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ESTIMATE OF FOURTH HANKEL DETERMINANT FOR A NEW SUBCLASS OF MULTIVALENT FUNCTIONS

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Abstract: The paper introduces a new subclass of *p*-valent functions in the open unit disc $E=\{z:|z|<1\}$ and investigates the upper bound of the third and fourth Hankel determinants for the functions in this class. This work leads towards the investigation of fourth Hankel determinant for several other classes of multivalent functions.

Keywords: Analytic functions, Univalent functions, Hankel determinant, Coefficient bounds, Functions with bounded boundary rotation.

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1. Introduction

Let A_p denote the class of analytic functions of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_k z^k, \left(p\in N=\{1,2,3,...\}\right) \tag{1}$$

in the unit disc $E=\{z:|z|<1\}$ which are further normalized by $f(0)=f'(0)-1=0$.

Let P denote the class of analytic functions p of the form

$$p(z)=1+\sum_{n=1}^{\infty}c_n z^n$$

whose real parts are positive in E.

For understanding of the main content, let us recall the following definitions:

$R=\{f : f\in A, \mathrm{Re}(f'(z))>0, z\in E\}$, the class of bounded turning functions introduced and studied by MacGregor [12].

$R'(\alpha)=\{f : f\in A, \mathrm{Re}\{f'(z)+\alpha z f''(z)\}>0, \alpha\geq 0, z\in E\}$, the class studied by Sahoo [15].

In particular $R'(0)\equiv R.$

Later on, Vamshee Krishna et al. [18] introduced a subclass of *p*-valent functions as follows:

$$RT_p=\left\{f : f\in A_p, \mathrm{Re}\left(\frac{f'(z)}{pz^{p-1}}\right)>0, z\in E\right\}.$$

For *p* = 1, $RT_1\equiv R.$

Motivated by the above defined classes, we define the following subclass of *p*-valent functions:

$$R'_p(\alpha)=\left\{f : f\in A_p, \mathrm{Re}\left\{(p+(1-p)\alpha)\frac{f'(z)}{p^2z^{p-1}}+\alpha\frac{f''(z)}{p^2z^{p-2}}\right\}>0, \alpha\geq 0, z\in E\right\}.$$

The following observations are obvious:

(i) $R'_1(\alpha)\equiv R'(\alpha).$

(ii) $R'_p(0)\equiv RT_p.$

(iii) $R'_1(0)\equiv R.$

Noonan and Thomas [14] stated the *qth* Hankel determinant for $q\geq 1$ and $n\geq 1$ as

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For $q=2, n=p, a_1=1$ and $q=2, n=p+1,$ the Hankel determinant takes the form of

$$H_2(p)=\left|a_{p+2}-a_{p+1}^2\right| \text{ and } H_2(p+1)=\left|a_{p+1}a_{p+3}-a_{p+2}^2\right|.$$

Also for *q* = 3 and *n* = *p* , the Hankel determinant becomes

$$H_3(p)=\begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix},$$

which is known as Hankel determinant of order 3.

For $f \in A_p, a_p = 1$, we have

$$H_3(p) = a_{p+2}\left(a_{p+1}a_{p+3} - a_{p+2}^2\right) - a_{p+3}\left(a_{p+3} - a_{p+1}a_{p+2}\right) + a_{p+4}\left(a_{p+2} - a_{p+1}^2\right)$$

and by using the triangle inequality, we have

$$|H_3(p)| \leq |a_{p+2}||a_{p+1}a_{p+3} - a_{p+2}^2| + |a_{p+3}||a_{p+3} - a_{p+1}a_{p+2}| + |a_{p+4}||a_{p+2} - a_{p+1}^2|. \tag{2}$$

For any $f \in A_p$ of the form (1), we can represent the fourth Hankel determinant as

$$H_{4,p}(f) = a_{p+6}H_3(p) - a_{p+5}D_1 + a_{p+4}D_2 - a_{p+3}D_3, \tag{3}$$

where D_1, D_2 and D_3 are determinants of order 3 given by

$$D_1 = \left(a_{p+2}a_{p+5} - a_{p+3}a_{p+4}\right) - a_{p+1}\left(a_{p+1}a_{p+5} - a_{p+2}a_{p+4}\right) + a_{p+3}\left(a_{p+1}a_{p+3} - a_{p+2}^2\right), \tag{4}$$

$$D_2 = \left(a_{p+3}a_{p+5} - a_{p+4}^2\right) - a_{p+1}\left(a_{p+2}a_{p+5} - a_{p+3}a_{p+4}\right) + a_{p+2}\left(a_{p+2}a_{p+4} - a_{p+3}^2\right), \tag{5}$$

$$D_3 = a_{p+1}\left(a_{p+3}a_{p+5} - a_{p+4}^2\right) - a_{p+2}\left(a_{p+2}a_{p+5} - a_{p+3}a_{p+4}\right) + a_{p+3}\left(a_{p+2}a_{p+4} - a_{p+3}^2\right). \tag{6}$$

