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ON ARRESTING THE COMPLEX GROWTH RATE OF A PERTURBATION IN FERROTHERMOSOLUTAL CONVECTION IN A POROUS MEDIUM: DARCY MODEL

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Abstract: In the present paper upper bounds for the complex growth rate of a disturbance in ferrothermosolutal convection in a porous medium have been obtained by using Darcy model. It is proved that the complex growth rate $\sigma = \sigma_r + i\sigma_i$($\sigma_r$ and σ_i are respectively the real and imaginary parts of σ) of an arbitrary oscillatory perturbation of growing amplitude in ferrothermosolutal convection in a densely packed porous medium, for the case of free boundaries, lies inside a semicircle in the right half of the $\sigma_r\sigma_i$-plane whose center is at the origin and radius $= \sqrt{\frac{\varepsilon R_s\left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]}{P_s}}$, where R_s is the solutal Rayleigh number, ε is the porosity of the medium, P_s is the solutal Prandtl number, M_1' is the ratio of magnetic flux due to concentration fluctuation to the gravitational force and M_5 is the ratio of concentration effect on magnetic field to pyromagnetic coefficient. Further, the case of deriving upper bounds for rigid boundaries is also dealt separately.

Keywords: Ferrofluid, Ferrothermosolutal Convection, Porous Medium, Darcy Model.

1. Introduction

A ferrofluid also known as magnetic Nano fluid is composed of small particles (dimensions of 3-15 nm) of solid, magnetic particles coated with some surfactant to prevent their aggregation and dispersed in a liquid carrier such as water, kerosene or ester. A ferrofluid does not exist in nature but are synthesized. Being highly interdisciplinary in nature, the ferrofluid researches has brought engineers, applied Mathematicians, Chemists, Physicians, theoretical and experimental Physicist together which ultimately resulted into wide range of practical applications e. g. energy conversion devices,noiseless jet printing system, high speed silent printers, liquid cooled loudspeakers, sealing of computer hard disc drives, the treatment of ulcers and brain tumours, destroying cancer cells, contrast enhancement of magnetic resonance imaging (MRI) etc. (Rosensweig [18], Odenbach [7-8]).

The concept of thermal convection in ferrofluids in the presence of magnetic field is similar to Bénard convection. These manifold practical applications of ferrofluids have attracted many scientists towards the study of thermal convection in ferrofluids in porous/non-porous media. In a fundamental paper Finlayson [4] studied the convective instability of ferromagnetic fluids and explained the concept of thermomechanical interactions in ferrofluids. Lalas and Carmi [6] investigated the thermoconvective stability of ferrofluids without considering buoyancy effects. Rosensweig et al. [19] investigated experimentally the penetration of ferrofluids in a Hele-Shaw cell. Gupta and Gupta [5] studied the effect of rotation on the convective instability of a layer of a ferromagnetic fluid. The effect of magnetic field dependent (MFD) viscosity on the thermal convection in a ferrofluid layer saturating a sparsely distributed porous medium has been studied by Prakash et al. [14].

For a broad view of the study of ferroconvection with heat as a single diffusive component one may be referred to Rahman and Suslov [16], Sekar and Vaidyanathan [22], Prakash [12], Prakash et al.[14], Ekaterina and Ekaterina [3], Prakasha [15], Ram et al. [17].

Moreover, since ferrofluids are mostly organic solvent carriers having suspensions of ferromagnetic salts acting as solute, it is also important to study the convective instability in ferrofluid layer having two diffusive components (also known as ferrothermosolutal convection). Several researchers have contributed in the development of ferrothermosolutal convection problem in porous/non-porous media. Vaidyanathan et al. [26] have studied the effect of a concentration gradient on double diffusive ferrofluid layer saturating a porous medium heated from below and salted from above for fixed particle suspensions causing various porosities. Sekar and Raju [21] investigated the effect of sparse distribution pores in thermohaline convection in a micropolar ferromagnetic fluid. Sekar and Murugan [25] studied the linear stability effect of densely distributed porous medium and coriolis force on Soret driven ferrothermohaline convection.For more studies one may be referred to Sekar et al. [23-24].

Since for a double diffusive ferroconvection problem the exact solutions in closed form are not possible for the cases where both the boundaries are not free, therefore the problem of obtaining the upper bounds for the complex growth rate of an arbitrary disturbance has its own importance[1]. Recently, Prakash [9-10] has derived upper bounds for the complex growth rates in some ferromagnetic convection problems in porous/non-porous medium. Prakash and Gupta [11] have extended his work to include the effect of rotation and magnetic field dependent viscosity on ferromagnetic convection. Recently, Prakash and Bala [13] also derived the upper bounds for complex growth rates in ferromagnetic convection with magnetic field dependent viscosity in a rotating sparsely distributed porous medium. To the best of our knowledge no attempt has been made, so far, to find such bounds for the case of ferrothermosolutal convection configurations.

Thus as a further step, we have derived the upper bounds for the complex growth rates in ferrothermosolutal convection in a densely packed porous medium in the presence of a uniform vertical magnetic field for the cases of free and rigid boundaries separately. The present work will definitely pave the way for further theoretical and experimental investigations in this field of enquiry.

2. Mathematical Formulation

Consider a ferromagnetic Boussinesq fluid layer of infinite horizontal extension and finite vertical thickness d, saturating a densely packed porous medium heated and salted from below. The fluid layer is statically confined between the horizontal boundaries $z = 0$ and $z = d$ which are respectively maintained at uniform temperatures T_0 and $T_1 (< T_0)$ and concentrations S_0 and $S_1 (< S_0)$ under the action of a uniform vertical magnetic field $\boldsymbol{H}$ (see Fig.1). Darcy model has been used to analyze this problem.

