1,1,1), v_2 = (-0.2-0.979796i, -0.8+0.979796i,1) v_3 = (-0.2+ 0.979796i, -0.8-0.979796i,1)

Discussion on Example 3.2

M(A) = 54

λ_1 =54, λ_2 = 14.6969i, λ_3 = -14.6969i

X_3 = 1,

v_1 = (1,1,1), v_2 = (-0.2-0.979796i, -0.8+0.979796i,1) v_3 = (-0.2+ 0.979796i, -0.8-0.979796i,1)

M(A_{10}) = 84

λ_1 =84, λ_2 = 14.6969i, λ_3 = -14.6969i

X_3 = 1,

v_1 = (1,1,1), v_2 = (-0.2-0.979796i, -0.8+0.979796i,1) v_3 = (-0.2+ 0.979796i, -0.8-0.979796i,1)

M(A_{20}) = 114

λ_1 =114, λ_2 = 14.6969i, λ_3 = -14.6969i

X_3 = 1,

v_1 = (1,1,1), v_2 = (-0.2-0.979796i, -0.8+0.979796i,1) v_3 = (-0.2+ 0.979796i, -0.8-0.979796i,1)

M(A24) = 126

λ_1 = 126, λ_2 = 14.6969i, λ_3 = -14.6969i

X_3 = 1,

v_1 =(1,1,1), v_2 = (-0.2-0.979796i, -0.8+0.979796i,1) v_3 = (-0.2+ 0.979796i, -0.8-0.979796i,1)

M(A_{-10}) = 24

λ_1 =24, λ_2 = 14.6969i, λ_3 = -14.6969i

X_3 = 1,

v_1 = (1,1,1), v_2 = (-0.2-0.979796i, -0.8+0.979796i,1) v_3 = (-0.2+ 0.979796i, -0.8-0.979796i,1)

M(A_{-20}) = –6

$\lambda_1 = -6$, λ_2 = 14.6969i, λ_3 = -14.6969i

X_3 = 1,

v_1 =(1,1,1), v_2 =(-0.2-0.979796i, -0.8+0.979796i,1) v_3 =(-0.2+ 0.979796i, -0.8-0.979796i,1)

M(A-24)= -18

λ_1 = -18, λ_2 = 14.6969i, λ_3 = -14.6969i

X_3 = 1,

v_1 =(1,1,1), v_2 =(-0.2-0.979796i, -0.8+0.979796i,1) v_3 =(-0.2+ 0.979796i, -0.8-0.979796i,1)

(iii) Concavo-convex Property of Magic Constant

For example 3.1:

M(A30/54)= 54+3.30/54= 55.67

M(A24/54)= 54+3.24/54= 55.33 for any $\lambda \in [0,1]$.

$$M[A_{[\lambda M(A_{30/54})+(1-\lambda)M(A_{24/54})]}] = M[A_{55.67\lambda+55.33(1-\lambda)}]$$
$$= 54 + 3(55.3 + 0.34\lambda) = 220 + \lambda \quad (1)$$

$$\lambda_{M(A_{30/54})} = \lambda_{[54+3(30/54)]} = 55.67\lambda$$

$$(1-\lambda)_{M(A_{24/54})} = 55.33(1-\lambda)$$

Therefore $\lambda_{M(A_{30/54})} + (1-\lambda)_{M(A_{24/54})} = 5.33 + 0.34\lambda$

$$\lambda M[A_{M(A_{30/54})}] + (1-\lambda)M[A_{M(A_{24/54})}] = 220 + \lambda \quad (2)$$

From (1) and (2)

$$M[A_{[\lambda M(A_{30/54})+(1-\lambda)M(A_{24/54})]}]$$
$$= \lambda M[A_{M(A_{30/54})}] + (1-\lambda)M[A_{M(A_{24/54})}]$$

For Example 3.2 :

$$M(A_{10/54}) = 54.55,\ M(A_{20/54}) = 55.11$$

M(A24/54)= 54+3.24/54= 55.33 for any $\lambda \in [0,1]$. Take $\lambda_1 + \lambda_2 + \lambda_3 = 1$

$$M[A_{[\lambda_1 M(A_{10/54})+\lambda_2 M(A_{20/54})+\lambda_3 M(A_{24/54})]}] = M[A_{54.55\lambda_1+55.11\lambda_2+55.33\lambda_3)}]$$

$$= 54 + 3(54.55 + 0.34\lambda_1) + 3(55.11 + 0.34\lambda_2) + 3(55.33 + 0.34\lambda_3)$$
$$= 220 + \lambda \quad (3)$$

$$\lambda_{1M(A_{10/54})} = \lambda_{[54+3(10/54)]} = 54.55\lambda_1$$

$$\lambda_{2M(A_{20/54})} = \lambda_{[54+3(20/54)]} = 55.11\lambda_2$$

$$\lambda_{3M(A_{24/54})} = \lambda_{[54+3(24/54)]} = 55.33\lambda_3$$

Therefore

$$\lambda_{1M(A_{10/54})} + \lambda_{2M(A_{20/54})} + \lambda_{3M(A_{24/54})} = 54.55\lambda_1 + 55.11\lambda_2 + 55.33\lambda_3$$

$$\lambda_1 M[A_{M(A_{10/54})}] + \lambda_2 M[A_{M(A_{20/54})}] + \lambda_3 M[A_{M(A_{24/54})}] = 220 + \lambda \qquad (4)$$

From (3) and (4)

$$M[A_{[\lambda M(A_{30/54}) + (1-\lambda)M(A_{24/54})]}]$$
$$= \lambda M[A_{M(A_{30/54})}] + (1-\lambda) M[A_{M(A_{24/54})}]$$

(iv) Arithmetic Mean & Geometric Mean

For Example 3.1.

A.M. of M (A30/54) and M (A24/54) is ½[M(A30/54) + M(A24/54)]= 55.5

And G.M. of M (A30/54) and M (A24/54) is [M (A30/54) x M (A24/54)]1/2=55.203.

Hence A.M. $\geq$ G.M.

For Example 3.2.

A.M. of $M(A_{10/54}), M(A_{20/54})$ and $M(A_{24/54})$ is

$$\frac{1}{3}[M(A_{10/54}) + M(A_{20/54}) + M(A_{24/54})] = \frac{1}{3}[54.55 + 55.11 + 55.33] = 54.997$$

And G.M. of $M(A_{10/54}), M(A_{20/54})$ and $M(A_{24/54})$ is 54.990

Hence A.M. $\geq$ G.M.

5. Conclusion

In short, adding matrices together will seriously change the geometry of their corresponding linear maps and linear subspaces. The places where they overlap will stay the same but everywhere they differ gets thrown out.

But in our methodology they always overlap.

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