In general, the problem of Hankel determinants for various subclasses of analytic functions has earlier been taken up by Noor [14], Ehrenborg [5] and Layman [8] etc. Particularly, for *p*-valent functions, Hankel determinants were studied by various authors including Vamshee et al. [18], Hayami and Owa [6] and Amourah et al.[1].

Moreover Singh et al. [16,17] has recently investigated the upper bound of the fourth Hankel determinant for the certain subclasses of analytic functions.

The early coefficients of the inverse of regular function are studied by Libera and Zlotkiewiez [9, 10] and the coefficient of multivalent close to convext function is investigated by Livingston [11].

The prime objective of this study is to investigate the upper bound for the functional $H_{4,p}(f)$ for the class $R'_p(\alpha)$. This work will pave the way for the future researchers to work in this direction.

2. Preliminary results

Lemma 2.1 [4,11] If $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$, then for $n, k \in N = \{1,2,3,...\}$, we have the following inequalities:

$$|c_{n+k} - \lambda c_n c_k| \leq 2 \text{ for } 0 \leq \lambda \leq 1$$

and

$$|c_n| \le 2.$$

Lemma 2.2 If $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$, then for $n, k \in N = \{1,2,3,...\}$, then

$$|c_{n+k} - \lambda c_n c_k| \le 4\lambda - 2 \text{ for } \lambda \ge 1.$$

Proof. For $\lambda \ge 1,$ we have

$$|c_{n+k} - \lambda c_n c_k| \le |c_n c_k - c_{n+k}| + (\lambda - 1)|c_n c_k|.$$

Using Lemma 2.1, the above inequality yields

$$|c_{n+k} - \lambda c_n c_k| \le 4\lambda - 2.$$

Lemma 2.3 [12,13] If $p \in P$, then

$$2c_2 = c_1^2 + (4 - c_1^2)x,$$

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z,$$

for some *x* and *z* satisfying $|x| \le 1, |z| \le 1$ and $c_1 \in [0,2]$.

Lemma 2.4 [3] If $p \in P$, then

$$\left|c_2 - \sigma \frac{c_1^2}{2}\right| \le \begin{cases} 2(1-\sigma) & if \quad \sigma \le 0, \\ 2 & if \quad 0 \le \sigma \le 2, \\ 2(\sigma - 1) & if \quad \sigma \ge 2. \end{cases}$$

3. Main Results

Theorem 3.1 If $f \in R'_p(\alpha),$ then

$$|a_{p+j}| \le \frac{2p^2}{(p + j\alpha)(p + j)}. \tag{7}$$

Proof. As $f \in R'_p(\alpha),$ therefore by definition, there exists a function $p(z) \in P$ such that

$$(p + (1-p)\alpha)\frac{f'(z)}{p^2 z^{p-1}} + \alpha \frac{f''(z)}{p^2 z^{p-2}} = p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

On expanding and equating the coefficients in the above equation, it yields

$$a_{p+j} = \frac{p^2 c_j}{(p+j\alpha)(p+j)}. \tag{8}$$

Using Lemma 2.1 in (8), the result (7) is obvious.

For p = 1, Theorem 3.1 yields the following result due to Sahoo [15]:

Corollary 3.1.1 If $f \in R'(\alpha)$, then

$$|a_k| \leq \frac{2}{k(1+(k-1)\alpha)}, k \geq 2.$$

For $\alpha = 0$, Theorem 3.1 gives the following result proved by V. Krishna et al.[8]:

Corollary 3.1.2 If $f \in RT_p$, then

$$|a_{p+j}| \leq \frac{2p}{p+j}.$$

For $p=1, \alpha=0$, Theorem 3.1 gives the following well known result mentioned in [19]:

