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SOME CURVATURE PROPERTIES OF TRANS-SASAKIAN MANIFOLDS WITH RESPECT TO GENERALIZED TANAKA WEBSTER OKUMURA CONNECTION

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Abstract: The main object of the present paper is to study trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. We established a relation between the curvature tensors $\tilde{R}$ and R with respect to the generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ respectively. We have studied locally symmetric as well as ϕ-symmetric trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection.

Keywords and phrases: Locally ϕ-symmetric, trans-Sasakian manifolds

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1. Introduction

Let M be an n-dimensional, $n \geq 3$, connected smooth Riemannian manifold endowed with the Riemannian metric g. Let ∇, R, S and r be the Levi-Civita connection, curvature tensor, Ricci tensor and the scalar curvature of M respectively. The manifold M is called locally symmetric due to cartan [4], [5], if the global geodesic symmetry at $p \in M$ is an isometry, which is equivalent to the fact that ∇R=0.

In 1977 Takahashi [15] introduced the notion of locally ϕ-symmetry on Sasakian manifolds. A Sasakian manifold is said to be locally ϕ-symmetric if $\phi^2(\nabla_W R)(X,Y)Z=0$ for any vector fields *X, Y, Z, W* on M, where ϕ is the structure tensor of the manifold M. The concept of locally ϕ-symmetry on various structures and their generalizations or extension are studied in [14]. In 1985, Oubina [13] introduced a new class of almost contact metric manifolds [11] called trans-Sasakian manifolds, which includes Sasakian, Kenmotso and Cosymplectic structures. The authors in the paper [1], [3] and [7] studied such manifolds and obtained some interesting results. The notion of generalized Tanaka Webster Okumura connection was introduced and studied by the authors in the paper [10]. The local structure of trans-Sasakian manifold is given by Marrero [12]. In the present paper we have studied trans-Sasakian manifolds with generalized Tanaka Webster Okumura connection.

The present paper is organized as follows.

After introduction in Section 1, we give some preliminaries in Section 2. In section 3 we have established a relation between the curvature tensors $\tilde{R}$ and R with respect to the generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ respectively. Section 4 is devoted to the study of locally symmetric as well as locally ϕ-symmetric trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection.

2. Preliminaries

Let M be a (2n+1)-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is a tensor field of type (1,1), ξ is a vector field, η is an 1-form and g is a Riemannian metric on M such that [2]

$$\phi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1 \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ X, Y \in T(M) \tag{2}$$

Then also

$$\phi\xi = 0 \eta(\phi X) = 0, \eta(X) = g(X, \eta) \tag{3}$$

$$g(\phi X, X) = 0 \tag{4}$$

An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be a trans-Sasakian manifold (13) if $(M^{2n+1} \times R, J, G)$ belongs to the class W_4 [9] of the Hermitian manifolds, where J is the almost complex structure on $M^{2n+1} \times R$ defined by [8]

$$J\left(Z, f\frac{d}{dt}\right) = \left[(\phi Z - f\xi), \eta(Z)\frac{d}{dt}\right] \tag{5}$$

for any vector field Z on M^{2n+1} and smooth function f on $M^{2n+1} \times R$ and G is the Hermitian metric on the product $M^{2n+1} \times R$.

This may be expressed by the condition [13]

$$(\nabla_X \phi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X] \tag{6}$$

for some smooth functions α and β on M^{2n+1} and we say that the trans-Sasakian structure is of type $((\alpha, \beta))$.

It follows from equation (6) that

$$\nabla_X \xi = -\alpha\phi X - \beta[X - \eta(X)]\xi \tag{7}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y)\xi + \beta[g(\phi X, \phi Y)] \tag{8}$$

In a *(2n+1)*-dimensional trans-Sasakian manifold from (6), (7) and (8) we can write [6]

$$R(X, Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] - (X\alpha)\phi Y + (Y\alpha)\phi X - (X\beta)\phi^2 Y - (Y\beta)\phi^2 X \tag{9}$$

$$S(X, \xi) = [2n(\alpha^2 - \beta^2) - \xi\beta]\eta X - (2n-1)X\beta - (\phi X)\alpha \tag{10}$$

where S is the Ricci tensor.

