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SOLUTIONS OF CAUCHY PROBLEMS WITH CAPUTO-HADAMARD FRACTIONAL DERIVATIVES

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Abstract: This work is dedicated to obtain explicit solutions of Cauchy problems incorporating a left side Caputo Hadamard fractional derivative, in continuously differentiable function spaces. We obtain law of exponents for the derivative which is available for almost all ancient fractional derivatives but hasn't been developed for this derivative yet. We also obtain compact form solutions of a few problems are obtained using the Adomain decomposition method in terms of Mittag-Leffler type functions.

Keywords: Caputo Hadamard fractional derivative, Cauchy problems, Adomian decomposition method.

1. Introduction

In the study of fractional operators, the fractional derivatives are defined via fractional integrals. The journey to the development of fractional operators is considerably long and witnesses the advent of many operators with versatile properties, suiting to the need of the problem under consideration for modeling. Although the fractional operators of Riemann-Liouville type are the initial points of this voyage, while looking for the viable properties of the fractional derivative of Caputo, it has been the more popular as compared to the Riemann-Liouville fractional derivative. Both these fractional derivative operators find their base in Riemann-Liouville fractional integral; hence this integral plays a major role in fractional calculus. Hadamard fractional integrals and the derivatives came into picture with the kernel of the integral implicating the logarithmic function of an arbitrary exponent. To have more applicability of these operators on real-world problems, Caputo modification to the Hadamard fractional derivative, known as the Caputo Hadamard derivative were devised in [9], on the same lines as Caputo fractional derivative followed the Riemann-Liouville fractional derivatives. For more properties of the Caputo Hadamard fractional derivative, we refer to [1, 2, 5, 15, 16].

The Adomian decomposition method is used for obtaining solutions iteratively in series form, it has been introduced and developed by Adomian [3, 4]. The advantage over other methods is that solution in series form provides an opportunity for writing the solution in the closed-form if possible otherwise we may truncate the series keeping in view our permissible truncation error. However, in the present paper, the solutions are written in terms of closed-form series. Without going into many details, we refer [3, 4, 14, 7, 8] and the references therein.

The solution to Cauchy problems with some fractional derivatives has also been dealt in [2, 11]. In this paper, we obtain law of exponents for Caputo Hadamard fractional derivative and explicit solutions to Cauchy problems with a left side Caputo Hadamard fractional derivative into the weighted space of continuously differentiable functions.

2. Law of exponents for Caputo Hadamard fractional derivative

For finite or infinite interval of the half-axis $\mathbb{R}^+$, (a,b), $(0 \le a < b \le \infty)$, the left side Hadamard fractional integral and derivative J_{a+}^{α} and D_{a+}^{α} of order $\alpha \in \mathbb{C}, \Re(\alpha) > 0$, $n = [\Re(\alpha)] + 1$ are defined in [11]. Jarad *et al.* [9] provided the Caputo Hadamard fractional derivative as ${}^{C}\mathrm{D}_{a+}^{\alpha}$. The law of exponents of certain fractional derivatives may be found in [6, 12]. Here, we obtain the law for Caputo-Hadamard fractional derivative ${}^{C}\mathrm{D}_{a+}^{\alpha}$.

Theorem 2.1. For $f(x) = \left(\log \frac{x}{a}\right)^{\lambda} g(x)$, with $a, \lambda > 0$, generalized series expansion $g(x) = \sum_{n=0}^{\infty} a_n \left(\log \frac{x}{a}\right)^{n\alpha}$, where convergence radius $R > 0, 0 < \alpha \le 1$,

$$ {}^{C}\mathrm{D}_{a+}^{\gamma}\, {}^{C}\mathrm{D}_{a+}^{\rho} f(x) = {}^{C}\mathrm{D}_{a+}^{\gamma+\rho} f(x), \tag{1} $$

for all $\gamma, \rho > 0, \log \frac{x}{a} \in (0, R)$, $\mu = \max\left(\rho + -[-\gamma] - 1, -[-\rho-\gamma] - 1\right)$ and either

$\lambda > \mu$, or

$\lambda = \mu, a_0 = 0$, or

$\lambda \le \mu, a_k = 0$, for k=0,1,...,$-\left[-\frac{\mu-\lambda}{\alpha}\right] - 1$.