Corollary 3.1.3 If $f \in R$, then

$$|a_k| \leq \frac{2}{k}.$$

Theorem 3.2 If $f \in R'_p(\alpha)$, then

$$|a_{p+2} - a_{p+1}^2| \leq \frac{2p^2}{(p+2\alpha)(p+2)}. \tag{9}$$

Proof. Using (8), we find that

$$|a_{p+2} - a_{p+1}^2| = \left| \frac{p^2 c_2}{(p+2\alpha)(p+2)} - \frac{p^4 c_1^2}{(p+\alpha)^2 (p+1)^2} \right|,$$

which implies

$$|a_{p+2} - a_{p+1}^2| = \frac{p^2}{(p+2\alpha)(p+2)} \left| c_2 - \frac{2p^2(p+2\alpha)(p+2)}{(p+\alpha)^2(p+1)^2} . \frac{c_1^2}{2} \right|.$$

Now, since $0 \leq \sigma = \dfrac{2p^2(p+2\alpha)(p+2)}{(p+\alpha)^2(p+1)^2} \leq 2,$ then by Lemma 2.4, the result (9) is obvious.

For p = 1, Theorem 3.2 yields the following result due to Sahoo [15]:

Corollary 3.2.1 If $f \in R'(\alpha)$, then

$$\left|a_3 - a_2^2\right| \le \frac{2}{3(1+2\alpha)}.$$

For $\alpha = 0$, Theorem 3.2 gives the following result proved by Vamshee et al.[18]:

Corollary 3.2.2 If $f \in RT_p$, then

$$\left|a_{p+2} - a_{p+1}^2\right| \le \frac{2p}{p+2}.$$

For $p = 1, \alpha = 0$, Theorem 3.1 gives the following well known result mentioned in [9]:

Corollary 3.2.3 If $f \in R$, then

$$\left|a_3 - a_2^2\right| \le \frac{2}{3}.$$

Theorem 3.3 If $f \in R_p'(\alpha)$, then

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \le \frac{4p^4}{(p+2)^2(p+2\alpha)^2}. \tag{10}$$

Proof. Using equation (8), we have

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| = \left|\frac{p^4 c_1 c_3}{(p+1)(p+3)(p+\alpha)(p+3\alpha)} - \frac{p^4 c_2^2}{(p+2)^2(p+2\alpha)^2}\right|.$$

Using Lemma 2.3, rearranging the terms and applying triangle inequality along with the inequality $|z| \le 1$, it yields

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \le \frac{T}{4}\left[\begin{array}{l} \left[(p+2)^2(p+2\alpha)^2 - (p+1)(p+3)(p+\alpha)(p+3\alpha)\right]c_1^4 \\ +2\left[(p+2)^2(p+2\alpha)^2 - (p+1)(p+3)(p+\alpha)(p+3\alpha)\right]c_1^2 x\left(4-c_1^2\right) \\ +\left[\begin{array}{l}\left\{(p+2)^2(p+2\alpha)^2 - (p+1)(p+3)(p+\alpha)(p+3\alpha)\right\}c_1^2 \\ +4(p+1)(p+3)(p+\alpha)(p+3\alpha)\end{array}\right]\left(4-c_1^2\right)x^2 \\ +2(p+2)^2(p+2\alpha)^2\left(4-c_1^2\right)c_1\left(1-|x|^2\right) \end{array}\right]$$

where $T = \dfrac{p^4}{(p+1)(p+2)^2(p+3)(p+\alpha)(p+2\alpha)^2(p+3\alpha)}$.

For $c_1 = c \in [0,2]$ and $|x| = \mu$, we have

$$\left|a_{p+1}a_{p+3}-a_{p+2}^{2}\right|\le\frac{T}{4}\left[\begin{array}{l}\left[(p+2)^{2}(p+2\alpha)^{2}-(p+1)(p+3)(p+\alpha)(p+3\alpha)\right]c^{4}\\+2(p+2)^{2}(p+2\alpha)^{2}\left(4-c^{2}\right)c\\+2\left[(p+2)^{2}(p+2\alpha)^{2}-(p+1)(p+3)(p+\alpha)(p+3\alpha)\right]c^{2}\left(4-c^{2}\right)\mu\\+\left\{(p+2)^{2}(p+2\alpha)^{2}-(p+1)(p+3)(p+\alpha)(p+3\alpha)\right\}\left(4-c^{2}\right)(c-2)(c-\beta)\mu^{2}\end{array}\right]=F(c,\mu),$$

where $\beta=\beta(\alpha)=\dfrac{2(p+1)(p+3)(p+\alpha)(p+3\alpha)}{(p+2)^{2}(p+2\alpha)^{2}-(p+1)(p+3)(p+\alpha)(p+3\alpha)}$.

