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CERTAIN RESULTS ON GENERALIZED SASAKIAN-SPACE-FORMS WITH SEMI-SYMMETRIC METRIC CONNECTIONS

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Abstract: The aim of present paper is to study the generalized Sasakian-space-forms endowed with semi-symmetric metric connections. Some results of curvature tensor admitting semi-symmetric metric connections have been derived.

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1. Introduction

In 1924, Friedmann and Schouten [6] introduce the idea of semi-symmetric linear connection in differentiable manifold. The idea of metric connection with torsion in a Riemannian manifold is introduced by Hayden [7]. In [11], Yano studied semi-symmetric metric connection in Riemannian manifold.

Furthermore, the notation of generalized Sasakian-space-forms were introduced by Alegre et al.[1]. A Sasakian manifold with constant ϕ-sectional curvature is a Sasakian-space-forms and it has a specific form of curvature tensor. An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be generalized Sasakian-space-forms if there exist three differentiable function f_1, f_2, f_3 on M such that the curvature tensor R is given by

$$\begin{aligned}R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\},\end{aligned} \tag{1}$$

for any vector fields X, Y, Z on M. In such case we denote the manifold as $M(f_1, f_2, f_3)$. In [4] the authors gives several examples of generalized Sasakian-space-forms. If $f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}$ then generalized Sasakian-space-forms with Sasakian structure becomes a Sasakian-space-form, if $f_1 = \frac{c-3}{4}, f_2 = f_3 = \frac{c+1}{4}$ then generalized Sasakian-space-forms becomes Kenmotsu space form and if $f_1 = f_2 = f_3 = \frac{c}{4}$ then generalized Sasakian-space-forms become cosympletic space forms. A large number of geometers have studied the generalized Sasakian-space-form in the papers [2], [3], [4], [5], [8]. In [9] and [10] Sular and Ozgur studied the generalized Sasakian-space-form with semi-symmetric metric and non metric connections respectively. Motivated by these studies in this paper, we have studied generalized Sasakian-space-forms with semi-symmetric metric connection.

This paper contains four sections, first section is introductory. In section 2, we give introduction of semi-symmetric metric connections. The definition of generalized Sasakian-space-forms admitting semi-symmetric metric connections is given in section 3. Finally in section 4, we establish some important results endowed with semi-symmetric metric connections.

2. Semi-Symmetric Metric Connection

Let M be an n-dimensional Riemannian manifold with Riemannian metric g. A linear connection $\tilde{\nabla}$ on a Riemannian manifold M is called a semi-symmetric connection if the torsion tensor T of the connection $\tilde{\nabla}$

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \tag{2}$$

satisfies

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{3}$$

where η is a 1-form associated with the vector field ξ on M defined by

$$\eta(X) = g(X,\xi). \tag{4}$$

The connection $\tilde{\nabla}$ is called a semi-symmetric metric connection if it satisfies $\tilde{\nabla}g = 0$. If ∇ is the Levi-Civita connection of a Riemannian manifold M, a semi-symmetric metric connection $\tilde{\nabla}$ is given by [11]

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi. \tag{5}$$

Let R and $\tilde{R}$ be curvature tensor of ∇ and $\tilde{\nabla}$ of a Riemannian manifold M respectively. Then R and $\tilde{R}$ are related by

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y - g(Y,Z)AX + g(X,Z)AY, \tag{6}$$

for all vector fields X, Y, Z on M, where α is the (0,2) tensor field defined by [11],

$$\alpha(X,Y) = (\nabla_X \eta)Y - \eta(X)\eta(Y) + \frac{1}{2}\eta(\xi)g(X,Y)$$

and $\quad g(AX,Y) = \alpha(X,Y).$

3. Generalized Sasakian Space Forms

Let M be an n-dimensional almost contact metric manifold with an almost contact structure (ϕ, ξ, η, g) consisting of (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g on M satisfying

$$\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \tag{7}$$

$$\phi\xi = 0, \quad g(X, \xi) = \eta(X), \ \eta(\phi X) = 0, \tag{8}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{9}$$

$$g(\phi X, Y) + g(X, \phi Y) = 0, \tag{10}$$

for all vector fields X, Y on M. Such a manifold is said to be contact metric manifold if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \phi Y)$ is called fundamental 2-form of M.