Proof. For part (a), we have by definition of Caputo Hadamard fractional derivative [9]

$$ \left({}^{C}\mathrm{D}_{a+}^{\rho} f\right)(x) = \mathrm{J}_{a+}^{(-[-\rho]-\rho)} \delta^{-[-\rho]} \sum_{n=0}^{\infty} a_n \left(\log \frac{x}{a}\right)^{n\alpha+\lambda} \tag{2} $$

Using $\lambda > \mu \ge -1$, uniform convergence of the series for $\log \frac{x}{a} \in (0, R)$, we obtain

$$\left({}^{C}\mathrm{D}_{a+}^{\rho} f\right)(x) = \mathrm{J}_{a+}^{(-[-\rho]-\rho)} \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda+1+[-\rho])} \left(\log\frac{x}{a}\right)^{n\alpha+\lambda+[-\rho]}$$

Given $\lambda > \mu \geq \left(-[-\rho]\right)-1$, using the following result [11],

$$\mathrm{J}_{a+}^{\alpha} \left(\log\frac{x}{a}\right)^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda+\alpha)} \left(\log\frac{x}{a}\right)^{\lambda+\alpha-1}, (a < x < b), \tag{3}$$

the uniform convergence and changing the order of summation and integration, we arrive at

$$\left({}^{C}\mathrm{D}_{a+}^{\rho} f\right)(x) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\rho+1)} \left(\log\frac{x}{a}\right)^{n\alpha+\lambda-\rho}. \tag{4}$$

Applying the same argument as above with $\lambda > \mu \geq \rho - 1$, $\lambda > \mu \geq \rho - [-\gamma] - 1$, we now have

$$\begin{aligned} {}^{C}\mathrm{D}_{a+}^{\gamma}\, {}^{C}\mathrm{D}_{a+}^{\rho} f(x) &= {}^{C}\mathrm{D}_{a+}^{\gamma} \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\rho+1)} \left(\log\frac{x}{a}\right)^{n\alpha+\lambda-\rho} \\ &= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\gamma-\rho+1)} \left(\log\frac{x}{a}\right)^{n\alpha+\lambda-\gamma-\rho}. \end{aligned} \tag{5}$$

Next, for $\lambda > \mu \geq -1$ and $\lambda > \mu \geq \rho + \gamma + \left(-[-\gamma-\rho] - \gamma - \rho\right) - 1$

$$\begin{aligned} {}^{C}\mathrm{D}_{a+}^{\gamma+\rho} f(x) &= {}^{C}\mathrm{D}_{a+}^{\gamma+\rho} \sum_{n=0}^{\infty} a_n \left(\log\frac{x}{a}\right)^{n\alpha+\lambda} \\ &= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\gamma-\rho+1)} \left(\log\frac{x}{a}\right)^{n\alpha+\lambda-\gamma-\rho}, \end{aligned} \tag{6}$$

which is precisely $\mathrm{D}_{a+}^{\gamma,\beta}\mathrm{D}_{a+}^{\rho,\beta} f(x)$, as given by (5).

We can write the conditions with (5) and (6), as conditions given in part (a).

For part (b), i.e. $\lambda \leq \mu$, by taking $a_k = 0$, for $0,1,...,l-1$, where $l = -\left[-\frac{\mu-\lambda}{\alpha}\right]$, because of the uniform convergence of derived series up to the order $-[-\rho]$, we have

$$\begin{aligned} \left({}^{C}\mathrm{D}_{a+}^{\rho} f\right)(x) &= {}^{C}\mathrm{D}_{a+}^{\rho} \sum_{n=0}^{\infty} a_n \left(\log\frac{x}{a}\right)^{n\alpha+\lambda} \\ &= \sum_{n=l}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\rho+1)} \left(\log\frac{x}{a}\right)^{n\alpha+\lambda-\rho} \end{aligned}$$

$$=\sum_{r=0}^{\infty} a_{r+l}\frac{\Gamma((r+l)\alpha+\lambda+1)}{\Gamma((r+l)\alpha+\lambda-\rho+1)}\left(\log\frac{x}{a}\right)^{(r+l)\alpha+\lambda-\rho} \tag{7}$$

If we let $\lambda' = l\alpha+\lambda$, then (7) is the same as (4) and the proof can be obtained similar to (a).