Further we have

$$2\alpha\beta + \xi\alpha = 0 \tag{11}$$

3. Relation between the curvature tensors $\tilde{R}$and R with respect to the generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ and the Levi-Civita connection∇ respectively

The generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ and the Levi-Civita connection∇ are related by [10]

$$\tilde{\nabla}_X Y = \nabla_X Y + A(X,Y) \tag{12}$$

for all vectors fields ***X, Y*** on ***M***. Here

$$A(X,Y) = \alpha[g(X,\phi Y)\xi + \eta(Y)\phi X] + \beta[g(X,Y)\xi - \eta(Y)X] - l\eta(X)\phi Y \tag{13}$$

where ***l*** is a real constant.

The Torsion $\tilde{T}$ of the gTWO -connection $\tilde{\nabla}$ is given by

$$+\eta(X)(\beta Y - l\phi Y) - \eta(Y)(\beta X - l\phi X) \tag{14}$$

From (12) and (13) we get

$$\tilde{\nabla}_Y Z = \nabla_Y Z + \alpha[g(Y,\phi z)\xi + \eta(Z)\phi Y] + \beta[g(Y,Z)\xi - \eta(Z)Y] - l\eta(Y)\phi X \tag{15}$$

Applying $\tilde{\nabla}_X$ on both side of (15) we get

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_Y Z = \tilde{\nabla}_X\nabla_Y Z + \alpha\big[\big(\tilde{\nabla}_X g(Y,\phi Z)\big)\xi + g(Y,\phi Z)\tilde{\nabla}_X\xi + \eta(Z)(\tilde{\nabla}_X\phi Y) + \big(\tilde{\nabla}_X\eta(Z)\big)\phi Y\big]\\ +\beta\big[\big(\tilde{\nabla}_X g(Y,Z)\big)\xi + g(Y,Z)\tilde{\nabla}_X\xi - \eta(Z)(\tilde{\nabla}_X Y) - \big(\tilde{\nabla}_X\eta(Z)\big)Y\big]\\ -l\big[\eta(Z)\tilde{\nabla}_X\phi Z + \big(\tilde{\nabla}\eta(Y)\big)\phi Z\big]\end{aligned} \tag{16}$$

We supposed that the generalized Tanaka Webster Okumura connection $\tilde{\nabla}_X$is metric compatible. This implies that

$$(\tilde{\nabla}_X g)(Y,Z) = 0$$

or, $\tilde{\nabla}_X g(Y,Z) = g(\tilde{\nabla}_X Y,Z) + g(Y,\tilde{\nabla}_X Y)$ (17)

Using (15) on right hand side of (17) we get

$$\tilde{\nabla}_X g(Y,Z) = g(\nabla_X Y,Z) + g(Y,\nabla_X Z) \tag{18}$$

Similarly we can write from (18)

$$\tilde{\nabla}_X g(Y,\phi Z) = g(\nabla_X Y,\phi Z) + g(Y,\nabla_X\phi Z) \tag{19}$$

Using relation (12) we get the following relations

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha[g(X,\phi Y)\xi + \eta(Y)\phi X] + \beta[g(X,Y)\xi - \eta(Y)X] - l\eta(X)\phi Y \tag{20}$$

$\tilde{\nabla}_X \nabla_Y Z = \nabla_X \nabla_Y Z + \alpha[g(X, \phi\nabla_Y Z)\xi + \eta(\nabla_Y Z)\phi X] + \beta[g(X, \nabla_Y Z)\xi - \eta(\nabla_Y Z)\xi] - l\eta(X)\phi\nabla_Y Z$ (21)

$\tilde{\nabla}_X \xi = \nabla_X \xi + \alpha\phi X + \beta[\eta(X)\xi - X]$ (22)