Let $\tilde{\nabla}$ be the semi-symmetric metric connection on an almost contact metric manifold M with closed 1-form η then M is said to be a generalized Sasakian-space-forms with semi-symmetric metric connections [9] if there exist three function $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$ on M such that

$$\begin{aligned}\tilde{R}(X, Y)Z = {} & \tilde{f}_1\{g(Y, Z)X - g(X, Z)Y\} \\ & + \tilde{f}_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ & + \tilde{f}_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}\end{aligned} \tag{11}$$

for any vector fields X, Y, Z on M, where $\tilde{R}$ denotes the curvature tensor of M with respect to semi-symmetric metric connection $\tilde{\nabla}$.

Contracting (11) with respect to X, we get

$$\tilde{S}(Y, Z) = \left((n-1)\tilde{f}_1 + 3\tilde{f}_2 - \tilde{f}_3\right) g(Y, Z) - \{3\tilde{f}_2 + (n-2)\tilde{f}_3\}\eta(Y)\eta(Z), \tag{12}$$

Equation (12) gives

$$\tilde{Q}Y = \left((n-1)\tilde{f}_1 + 3\tilde{f}_2 - \tilde{f}_3\right) Y - \{3\tilde{f}_2 + (n-2)\tilde{f}_3\}\eta(Y)\xi, \tag{13}$$

$$\tilde{r} = n(n-1)\tilde{f}_1 + 3(n-1)\tilde{f}_2 - (2n-2)\tilde{f}_3. \tag{14}$$

4. Main Results

Taking cyclic permutation of X, Y, Z in equation (11), we get

$$\begin{aligned}\tilde{R}(Y, Z)X = {} & \tilde{f}_1\{g(Z, X)Y - g(Y, X)Z\} \\ & + \tilde{f}_2\{g(Y, \phi X)\phi Z - g(Z, \phi X)\phi Y + 2g(Y, \phi Z)\phi X\} \\ & + \tilde{f}_3\{\eta(Y)\eta(X)Z - \eta(Z)\eta(X)Y + g(Y, X)\eta(Z)\xi - g(Z, X)\eta(Y)\xi\},\end{aligned} \tag{15}$$

$$\begin{aligned}\tilde{R}(Z, X)Y = {} & \tilde{f}_1\{g(X, Y)Z - g(Z, Y)X\} \\ & + \tilde{f}_2\{g(Z, \phi Y)\phi X - g(X, \phi Y)\phi Z + 2g(Z, \phi X)\phi Y\}\end{aligned}$$

$$+\tilde{f}_3\{\eta(Z)\eta(Y)X - \eta(X)\eta(Y)Z + g(Z,Y)\eta(X)\xi - g(X,Y)\eta(Z)\xi\}. \tag{16}$$

Adding equations (11), (15), (16) and using equation (10), we get

$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0.$

Thus we state the following result:

Theorem 4.1. In a generalized Sasakian-space-forms admitting semi-symmetric metric connection the relation $\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0$ holds.

Further, taking inner product of (11) with U, we obtain

$$\begin{aligned}\tilde{R}(X,Y,Z,U) = {} & \tilde{f}_1\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} \\ & +\tilde{f}_2\{g(X,\phi Z)g(\phi Y,U) - g(Y,\phi Z)g(\phi X,U) + 2g(X,\phi Y)g(\phi Z,U)\} \\ & +\tilde{f}_3\{\eta(X)\eta(Z)g(Y,U) - \eta(Y)\eta(Z)g(X,U) + g(X,Z)\eta(Y)\eta(U) \\ & -g(Y,Z)\eta(X)\eta(U)\},\end{aligned} \tag{17}$$