Theorem 2.2. For $f(x)=\left(\log\frac{b}{x}\right)^{\lambda} g(x)$, with $b,\lambda>0$, generalized series expansion $g(x)=\sum_{n=0}^{\infty} a_n\left(\log\frac{b}{x}\right)^{n\alpha}$ where convergence radius $R>0, 0<\alpha\le 1$. We have

$$ {}^{C}\mathrm{D}_{b-}^{\gamma}\, {}^{C}\mathrm{D}_{b-}^{\rho} f(x) = {}^{C}\mathrm{D}_{b-}^{\gamma+\rho} f(x), \tag{8}$$

for all $\log\frac{b}{x}\in(0,R)$, $\mu=\max\left(\rho-[-\gamma]-1,-[-\rho-\gamma]-1\right)$ and either

$\lambda>\mu,$ or

$\lambda=\mu, a_0=0,$ or

$\lambda<\mu, a_k=0,$ for $k=0,1,...,-\left[-\frac{\mu-\lambda}{\alpha}\right]-1$.

The proof may be obtained on the same lines of Theorem 2.1.

3. Cauchy Type Problems involving Caputo-Hadamard Fractional Derivatives

In this section, we will use the Adomian decomposition method to find explicit solutions to linear fractional differential equations with the left-side Caputo-Hadamard fractional derivative. In the weighted Banach space $C^{n}_{\delta,\gamma,\log}[a,b]$, $(0<\gamma\le 1)$ of functions, defined in [2]. To understand the working, we will solve two Cauchy type problems here.

Problem 3.1. We take the following problem:

$$\left({}^{C}\mathrm{D}_{a+}^{\alpha} y\right)(x)-\lambda y(x)= f(x)\left(a<x\le b; n-1<\alpha\le n; n\in\mathbb{N}; \lambda\in\mathbb{R}\right), \tag{9}$$

$$\left(\delta^{k} y\right)(a+)=b_k\left(b_k\in\mathbb{R}; k=0,1,...,n-1\right), \tag{10}$$

with $f(x)\in C_{\gamma,\log}[a,b](0\le\gamma<1)$ and $\gamma\le\alpha$.

On applying J_{a+}^{α} both the sides of (9) and using initial condition (10), in view of the result for compositions of Hadamard fractional integral J_{a+}^{α} with Caputo Hadamard fractional derivative ${}^{C}\mathrm{D}_{a+}^{\alpha}$, given in [9], we have

$$y(x)=\sum_{j=0}^{n-1}\frac{b_j\left(\log\frac{x}{a}\right)^{j}}{j!}+\lambda \mathrm{J}_{a+}^{\alpha} y(x)+\mathrm{J}_{a+}^{\alpha} f(x) \tag{11}$$

On decomposing the function $y(x)$ into:

$$y(x)=\sum_{k=0}^{\infty} y_k(x). \tag{12}$$

Applying the Adomian decomposition method, these components can recursively be obtained as

$$y_0=\sum_{j=0}^{n-1}\frac{b_j\left(\log\frac{x}{a}\right)^j}{j!}+\mathrm{J}_{a+}^{\alpha} f(x) \tag{13}$$

and $y_{k+1}(x)=\lambda \mathrm{J}_{a+}^{\alpha} y_k(x)$ (14)

Using (3) in recursive formulae (13) and (14), we obtain these components as

$$y_k=\sum_{j=0}^{n-1}b_j\frac{\lambda^k\left(\log\frac{x}{a}\right)^{\alpha k+j}}{\Gamma(\alpha k+j+1)}+\int_a^x\frac{\lambda^k\left(\log\frac{x}{t}\right)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)}\frac{f(t)}{t}dt. \tag{15}$$

Therefore, by using (15) in (12), the solution to (9) is given by

$$y(x)=\sum_{j=0}^{n-1}b_j\sum_{k=0}^{\infty}\frac{\lambda^k\left(\log\frac{x}{a}\right)^{\alpha k+j}}{\Gamma(\alpha k+j+1)}+\int_a^x\frac{f(t)}{t}\sum_{k=0}^{\infty}\frac{\lambda^k\left(\log\frac{x}{t}\right)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)}dt.$$