Now $\dfrac{\partial F}{\partial\mu}=$

$$\frac{p^{4}\left\{(p+2)^{2}(p+2\alpha)^{2}-(p+1)(p+3)(p+\alpha)(p+3\alpha)\right\}\left(4-c^{2}\right)\left\{c^{2}+(c-2)(c-\beta)\mu\right\}}{2(p+1)(p+2)^{2}(p+3)(p+\alpha)(p+2\alpha)^{2}(p+3\alpha)}>0.$$

So $\max.F(c,\mu)=F(c,1)=G(c)$.

Now

$$G'(c)=-\frac{2p^{4}}{(p+1)(p+2)^{2}(p+3)(p+\alpha)(p+2\alpha)^{2}(p+3\alpha)}\left[\begin{array}{l}\left\{(p+2)^{2}(p+2\alpha)^{2}-(p+1)(p+3)(p+\alpha)(p+3\alpha)\right\}c^{3}\\+\left[4(p+1)(p+3)(p+\alpha)(p+3\alpha)-3p^{2}(p+2)^{2}(p+2\alpha)^{2}\right]\end{array}\right]<0.$$

Therefore $\max.G(c)=G(0)$.

Hence the result (10).

For p = 1, Theorem 3.3 yields the following result due to Sahoo [15]:

Corollary 3.3.1 If $f\in R'(\alpha)\left(0\le\alpha\le\frac{1}{2}\right)$, then

$$\left|a_{2}a_{4}-a_{3}^{2}\right|\le\frac{4}{9(1+2\alpha)^{2}}.$$

For $\alpha=0$, Theorem 3.3 gives the following result proved by Vamshee et al.[18]:

Corollary 3.3.2 If $f\in RT_{p}$, then

$$\left|a_{p+1}a_{p+3}-a_{p+2}^{2}\right|\le\frac{4p^{2}}{(p+2)^{2}}.$$

For $p=1,\alpha=0$, Theorem 3.3 gives the following result due to Janteng et al.[7]:

Corollary 3.3.3 If $f\in R$, then

$$\left|a_{2}a_{4}-a_{3}^{2}\right|\le\frac{4}{9}.$$

Theorem 3.4 If $f \in R'_p(\alpha)$, then

$$\left|a_{p+1}a_{p+2}-a_{p+3}\right| \leq \frac{2p^2\left[3(p+1)(p+2)(p+\alpha)(p+2\alpha)-2p^2(p+3)(p+3\alpha)\right]^{\frac{3}{2}}}{3(p+1)(p+2)(p+3)(p+\alpha)(p+2\alpha)(p+3\alpha)\sqrt{3\{(p+1)(p+2)(p+\alpha)(p+2\alpha)-p^2(p+3)(p+3\alpha)\}}}. \quad (11)$$

Proof. Using equation (8), we have

$$\left|a_{p+1}a_{p+2}-a_{p+3}\right| = \left|\frac{p^4c_1c_2}{(p+1)(p+2)(p+\alpha)(p+2\alpha)} - \frac{p^2c_3}{(p+3)(p+3\alpha)}\right|.$$

Using Lemma 2.3, rearranging the terms and applying triangle inequality along with the inequality $|z| \leq 1$, it yields

$$\left|a_{p+1}a_{p+2}-a_{p+3}\right| \leq \frac{S}{4}\begin{bmatrix}\{2p^2(p+3)(p+3\alpha)-(p+1)(p+2)(p+\alpha)(p+2\alpha)\}c_1^3 \\ +2(p+1)(p+2)(p+\alpha)(p+2\alpha)(4-c_1^2) \\ +2\left[-p^2(p+3)(p+3\alpha)+(p+1)(p+2)(p+\alpha)(p+2\alpha)\right]c_1(4-c_1^2)|x| \\ +(p+1)(p+2)(p+\alpha)(p+2\alpha)(c-2)(4-c_1^2)|x|^2\end{bmatrix},$$

where $S = \dfrac{p^2}{(p+1)(p+2)(p+3)(p+\alpha)(p+2\alpha)(p+3\alpha)}$.