Interchanging X and Y in equation (17), we get

$$\begin{aligned}\tilde{R}(Y,X,Z,U) = {} & \tilde{f}_1\{g(X,Z)g(Y,U) - g(Y,Z)g(X,U)\} \\ & +\tilde{f}_2\{g(Y,\phi Z)g(\phi X,U) - g(X,\phi Z)g(\phi Y,U) + 2g(Y,\phi X)g(\phi Z,U)\} \\ & +\tilde{f}_3\{\eta(Y)\eta(Z)g(X,U) - \eta(X)\eta(Z)g(Y,U) + g(Y,Z)\eta(X)\eta(U) \\ & -g(X,Z)\eta(Y)\eta(U)\},\end{aligned} \tag{18}$$

Adding equations (17), (18) and then using equation (10), we obtain

$$\tilde{R}(X,Y,Z,U) + \tilde{R}(Y,X,Z,U) = 0. \tag{19}$$

Again interchanging $Z$and U in equation (17), we get

$$\begin{aligned}\tilde{R}(X,Y,U,Z) = {} & \tilde{f}_1\{g(Y,U)g(X,Z) - g(X,U)g(Y,Z)\} \\ & +\tilde{f}_2\{g(X,\phi U)g(\phi Y,Z) - g(Y,\phi U)g(\phi X,Z) + 2g(X,\phi Y)g(\phi U,Z)\} \\ & +\tilde{f}_3\{\eta(X)\eta(U)g(Y,Z) - \eta(Y)\eta(U)g(X,Z) + g(X,U)\eta(Y)\eta(Z) \\ & -g(Y,U)\eta(X)\eta(Z)\},\end{aligned} \tag{20}$$

Adding equations (17), (20) and then using equation (10), we obtain

$$\tilde{R}(X,Y,Z,U) + \tilde{R}(X,Y,U,Z) = 0. \tag{21}$$

Now, interchanging pair of slots in (17), we get

$$\tilde{R}(Z,U,X,Y) = \tilde{f}_1\{g(U,X)g(Z,Y) - g(Z,X)g(U,Y)\}$$
$$+\tilde{f}_2\{g(Z,\phi X)g(\phi U,Y) - g(U,\phi X)g(\phi Z,Y) + 2g(Z,\phi U)g(\phi X,Y)\}$$
$$+\tilde{f}_3\{\eta(Z)\eta(X)g(U,Y) - \eta(U)\eta(X)g(Z,Y) + g(Z,X)\eta(U)\eta(Y)$$
$$-g(U,X)\eta(Z)\eta(Y)\}, \tag{22}$$

Subtracting equation (22) from equation (17) and then using equation (10), we obtain

$$\tilde{R}(X,Y,Z,U) - \tilde{R}(Z,U,X,Y) = 0. \tag{23}$$

Thus, in view of equations (19), (21) and (23), we state the following result:

Theorem 4.2. In a generalized Sasakian-space-forms admitting semi-symmetric metric connection, we have

(i) $\tilde{R}(X,Y,Z,U) + \tilde{R}(Y,X,Z,U) = 0.$

(ii) $\tilde{R}(X,Y,Z,U) + \tilde{R}(X,Y,U,Z) = 0.$

(iii) $\tilde{R}(X,Y,Z,U) - \tilde{R}(Z,U,X,Y) = 0.$

Further, taking $Z = \xi$ in equation (11) and then using equation (8), we obtain

$$\tilde{R}(X,Y)\xi = (\tilde{f}_1 - \tilde{f}_3)\{\eta(Y)X - \eta(X)Y\}. \tag{24}$$

Again taking $X = \xi$ in equation (11) and then using equation (8), we have

$$\tilde{R}(\xi,Y)Z = (\tilde{f}_1 - \tilde{f}_3)\{g(Y,Z)\xi - \eta(Z)Y\}. \tag{25}$$