(16)

$$=\sum_{j=0}^{n-1}b_j\left(\log\frac{x}{a}\right)^j E_{\alpha,j+1}\left[\lambda\left(\log\frac{x}{a}\right)^{\alpha}\right]+\int_a^x\left(\log\frac{x}{t}\right)^{\alpha-1}E_{\alpha,\alpha}\left[\lambda\left(\log\frac{x}{t}\right)^{\alpha}\right]\frac{f(t)}{t}dt.$$

where $E_{\alpha,j+1}\left[\lambda\left(\log\frac{x}{a}\right)^{\alpha}\right]$ and $E_{\alpha,\alpha}\left[\lambda\left(\log\frac{x}{t}\right)^{\alpha}\right]$ are two parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ given by Wiman [16].

Lipschitz condition given in [2], is satisfied by the function $f(x,y)=\lambda y(x)+f(x)$, so if $\gamma\le\alpha$, by [Theorem 4.2, 2], problem (9)-(10) possess a unique solution in the space $C_{\delta,\gamma,\log}^{\alpha,n-1}[a,b]$.

In particular, the problem involving the homogeneous differential equation

$$\left({}^{C}\mathrm{D}_{a+}^{\alpha}y\right)(x)-\lambda y(x)=0\left(a<x\le b;\alpha>0;\lambda\in\mathbb{R}\right), \tag{17}$$

with the initial conditions (10), has a unique solution $y(x)$ in the space $C_{\delta,\log}^{\alpha,n-1}[a,b] = C_{\delta,0,\log}^{\alpha,n-1}[a,b]$ of the form

$$y(x) = \sum_{j=0}^{n-1} b_j \left(\log\frac{x}{a} \right)^j E_{\alpha,j+1}\left[\lambda \left(\log\frac{x}{a} \right)^\alpha \right]. \tag{18}$$

Following are the two examples based on this problem.

(a) The problem

$$\left({}^C\mathrm{D}_{a+}^{\alpha} y \right)(x) - \lambda y(x) = f(x), y(a+) = b(b \in \mathbb{R}), \tag{19}$$

with $0 < \alpha < 1$ and $\lambda \in \mathbb{R}$ has solution of the form

$$y(x) = bE_\alpha\left[\lambda \left(\log\frac{x}{a} \right)^\alpha \right] + \int_a^x \left(\log\frac{x}{t} \right)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda \left(\log\frac{x}{t} \right)^\alpha \right] \frac{f(t)}{t} dt. \tag{20}$$

while the solution to the problem

$$\left({}^C\mathrm{D}_{a+}^{\alpha} y \right)(x) - \lambda y(x) = 0, y(a+) = b(b \in \mathbb{R}), \tag{21}$$

will be

$$y(x) = bE_\alpha\left[\lambda \left(\log\frac{x}{a} \right)^\alpha \right]. \tag{22}$$

Particularly, the problem

$$\left({}^C\mathrm{D}_{a+}^{1/2} y \right)(x) - \lambda y(x) = f(x), y(a+) = b(b \in \mathbb{R}), \tag{23}$$

has the solution as

$$y(x) = bE_{1/2}\left[\lambda \left(\log\frac{x}{a} \right)^{1/2} \right] + \int_a^x \left(\log\frac{x}{t} \right)^{-1/2} E_{1/2,1/2}\left[\lambda \left(\log\frac{x}{t} \right)^{1/2} \right] \frac{f(t)}{t} dt, \tag{24}$$

the solution to the problem

$$\left({}^C\mathrm{D}_{a+}^{1/2} y \right)(x) - \lambda y(x) = 0, y(a+) = b(b \in \mathbb{R}), \tag{25}$$

is given by

$$y(x) = bE_{1/2}\left[\lambda \left(\log\frac{x}{a} \right)^{1/2} \right], \tag{26}$$

and the problem with ordinary derivative

$$\left(x\frac{d}{dx}y\right)(x)-\lambda y(x)=0, y(a+)=b\left(b\in\mathbb{R}\right), \tag{27}$$