For $c_1 = c \in [0,2]$ and $|x| = \mu$, we have

$$\left|a_{p+1}a_{p+2}-a_{p+3}\right| \leq \frac{S}{4}\begin{bmatrix}\{2p^2(p+3)(p+3\alpha)-(p+1)(p+2)(p+\alpha)(p+2\alpha)\}c^3 \\ +2(p+1)(p+2)(p+\alpha)(p+2\alpha)(4-c^2) \\ +2\left[-p^2(p+3)(p+3\alpha)+(p+1)(p+2)(p+\alpha)(p+2\alpha)\right]c(4-c^2)\mu \\ +(p+1)(p+2)(p+\alpha)(p+2\alpha)(c-2)(4-c^2)\mu^2\end{bmatrix} = F(c,\mu).$$

Now $F(\mu) \leq F(1)$ and

$$F(1) = \frac{p^2\left\{\{3(p+1)(p+2)(p+\alpha)(p+2\alpha)-2p^2(p+3)(p+3\alpha)\}c-\{(p+1)(p+2)(p+\alpha)(p+2\alpha)-p^2(p+3)(p+3\alpha)\}c^3\right\}}{(p+1)(p+2)(p+3)(p+\alpha)(p+2\alpha)(p+3\alpha)} = G_1(c).$$

$$G_1'(c) = \frac{p^2\left\{\{3(p+1)(p+2)(p+\alpha)(p+2\alpha)-2p^2(p+3)(p+3\alpha)\}-\{(p+1)(p+2)(p+\alpha)(p+2\alpha)-p^2(p+3)(p+3\alpha)\}3c^2\right\}}{(p+1)(p+2)(p+3)(p+\alpha)(p+2\alpha)(p+3\alpha)}.$$

Now $G_1'(c) = 0$, gives

$$c = \sqrt{\frac{3(p+1)(p+2)(p+\alpha)(p+2\alpha)-2p^2(p+3)(p+3\alpha)}{3\{(p+1)(p+2)(p+\alpha)(p+2\alpha)-p^2(p+3)(p+3\alpha)\}}} = c_0.$$

So max. $G_1(c) = G_1(c_0)$.

Hence, the result (11) is obvious.

For p = 1, Theorem 3.4 yields the following result due to Singh et al. [16]:

Corollary 3.4.1 If $f \in R'(\alpha)$, then

$$\left|a_2 a_3 - a_4\right| \le \frac{\left(5 + 15\alpha + 18\alpha^2\right)^{\frac{3}{2}}}{18(1+\alpha)(1+2\alpha)(1+3\alpha)\sqrt{3\left(1+3\alpha+6\alpha^2\right)}}.$$

For $\alpha = 0$, Theorem 3.4 gives the following result proved by V. Krishna et al.[18]:

Corollary 3.4.2 If $f \in RT_p$, then

$$\left|a_{p+1} a_{p+2} - a_{p+3}\right| \le \frac{\sqrt{2}\,p\left[p^2 + 3p + 6\right]^{\frac{3}{2}}}{3\sqrt{3}(p+1)(p+2)(p+3)}.$$

For $p = 1, \alpha = 0$, Theorem 3.4 gives the following result due to Babalola [2,3]:

Corollary 3.4.3 If $f \in R$, then

$$\left|a_2 a_3 - a_4\right| \le \frac{5\sqrt{5}}{18\sqrt{3}}.$$

Theorem 3.5 If $f \in R'_p(\alpha)$, then

$$\left|H_3(p)\right| \le \frac{4p^4}{(p+2)(p+2\alpha)}\left[\begin{array}{l} \dfrac{2p^2}{(p+2)^2(p+2\alpha)^2} + \dfrac{1}{(p+4)(p+4\alpha)} \\ + \dfrac{\left\{3(p+1)(p+2)(p+\alpha)(p+2\alpha) - 2p^2(p+3)(p+3\alpha)\right\}^{\frac{3}{2}}}{3(p+1)(p+3)^2(p+\alpha)(p+3\alpha)^2\sqrt{3\left\{(p+1)(p+2)(p+\alpha)(p+2\alpha) - p^2(p+3)(p+3\alpha)\right\}}} \end{array}\right].$$

Proof. Using Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4 in (2), the above result is obvious.