Now, interchanging X and Y in equation (11), we get

$$\tilde{R}(Y,X)Z = \tilde{f}_1\{g(X,Z)Y - g(Y,Z)X\}$$
$$+\tilde{f}_2\{g(Y,\phi Z)\phi X - g(X,\phi Z)\phi Y + 2g(Y,\phi X)\phi Z\}$$
$$+\tilde{f}_3\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi\}. \tag{26}$$

Taking $X = \xi$ in (26) and then using equation (8), we get

$$\tilde{R}(Y,\xi)Z = (\tilde{f}_1 - \tilde{f}_3)\{\eta(Z)Y - g(Y,Z)\xi\}. \tag{27}$$

Comparing equations (25) and (27), we obtain

$$\tilde{R}(\xi,Y)Z = -\tilde{R}(Y,\xi)Z. \tag{28}$$

Now, taking inner product of equation (11) with ξ and then using equation (8), we get

$$\eta(\tilde{R}(X,Y)Z) = (\tilde{f}_1 - \tilde{f}_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}. \tag{29}$$

Thus, in view of equations (24), (25), (27), (28) and (29), we state the following result:

Theorem 4.3. In a generalized Sasakian-space-forms admitting semi-symmetric metric connection, we have

(i) $\tilde{R}(X,Y)\xi = (\tilde{f}_1 - \tilde{f}_3)\{\eta(Y)X - \eta(X)Y\}$.

(ii) $\tilde{R}(\xi,Y)Z = (\tilde{f}_1 - \tilde{f}_3)\{g(Y,Z)\xi - \eta(Z)Y\}$.

(iii) $\tilde{R}(Y,\xi)Z = (\tilde{f}_1 - \tilde{f}_3)\{\eta(Z)Y - g(Y,Z)\xi\}$.

(iv) $\tilde{R}(\xi,Y)Z = -\tilde{R}(Y,\xi)Z$.

(v) $\eta(\tilde{R}(X,Y)Z) = (\tilde{f}_1 - \tilde{f}_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}$.

Definition 4.1. An n-dimensional generalized Sasakain-space-forms is said to have η-recurrent Ricci tensor if there exist a non zero 1-form A such that

$$(\tilde{\nabla}_W\tilde{S})(\phi X,\phi Y) = A(W)\tilde{S}(\phi X,\phi Y). \tag{30}$$

If 1-form A vanishes on M then the space form is said to have η-parallel Ricci tensor.

Taking L.H.S. of equation (30), we have

$$(\tilde{\nabla}_W\tilde{S})(\phi X,\phi Y) = \tilde{\nabla}_W(\tilde{S}(\phi X,\phi Y) - \tilde{S}(\tilde{\nabla}_W\phi X,\phi Y) - \tilde{S}(\phi X,\tilde{\nabla}_W\phi Y). \tag{31}$$

Using equations (5), (8) and (12), we get

$$(\tilde{\nabla}_W\tilde{S})(\phi X,\phi Y) = 0. \tag{32}$$

By virtue of equation (32), equation (30) reduces to

$$A(W)\tilde{S}(\phi X,\phi Y) = 0. \tag{33}$$

Using equation (12) in (33), we get

$$\left((n-1)\tilde{f}_1 + 3\tilde{f}_2 - \tilde{f}_3\right)A(W)g(\phi X,\phi Y) = 0. \tag{34}$$

Since $A(W) \neq 0, g(\phi X,\phi Y) \neq 0$, therefore from equation (34), we have

$$\tilde{f}_1 = \frac{\tilde{f}_3 - 3\tilde{f}_2}{(n-1)}. \tag{35}$$

Thus in view of equation (35), we state the following result:

Theorem 4.4. An n-dimensional $(n > 1)$ generalized Sasakain-space-forms admitting semi-symmetric metric connection is η-recurrent Ricci tensor if $\tilde{f}_1 = \frac{\tilde{f}_3 - 3\tilde{f}_2}{(n-1)}$.

Again in view of equation (32), we have the following result:

Theorem 4.5. An n-dimensional generalized Sasakain-space-forms admitting semi-symmetric metric connection is η-parallel Ricci tensor.

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