$\tilde{\nabla}_X \phi Y = \nabla_X \phi Y + \alpha[g(X, Y) + \eta(X)\eta(Y)]\xi + \beta g(X, \phi Y)\xi - l\eta(X)[Y - \eta(Y)\xi]$ (23)

$$\tilde{\nabla}_X \eta(Y) = \nabla_X \eta(Y) + \alpha\big[g\big(X, \phi\eta(Y)\big)\xi + \eta\big(\eta(Y)\big)\phi X\big] + \beta\big[g\big(X, \eta(Y)\big)\xi - \eta\big(\eta(Y)\big)X\big] - l\eta(X)\phi\eta(Y) \quad (24)$$

Using relations (18) to (24) in (16) we get

$$\tilde{\nabla}_X \tilde{\nabla}_Y Z = \nabla_X \nabla_Y Z + \alpha[g(X, \phi\nabla_Y Z)\xi + \eta(\nabla_Y Z)\phi X] + \beta[g(X, \nabla_Y Z)\xi - \eta(\nabla_Y Z)X] - l\eta(X)\phi\nabla_Y Z$$
$$+\alpha[g(\nabla_X Y, \phi Z) + g(Y, \nabla_X \phi Z)]\xi + \beta[g(\nabla_X Y, Z) + g(Y, \nabla_X Z)]\xi$$
$$+ [\alpha g(Y, \phi Z) + \beta g(Y, Z)][\nabla_X \xi + \alpha\phi X + \beta(\eta(X)\xi - X)]+$$
$$\big[\nabla_X \eta(Z) + \alpha\big[g\big(X, \phi\eta(Z)\big)\xi + \eta\big(\eta(Z)\big)\phi X\big] + \beta\big[g\big(X, \eta(Z)\big)\xi - \eta\big(\eta(Z)\big)X\big] - l\eta(X)\phi\eta(Z)\big][\alpha\phi Y - \beta Y]$$
$$+\alpha\eta(Z)\big[\nabla_X \phi Y + \alpha[g(X, Y) + \eta(X)\eta(Y)\xi] + \beta g(X, \phi Y)\xi + l\eta(X)[Y - \eta(Y)\xi]\big]$$
$$-\beta\eta(Z)[\nabla_X Y + \alpha[g(X, \phi Y)\xi + \eta(Y)\phi X] + \beta[g(X, Y)\xi - \eta(Y)X] - l\eta(X)\phi Y]$$
$$-l\eta(Y)\big[\nabla_X \phi Z + \alpha[g(X, Y) + \eta(X)\eta(Z)]\xi + \beta g(X, \phi Z)\xi + l\eta(X)[Z - \eta(Z)\xi]\big]$$
$$- \ l\Big[\nabla_X \eta(Y) + \alpha\big[g\big(X, \phi\eta(Y)\big)\xi + \eta\big(\eta(Y)\big)\phi X\big] + \beta\big[g\big(X, \eta(Y)\big)\xi - \eta\big(\eta(Y)\big)X - l\eta(X)\phi\eta(Y)\big]\Big]\phi Z \quad (25)$$

Interchanging X and Y in (25)

$$\tilde{\nabla}_Y \tilde{\nabla}_X Z = \nabla_Y \nabla_X Z + \alpha[g(Y, \phi\nabla_X Z)\xi + \eta(\nabla_X Z)\phi Y] + \beta[g(Y, \nabla_X Z)\xi - \eta(\nabla_Y Z)Y] - l\eta(Y)\phi\nabla_X Z$$
$$+\alpha[g(\nabla_Y X, \phi Z) + g(X, \nabla_Y \phi Z)]\xi + \beta[g(\nabla_Y X, Z) + g(X, \nabla_Y Z)]\xi$$
$$+ [\alpha g(X, \phi Z) + \beta g(X, Z)][\nabla_Y \xi + \alpha\phi Y + \beta(\eta(Y)\xi - Y)]+$$
$$\big[\nabla_Y \eta(Z) + \alpha\big[g\big(Y, \phi\eta(Z)\big)\xi + \eta\big(\eta(Z)\big)\phi Y\big] + \beta\big[g\big(Y, \eta(Z)\big)\xi - \eta\big(\eta(Z)\big)Y\big] - l\eta(Y)\phi\eta(Z)\big][\alpha\phi X - \beta X]$$
$$+\alpha\eta(Z)\big[\nabla_Y \phi Y + \alpha[g(Y, X) + \eta(Y)\eta(X)\xi] + \beta g(Y, \phi X)\xi + l\eta(Y)[X - \eta(X)\xi]\big]$$
$$-\beta\eta(Z)[\nabla_Y X + \alpha[g(Y, \phi X)\xi + \eta(X)\phi Y] + \beta[g(Y, X)\xi - \eta(X)Y] - l\eta(Y)\phi X]$$
$$-l\eta(X)\big[\nabla_Y \phi Z + \alpha[g(Y, X) + \eta(Y)\eta(Z)]\xi + \beta g(Y, \phi Z)\xi + l\eta(Y)[Z - \eta(Z)\xi]\big]$$