For p = 1, Theorem 3.5 yields the following result due to Singh et al. [16]:

Corollary 3.5.1 If $f \in R'(\alpha)$, then

$$\left|H_3(1)\right| \le \frac{1}{3(1+2\alpha)}\left[\frac{8}{9(1+2\alpha)^2} + \frac{4}{5(1+4\alpha)} + \frac{\left(5+15\alpha+18\alpha^2\right)^{\frac{3}{2}}}{12(1+\alpha)(1+3\alpha)^2\sqrt{3\left(1+3\alpha+6\alpha^2\right)}}\right].$$

For $\alpha = 0$, Theorem 3.5 gives the following result proved by Vamshee et al.[8]:

Corollary 3.5.2 If $f \in RT_p$, then

$$\left|H_3(p)\right| \le \frac{2p^2}{p+2}\left[\frac{4p}{(p+2)^2} + \frac{\sqrt{2}\left[p^2+3p+6\right]^{\frac{3}{2}}}{3\sqrt{3}(p+1)(p+3)^2} + \frac{2}{p+4}\right].$$

For $p=1, \alpha=0$, Theorem 3.5 gives the following result due to Babalola [2]:

Corollary 3.5.3 If $f \in R$, then

$$\left|H_3(1)\right| \le 0.7422.$$

Theorem 3.6 If $f(z) \in R_p'(\alpha)$, then

$$\left|H_4(p)\right| \le \frac{8p^6}{(p+2)(p+6)(p+2\alpha)(p+6\alpha)}\left[\begin{array}{l}\dfrac{2p^2}{(p+2)^2(p+2\alpha)^2} + \dfrac{1}{(p+4)(p+4\alpha)} \\ + \dfrac{\left\{3(p+1)(p+2)(p+\alpha)(p+2\alpha) - 2p^2(p+3)(p+3\alpha)\right\}^{\frac{3}{2}}}{3(p+1)(p+3)^2(p+\alpha)(p+3\alpha)^2\sqrt{3\left\{(p+1)(p+2)(p+\alpha)(p+2\alpha) - p^2(p+3)(p+3\alpha)\right\}}}\end{array}\right] \tag{12}$$

$$+\frac{2p^2}{(p+5)(p+5\alpha)}u(p,\alpha) + \frac{2p^2}{(p+4)(p+4\alpha)}v(p,\alpha) + \frac{2p^2}{(p+3)(p+3\alpha)}w(p,\alpha)$$

where

$$u(p,\alpha) = 2p^4\left(4p^2-2\right)\left[\begin{array}{l}\dfrac{1}{(p+1)^2(p+5)(p+\alpha)^2(p+5\alpha)} + \dfrac{1}{(p+2)^2(p+3)(p+2\alpha)^2(p+3\alpha)} \\ + \dfrac{1}{(p+1)(p+3)^2(p+\alpha)(p+3\alpha)^2}\end{array}\right] \tag{13}$$

$$+\frac{172p^4\left(4p^2-2\right)+4p^4}{48(p+1)(p+2)(p+4)(p+\alpha)(p+2\alpha)(p+4\alpha)},$$

$$v(p,\alpha) = \left[\begin{array}{l}\dfrac{63p^4\left(4p^2-2\right)}{25(p+1)(p+2)(p+5)(p+\alpha)(p+2\alpha)(p+5\alpha)} \\ + \dfrac{18p^4\left(4p^2-2\right)}{5(p+4)(p+2)^2(p+4\alpha)(p+2\alpha)^2} + \dfrac{150p^4\left(4p^2-2\right)+4p^4}{75(p+2)(p+3)^2(p+2\alpha)(p+3\alpha)^2}\end{array}\right] \tag{14}$$

and

$$w(p,\alpha) = 2p^4\left(4p^2-2\right)\left[\begin{array}{l}\dfrac{1}{(p+2)^2(p+5)(p+2\alpha)^2(p+5\alpha)} + \dfrac{1}{(p+1)(p+3)(p+5)(p+\alpha)(p+3\alpha)(p+5\alpha)} \\ + \dfrac{2}{(p+3)^3(p+3\alpha)^3} + \dfrac{1}{(p+1)(p+4)^2(p+\alpha)(p+4\alpha)^2} \\ + \dfrac{17}{16(p+2)(p+3)(p+4)(p+2\alpha)(p+3\alpha)(p+4\alpha)}\end{array}\right] \tag{15}$$

$$+\frac{p^4}{(p+1)(p+2)^2(p+3)(p+4)^2(p+5)(p+\alpha)(p+2\alpha)^2(p+3\alpha)(p+4\alpha)^2(p+5\alpha)}.$$