has the solution given by

$$y(x)=b\left(\frac{x}{a}\right)^{\lambda}. \tag{28}$$

(b) Let $b,d\in\mathbb{R}$. The solution to the problem

$$\left({}^{C}\mathrm{D}_{a+}^{\alpha}y\right)(x)-\lambda y(x)=f(x), y(a+)=b, \delta y(a+)=d, \tag{29}$$

with $1<\alpha<2$ and $\lambda\in\mathbb{R}$ has the form

$$\begin{aligned} y(x)=&\, bE_{\alpha}\left[\lambda\left(\log\frac{x}{a}\right)^{\alpha}\right]+d\left(\log\frac{x}{a}\right)E_{\alpha,2}\left[\lambda\left(\log\frac{x}{a}\right)^{\alpha}\right] \\ &+\int_{a}^{x}\left(\log\frac{x}{t}\right)^{\alpha-1}E_{\alpha,\alpha}\left[\lambda\left(\log\frac{x}{t}\right)^{\alpha}\right]\frac{f(t)}{t}dt. \end{aligned} \tag{30}$$

Particularly, the problem

$$\left({}^{C}\mathrm{D}_{a+}^{\alpha}y\right)(x)-\lambda y(x)=0, y(a+)=b, \delta y(a+)=d, \tag{31}$$

with $1<\alpha<2$ and $\lambda\in\mathbb{R}$ has the following solution

$$y(x)=bE_{\alpha}\left[\lambda\left(\log\frac{x}{a}\right)^{\alpha}\right]+d\left(\log\frac{x}{a}\right)E_{\alpha,2}\left[\lambda\left(\log\frac{x}{a}\right)^{\alpha}\right]. \tag{32}$$

Problem 3.2. Next, we consider the following more general fractional differential equation of homogeneous type

$$\left({}^{C}\mathrm{D}_{a+}^{\alpha}y\right)(x)-\lambda\left(\log\frac{x}{a}\right)^{\beta}y(x)=0\left(a<x\le b; n-1<\alpha<n; n\in N; \lambda\in\mathbb{R}\right), \tag{33}$$

$$\left(\delta^{k}y\right)(a+)=b_{k}\left(b_{k}\in\mathbb{R}; k=0,\dots,n-1\right). \tag{34}$$

Again, in view of the result for compositions of Hadamard fractional integral J_{a+}^{α} with Caputo Hadamard fractional derivative ${}^{C}\mathrm{D}_{a+}^{\alpha}$, given in [9], on applying J_{a+}^{α} on both the sides of (33) and on using initial condition (34), we get

$$y(x)=\sum_{j=0}^{n-1}\frac{b_{j}\left(\log\frac{x}{a}\right)^{j}}{j!}+\lambda\mathrm{J}_{a+}^{\alpha}\left(\left(\log\frac{x}{a}\right)^{\beta}y(x)\right) \tag{35}$$

On decomposing the unknown function $y(x)$ as a sum of an infinite number of components:

$$y(x) = \sum_{k=0}^{\infty} y_k(x). \tag{36}$$

The Adomian decomposition method gives these components recursively as

$$y_0 = \sum_{j=0}^{n-1} \frac{b_j \left(\log \frac{x}{a}\right)^j}{j!} \tag{37}$$

and $y_{k+1}(x) = \lambda \mathrm{J}_{a+}^{\alpha}\left(\left(\log \frac{x}{a}\right)^{\beta} y_k(x)\right)$ (38)

Using (3) in recursive formulae (13) and (14), we obtain these components as

$$y_k(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!} c_k \lambda^k \left(\log \frac{x}{a}\right)^{(\alpha+\beta)k+j}, \tag{39}$$

where

$$c_k = \prod_{r=1}^{k} \frac{\Gamma[r(\alpha+\beta)-\alpha+j+1]}{\Gamma[r(\alpha+\beta)+j+1]} (k \in N). \tag{40}$$