$- l\Big[\nabla_Y\eta(X) + \alpha\big[g\big(Y,\phi\eta(X)\big)\xi + \eta\big(\eta(X)\big)\phi Y\big] + \beta\big[g\big(Y,\eta(X)\big)\xi - \eta\big(\eta(X)\big)Y - l\eta(Y)\phi\eta(X)\big]\Big]\phi Z$ (26)

In view of (12) and (13) we get

$(\tilde{\nabla})_{[X,Y]}Z = \nabla_{[X,Y]}Z + \alpha\big[g([X,Y],\phi Z)\phi + \eta(Z)\phi[X,Y]\big] + \beta\big[g([X,Y],Z)\xi - \eta(Z)[X,Y]\big] - l\eta([X,Y])\phi Z$ (27)

We know that

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \quad (28)$$

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \nabla_{[\widetilde{X,Y}]}Z \quad (29)$$

Using (25) , (26), (27), (28) in (29) and by straightforward calculation we get $\tilde{R}(X,Y)Z = R(X,Y)Z + \alpha\big[g(Y,\nabla_X\phi Z)\xi - g(X,\nabla_Y\phi)\xi + g(X,\phi\nabla_Y Z)\xi + \eta(\nabla_Y Z)\phi X - g(Y,\phi\nabla_X Z)\xi - \eta(\nabla_X Z)\phi Y - \eta(Z)\eta[X,Y]\big]$

$+\beta[\eta(\nabla_X Z)Y - \eta(\nabla_Y Z)X + g([X,Y],Z)\xi] - l[\eta(X)\phi\nabla_Y Z + \eta(Y)\phi\nabla_X Z + \eta([X,Y])\phi Z]+$

$$[\alpha g(Y,\phi Z) + \beta g(Y,Z)][\nabla_X\xi + \alpha\phi X + \beta(\eta(X)\xi - X)] - [\alpha g(X,\phi Z) + \beta g(X,Z)][\nabla_Y\xi + \alpha\phi Y + \beta(\eta(Y)\xi - Y)]$$

+

$-(\alpha\phi X - \beta X)\big[\nabla_Y\eta(Z) + \alpha\big(g(Y,\phi\eta(Z)\xi) + \eta\big(\eta(Z)\big)\phi Y\big) + \beta\big(g\big(Y,\eta(Z)\big)\xi\big) - \eta(\eta(Z)Y) - l\eta(Y)\phi\eta(Z)\big]$

+

$-l\eta(Y)\big[\nabla_X\phi Z + \alpha\big(\eta(X)\eta(Z) - g(X,Z)\big)\xi + \beta g(X,\phi Z)\xi + l\eta(X)(Z - \eta(Z)\xi)\big]$

$+ -\beta\eta(Z)[\alpha(2g(X,\phi Y)\xi + \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(Y)X - \eta(X)Y) + l(\eta(Y)\phi X - \eta(X)\phi Y)]$ $- l\Big[\nabla_X\eta(Y) - \nabla_Y\eta(X) - \alpha\big[g\big(X,\phi\eta(Y)\big)\xi + \eta\big(\eta(Y)\big)\phi X - g\big(Y,\phi\eta(X)\big)\xi - \eta\big(\eta(X)\big)\phi Y\big]\Big] + l\beta\big[g\big(X,\eta(Y)\big)\xi - \eta\big(\eta(Y)\big)X - g\big(Y,\eta(X)\big)\xi + \eta\big(\eta(X)\big)Y\big] + l^2[\eta(Y)\phi\eta(X) - \eta(X)\phi\eta(Y)]\phi Z$ (30)

This is the relation between the curvature tensors $\tilde{R}$ and R with respect to the generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ respectively.