Proof. Using (8) in (4), (5) and (6), it gives

$$D_1 = \frac{p^4 c_2 c_5}{(p+2)(p+5)(p+2\alpha)(p+5\alpha)} - \frac{p^4 c_3 c_4}{(p+3)(p+4)(p+3\alpha)(p+4\alpha)}$$
$$- \frac{p^6 c_1^2 c_5}{(p+1)^2(p+5)(p+\alpha)^2(p+5\alpha)} + \frac{p^6 c_1 c_2 c_4}{(p+1)(p+2)(p+4)(p+\alpha)(p+2\alpha)(p+4\alpha)} \tag{16}$$
$$+ \frac{p^6 c_1 c_3^2}{(p+1)(p+3)^2(p+\alpha)(p+3\alpha)^2} - \frac{p^6 c_2^2 c_3}{(p+3)(p+2)^2(p+3\alpha)(p+2\alpha)^2},$$

$$D_2 = \frac{p^4 c_3 c_5}{(p+3)(p+5)(p+3\alpha)(p+5\alpha)} - \frac{p^4 c_4^2}{(p+4)^2(p+4\alpha)^2}$$
$$- \frac{p^6 c_1 c_2 c_5}{(p+1)(p+2)(p+5)(p+\alpha)(p+2\alpha)(p+5\alpha)} + \frac{p^6 c_1 c_3 c_4}{(p+1)(p+3)(p+4)(p+\alpha)(p+3\alpha)(p+4\alpha)} \tag{17}$$
$$+ \frac{p^6 c_2^2 c_4}{(p+2)^2(p+4)(p+2\alpha)^2(p+4\alpha)} - \frac{p^6 c_2 c_3^2}{(p+2)(p+3)^2(p+2\alpha)(p+3\alpha)^2}$$

and

$$D_3 = \frac{p^6 c_1 c_3 c_5}{(p+1)(p+3)(p+5)(p+\alpha)(p+3\alpha)(p+5\alpha)} - \frac{p^6 c_1 c_4^2}{(p+1)(p+4)^2(p+\alpha)(p+4\alpha)^2}$$
$$- \frac{p^6 c_2^2 c_5}{(p+2)^2(p+5)(p+2\alpha)^2(p+5\alpha)} + \frac{2p^6 c_2 c_3 c_4}{(p+2)(p+3)(p+4)(p+2\alpha)(p+3\alpha)(p+4\alpha)} - \frac{p^6 c_3^3}{(p+3)^3(p+3\alpha)^3}. \tag{18}$$

On rearranging the terms in (16), (17) and (18), it yields

$$D_1 = \frac{p^4 c_5 (c_2 - p^2 c_1^2)}{(p+1)^2(p+5)(p+\alpha)^2(p+5\alpha)} + \frac{p^4 c_3 (c_4 - p^2 c_2^2)}{(p+3)(p+2)^2(p+3\alpha)(p+2\alpha)^2}$$
$$- \frac{p^4 c_3 (c_4 - p^2 c_1 c_3)}{(p+1)(p+3)^2(p+\alpha)(p+3\alpha)^2} - \frac{67 p^4 c_4 (c_3 - p^2 c_1 c_2)}{48(p+1)(p+2)(p+4)(p+\alpha)(p+2\alpha)(p+4\alpha)} \tag{19}$$
$$+ \frac{19 p^4 c_2 (c_5 - p^2 c_1 c_4)}{48(p+1)(p+2)(p+4)(p+\alpha)(p+2\alpha)(p+4\alpha)} + \frac{p^4 c_2 c_5}{48(p+1)(p+2)(p+4)(p+\alpha)(p+2\alpha)(p+4\alpha)},$$

$$D_2 = \frac{p^4 c_5 (c_3 - p^2 c_1 c_2)}{(p+1)(p+2)(p+5)(p+\alpha)(p+2\alpha)(p+5\alpha)} - \frac{p^4 c_4 (c_4 - p^2 c_2^2)}{(p+4)(p+2)^2(p+4\alpha)(p+2\alpha)^2}$$
$$+ \frac{p^4 c_3 (c_5 - p^2 c_2 c_3)}{(p+2)(p+3)^2(p+2\alpha)(p+3\alpha)^2} - \frac{4 p^4 c_4 (c_4 - p^2 c_1 c_3)}{5(p+2)^2(p+4)(p+2\alpha)^2(p+4\alpha)} \tag{20}$$
$$- \frac{13 p^4 c_3 (c_5 - p^2 c_1 c_4)}{50(p+1)(p+2)(p+5)(p+\alpha)(p+2\alpha)(p+5\alpha)} + \frac{p^4 c_3 c_5}{75(p+2)(p+3)^2(p+2\alpha)(p+3\alpha)^2}$$