Therefore by using (15) in (12), the solution of the Cauchy problem (9) is given by

$$y(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!} \sum_{k=0}^{\infty} c_k \lambda^k \left(\log \frac{x}{a}\right)^{(\alpha+\beta)k+j} = \sum_{j=0}^{n-1} \frac{b_j}{j!} \left(\log \frac{x}{a}\right)^j E_{\alpha,1+\beta/\alpha,(\beta+j)/\alpha}\left[\lambda \left(\log \frac{x}{a}\right)^{\alpha+\beta}\right]. \tag{41}$$

where $E_{\alpha,1+\beta/\alpha,(\beta+j)/\alpha}\left[\lambda\left(\log \frac{x}{a}\right)^{\alpha+\beta}\right]$ represents $E_{\alpha,m,l}(z)$ a generalized Mittag-Leffler function defined in [10,13].

If $\beta \geq 0$, $f(x,y) = \lambda\left(\log \frac{x}{a}\right)^{\beta} y(x)$ satisfies the Lipschitz condition given in [2]. If $\gamma \leq \alpha$, then by [Theorem 4.2, 2], there exists a unique solution (41) to the problem (33), (34) in the space $C_{\delta,\gamma,\log}^{\alpha,n-1}[a,b]$.

(a) The problem

$$\left({}^{C}\mathrm{D}_{a+}^{\alpha} y\right)(x) - \lambda\left(\log \frac{x}{a}\right)^{\beta} y(x) = 0, y(a+) = b, \tag{42}$$

with $0 < \alpha < 1, b, \beta \in \mathbb{R}$ $(\beta \geq 0)$ and $\lambda \in \mathbb{R}$ has solution of the form

$$y(x) = bE_{\alpha,1+\beta/\alpha,\beta/\alpha}\left[\lambda\left(\log\frac{x}{a}\right)^{\alpha+\beta}\right]. \tag{43}$$

In particular, the problem

$$\left({}^{C}\mathrm{D}_{a+}^{1/2}y\right)(x) - \lambda\left(\log\frac{x}{a}\right)^{\beta} y(x) = 0, \left({}^{C}\mathrm{D}_{a+}^{-1/2}y\right)(a+) = b\left(\lambda \in \mathbb{R}\right), \tag{44}$$

with $b \in \mathbb{R}$, has a unique solution given by

$$y(x) = bE_{1/2,1+2\beta,2\beta}\left[\lambda\left(\log\frac{x}{a}\right)^{\beta+1/2}\right]. \tag{45}$$

and solution of the problem with ordinary derivative

$$\left(x\frac{dy}{dx}\right)(x) - \lambda\left(\log\frac{x}{a}\right)^{\beta} y(x) = 0, y(a+) = b\left(\lambda, b \in \mathbb{R}\right), \tag{46}$$

is given by

$$y(x) = b\exp\left(\frac{\lambda}{\beta+1}\left(\log\frac{x}{a}\right)^{\beta+1}\right). \tag{47}$$

(b) The problem

$$\left({}^{C}\mathrm{D}_{a+}^{\alpha}y\right)(x) - \lambda\left(\log\frac{x}{a}\right)^{\beta} y(x) = 0, y(a+) = b, (\delta y)(a+) = d, \tag{48}$$

with $1 < \alpha < 2, b, d, \beta \in \mathbb{R}\ (\beta \geq 0)$ and $\lambda \in \mathbb{R}$ has solution of the form

$$y(x) = bE_{\alpha,1+\beta/\alpha,\beta/\alpha}\left[\lambda\left(\log\frac{x}{a}\right)^{\alpha+\beta}\right] + d\left(\log\frac{x}{a}\right)E_{\alpha,1+\beta/\alpha,(\beta+1)/\alpha}\left[\lambda\left(\log\frac{x}{a}\right)^{\alpha+\beta}\right]. \tag{49}$$

Solution to similar problems may also be obtained in the same way.

4. Conclusion

We have established the law of exponents for Caputo Hadamard fractional derivatives. The law may be useful while solving the fractional differential equations of higher order involving Caputo Hadamard fractional derivatives. We have obtained explicit solutions to several Cauchy problems with a left side Caputo Hadamard fractional derivative in spaces of continuously differentiable functions. The resulting solutions are expressed in compact form concerning the Mittag-Leffler functions. The lines may be followed for obtaining solutions of more such fractional differential equations involving the said derivatives.

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