4. Locally symmetric and locally ϕ-symmetric trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection

Definition 4.1. A (2n+1)-dimensional trans-Sasakian manifold will be called locally symmetric if

$$(\nabla_W R)(X,Y)Z = 0 \quad (31)$$

for any vector fields X, Y, Z and W orthogonal to ξ.

Definition 4.2. A (2n+1)-dimensional trans-Sasakian manifold will be called locally ϕ -symmetric if

$$\phi^2(\nabla_W R)(X,Y)Z = 0 \tag{32}$$

for any vector fields *X*, *Y*, *Z* and *W* orthogonal to ξ.

In this connection it should be mentioned that the notion of locally symmetric manifolds was introduced and studied in the paper ([4], [5]) and the locallyϕ-symmetric manifolds was introduced by Takahashi [15] in the context of Sasakian geometry. Gray and Hervella [9] have studied the sixteen classes of almost Hermitian manifolds and their linear invariants.

Analogous to the definition of locally symmetric as well as locally ϕ-symmetric trans-Sasakian manifolds with respect to Levi-Civita connection ∇ we define respectively locally symmetric and locally ϕ-symmetric trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ by

$$(\tilde{\nabla}_W R)(X,Y)Z = 0 \tag{33}$$

and

$$\phi^2(\tilde{\nabla}_W R)(X,Y)Z = 0 \tag{34}$$

for any vector fields *X*, *Y*, *Z* and *W* orthogonal to ξ.

We consider that the vector fields *X*, *Y*, *Z* and *W* orthogonal to ξ.

Then relation (30) reduces to

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z &+ \alpha[g(Y,\nabla_X\phi Z)\xi - g(X,\nabla_Y\phi Z)\xi + g(X,\phi\nabla_Y Z)\xi - g(Y,\phi\nabla_X Z)\xi] \\ &+ \beta g([X,Y],Z)\xi \\ &-[\alpha g(Y,\phi Z) + \beta g(Y,Z)][\nabla_X\xi + \alpha\phi X - \beta X] - [\alpha g(X,\phi Z) + \beta g(X,Z)][\nabla_Y\xi + \alpha\phi Y - \beta Y]\end{aligned} \tag{35}$$

Using relation (7) in (35) we get

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \alpha[g(Y,\nabla_X\phi Z)\xi - g(X,\nabla_Y\phi Z)\xi + g(X,\phi\nabla_Y Z)\xi - g(Y,\phi\nabla_X Z)\xi] + \beta g([X,Y],Z)\xi \tag{36}$$

Differentiating (35) covariantly by W with respect to Levi-Civita connection ∇we get

$$(\nabla_W\tilde{R})(X,Y)Z = (\nabla_W R)(X,Y)Z + \alpha\big[\big(g(Y,\nabla_X\phi Z) - g(X,\nabla_Y\phi Z)\big) + \beta g([X,Y],Z)\big]\nabla_W\xi$$

$$+\alpha[g(X,\nabla_W\phi\nabla_Y Z) - g(Y,\nabla_W\nabla_X Z)]\xi + \alpha[g(X,\phi\nabla_Y Z) - g(Y,\phi\nabla_X Z)]\nabla_W\xi \tag{37}$$

As *X* and *Y* are orthogonal to ξ, equation (12) reduces to

$$\tilde{\nabla}_X Y = \nabla_X Y + [\alpha g(X,\phi Y) + \beta g(X,Y)]\xi \tag{38}$$

for all vector fields *X*, *Y* on *M*.