and

$$D_3 = \frac{p^4 c_5 (c_4 - p^2 c_2^2)}{(p+2)^2 (p+5)(p+2\alpha)^2 (p+5\alpha)} - \frac{p^4 c_5 (c_4 - p^2 c_1 c_3)}{(p+1)(p+3)(p+5)(p+\alpha)(p+3\alpha)(p+5\alpha)} + \frac{p^4 c_3 (c_6 - p^2 c_3^2)}{(p+3)^3 (p+3\alpha)^3}$$
$$- \frac{p^4 c_3 (c_6 - p^2 c_2 c_4)}{(p+3)^3 (p+3\alpha)^3} + \frac{p^4 c_4 (c_5 - p^2 c_1 c_4)}{(p+1)(p+4)^2 (p+\alpha)(p+4\alpha)^2} - \frac{17 p^4 c_4 (c_5 - p^2 c_2 c_3)}{16(p+2)(p+3)(p+4)(p+2\alpha)(p+3\alpha)(p+4\alpha)} \tag{21}$$
$$+ \frac{p^4 c_4 c_5}{4(p+1)(p+2)^2 (p+3)(p+4)^2 (p+5)(p+\alpha)(p+2\alpha)^2 (p+3\alpha)(p+4\alpha)^2 (p+5\alpha)}.$$

Using Lemma 2.2 and applying triangle inequality in (19), (20) and (21), we obtain

$$|D_1| \le u(p,\alpha), \tag{22}$$

$$|D_2| \le v(p,\alpha) \tag{23}$$

and

$$|D_3| \le w(p,\alpha) \tag{24}$$

where $u(p,\alpha), v(p,\alpha)$ and $w(p,\alpha)$ are defined in (13), (14) and (15) respectively.

Hence using Theorem 3.1, Theorem 3.5, (22), (23) and (24) and applying triangle inequality in (3), the result (12) is obvious.

On putting $p = 1$, in Theorem 3.6, we obtain the following result due to Singh et al.[16]:

Corollary 3.6.1 If $f(z) \in R'(\alpha)$, then

$$|H_4(1)| \le$$
$$\frac{2}{21(1+2\alpha)(1+6\alpha)} \left[\frac{8}{9(1+2\alpha)^2} + \frac{4}{5(1+4\alpha)} + \frac{(5+15\alpha+18\alpha^2)^{\frac{3}{2}}}{12(1+\alpha)(1+3\alpha)^2 \sqrt{3(1+3\alpha+6\alpha^2)}} \right]$$
$$+ \frac{1}{3(1+5\alpha)} u(\alpha) + \frac{2}{5(1+4\alpha)} v(\alpha) + \frac{1}{2(1+3\alpha)} w(\alpha),$$

where

$$u(\alpha) = \frac{1}{6(1+\alpha)^2 (1+5\alpha)} + \frac{1}{9(1+3\alpha)(1+2\alpha)^2} + \frac{1}{8(1+\alpha)(1+3\alpha)^2} + \frac{29}{120(1+\alpha)(1+2\alpha)(1+4\alpha)},$$
$$v(\alpha) = \frac{7}{50(1+\alpha)(1+2\alpha)(1+5\alpha)} + \frac{4}{25(1+4\alpha)(1+2\alpha)^2} + \frac{19}{225(1+2\alpha)(1+3\alpha)^2}$$

and

$$w(\alpha)=\frac{2}{27(1+2\alpha)^2(1+5\alpha)}+\frac{1}{12(1+\alpha)(1+3\alpha)(1+5\alpha)}+\frac{1}{8(1+3\alpha)^3}+\frac{2}{25(1+\alpha)(1+4\alpha)^2}$$
$$+\frac{17}{240(1+2\alpha)(1+3\alpha)(1+4\alpha)}+\frac{1}{10800(1+\alpha)(1+2\alpha)^2(1+3\alpha)(1+4\alpha)^2(1+5\alpha)}.$$

On putting $p=1, \alpha=1$ in Theorem 3.6, we obtain the following result:

Corollary 3.6.2 Let $f(z)\in R$, then

$$\left|H_{4,1}(f)\right|\leq 0.7973.$$

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