In view of (38) we can write

$$(\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z + [\alpha g(W,\phi \tilde{R}(X,Y)Z) + \beta g(W,\tilde{R}(X,Y)Z)]\xi \quad (39)$$

In view of (37) we obtain from (39)

$$(\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\nabla_W R)(X,Y)Z + \alpha\big[\big(g(Y,\nabla_X \phi Z) - g(X,\nabla_Y \phi Z)\big) + \beta g([X,Y],Z)\big]\nabla_W \xi$$
$$+\alpha[g(X,\nabla_W \phi \nabla_Y Z) - g(Y,\nabla_W \nabla_X Z)]\xi + \alpha[g(X,\phi\nabla_Y Z) - g(Y,\phi\nabla_X Z)]\nabla_W \xi$$
$$+[\alpha g(W,\phi\tilde{R}(X,Y)Z) + \beta g(W,\tilde{R}(X,Y)Z)]\xi \quad (40)$$

Taking inner product on both side of (40) with respect to W we get

$$g\big((\tilde{\nabla}_W \tilde{R})(X,Y)Z,W\big)$$
$$= g\big((\nabla_W R)(X,Y)Z,W\big)$$
$$+ \alpha\big[\big(g(Y,\nabla_X \phi Z) - g(X,\nabla_Y \phi Z)\big) + \beta g([X,Y],Z)\big]g(\nabla_W \xi, W)$$
$$+\alpha[g(X,\phi\nabla_Y Z) - g(Y,\phi\nabla_X Z)]g(\nabla_W \xi, W) \quad (41)$$

In view of (7) we get from (41)

$$g\big((\tilde{\nabla}_W \tilde{R})(X,Y)Z,W\big)$$
$$= g\big((\nabla_W R)(X,Y)Z,W\big)$$
$$+ \alpha\big[\big(g(Y,\nabla_X \phi Z) - g(X,\nabla_Y \phi Z)\big) + \beta g([X,Y],Z)\big]g(W,\beta W)$$
$$+\alpha[g(X,\phi\nabla_Y Z) - g(Y,\phi\nabla_X Z)]g(W,\beta W) \quad (42)$$

Suppose $\alpha$is constant, then by (42) $\beta = 0$. In such a case from equation (42) we get

$$(\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\nabla_W R)(X,Y)Z \quad (43)$$

Thus we are in a position to state the following:

Theorem 1. A (2n+1)-dimensional trans-Sasakian manifold of type (α,β) with α as a constant is locally symmetric with respect to generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ if and only if it is so with respect to Levi- Civita connection ∇.

Now applying ϕ^2on both side of the equation (40) and using relation (1) and (7) we get

$$\phi^2(\nabla_W \tilde{R})(X,Y)Z,$$
$$= \phi^2(\nabla_W R)(X,Y)Z$$
$$+ \alpha[g(Y,\nabla_X \phi Z) - g(X,\nabla_Y \phi Z) + g(X,\phi\nabla_Y Z) - g(Y,\phi\nabla_X)]$$
$$+\beta\big(g([X,Y],Z)\big)(\alpha\phi W - \beta W) \quad (44)$$

Taking inner product on both side of above with respect to W we get

$$g(\phi^2(\nabla_W \tilde{R})(X,Y)Z,W) = g(\phi^2(\nabla_W R)(X,Y)Z,W)$$
$$+\alpha[g(Y,\nabla_X \phi Z) - g(X,\nabla_Y \phi Z) + g(X,\phi\nabla_Y Z) - g(Y,\phi\nabla_X)]$$
$$-\beta\big(g([X,Y],Z)\big)g(W,W) \quad (45)$$

Suppose α is a constant. Then by (11) we get $\beta = 0$. In such a case from equation (45) we get

$\phi^2(\nabla_W \tilde{R})(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z$ (46)

Thus we are in a position to state the following:

Theorem 1. A (2n+1)-dimensional trans-Sasakian manifold of type (α, β) with α as a constant is locally ϕ symmetric with respect to generalized Tanaka Webster Okumura connection $\bar{\nabla}$ if and only if it is so with respect to Levi- Civita connection ∇.

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