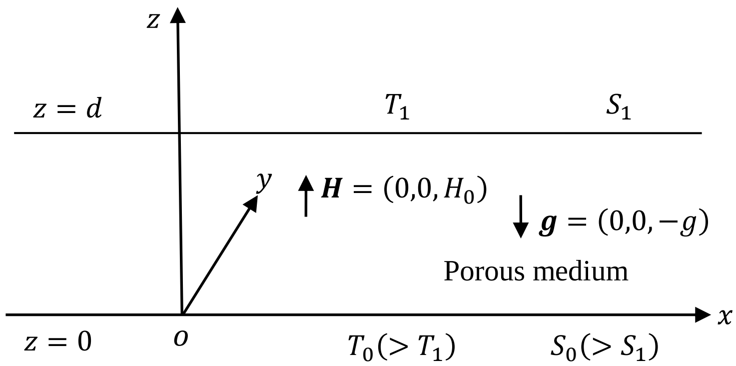

Fig 1 Geometrical configuration of the problem.

The Basic hydrodynamic equations governing the flow of the ferromagnetic fluid for the above model are given by (Finlayson [4] Divya et al. [2])

$$\nabla . \boldsymbol{q} = 0, \tag{1}$$

$$\frac{\rho_0}{\varepsilon}\left[\frac{\partial}{\partial t} + \frac{1}{\varepsilon}(\boldsymbol{q}.\nabla)\right]\boldsymbol{q} = -\nabla p + \rho \boldsymbol{g} + \nabla.(\boldsymbol{HB}) - \frac{\mu}{k_1}\boldsymbol{q}, \tag{2}$$

$$\varepsilon\left[\rho_0 C_{V,H} - \mu_0 \boldsymbol{H}.\left(\frac{\partial \boldsymbol{M}}{\partial T}\right)_{V,H}\right]\frac{DT}{Dt} + (1-\varepsilon)\rho_s C_s \frac{\partial T}{\partial t} + \mu_0 T \left(\frac{\partial \boldsymbol{M}}{\partial T}\right)_{V,H} . \frac{D\boldsymbol{H}}{Dt} = K_1 \nabla^2 T + \Phi_T, \tag{3}$$

$$\varepsilon\left[\rho_0 C_{V,H} - \mu_0 \boldsymbol{H}.\left(\frac{\partial \boldsymbol{M}}{\partial S}\right)_{V,H}\right]\frac{DS}{Dt} + (1-\varepsilon)\rho_s C_s \frac{\partial S}{\partial t} + \mu_0 S \left(\frac{\partial \boldsymbol{M}}{\partial S}\right)_{V,H} . \frac{D\boldsymbol{H}}{Dt} = K_1' \nabla^2 S + \Phi_S, \tag{4}$$

where $\boldsymbol{q}$, ε, t,p, $\boldsymbol{H}$,$\boldsymbol{B}$,μ, $\boldsymbol{g}(0,0-g)$, k_1, $C_{V,H}$, μ_0, T, S, $\boldsymbol{M}$, K_1 , K_1', $\Phi_T$and Φ_S denote respectively the velocity, porosity of the medium, time, pressure, magnetic field, magnetic induction, variable viscosity, acceleration due to gravity, permeability of the porous medium, heat capacity at constant volume and magnetic field, magnetic permeability, temperature, concentration, magnetization, thermal conductivity, solute conductivity, viscous dissipation function containing second order terms in velocity gradient and viscous dissipation function analogous to Φ_T but corresponding to solute.

The equation of state is given by

$$\rho = \rho_0[1 - \alpha(T - T_0) + \alpha'(S - S_0)], \tag{5}$$

where ρ is the fluid density, ρ_0 is the reference density, α is the coefficient of volume expansion, α' is an analogous solvent coefficient of expansion, T_0 is the temperature and S_0 is the concentration at the lower boundary.

Maxwell's equations for non-conducting fluid with no displacement current given by Finlayson (1970) as

$$\nabla.\boldsymbol{B} = 0, \nabla \times \boldsymbol{H} = \boldsymbol{0}. \tag{6a,b}$$

Further, the relation between $\boldsymbol{B}$and$\boldsymbol{H}$is expressed by

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}). \tag{7}$$

It is assumed that the magnetization is aligned with the magnetic field intensity and depends on the magnitude of magnetic field, temperature and salinity (Finlayson [4]), so that

$$\boldsymbol{M} = \frac{\boldsymbol{H}}{H} M(H, T, S). \tag{8}$$

Finally, the linearized magnetic equation of state is given by

$$M = M_o + \chi(H - H_0) - K_2(T - T_0) + K_3\,(S - S_0). \tag{9}$$

In the above equation $M_0 = M(H_0, T_0, S_0\,)$ is the magnetization when magnetic field is H_0, temperature is T_0 and concentration is S_0, $\chi = \left(\frac{\partial M}{\partial H}\right)_{H_0,T_0}$ is magnetic susceptibility, $K_2 = -\left(\frac{\partial M}{\partial T}\right)_{H_0,T_0}$ is the pyromagnetic coefficient, $K_3 = \left(\frac{\partial M}{\partial S}\right)_{H_0,S_0}$ is the salinity magnetic coefficient, H is the magnitude of $\boldsymbol{H}$, M is the magnitude of $\boldsymbol{M}$.

Now following Finlayson [4] and Divya et al. [2] using linear stability theory, we obtain the following set of non-dimensional equations

$$\left(\frac{\sigma}{\varepsilon} + \frac{1}{k_1}\right)(D^2 - a^2)w = a\,R_a{}^{1/2}\left[(M_1 - M_4\,)D\Phi_1 - (1 + M_1 - M_4\,)\theta\,\right] + a\,R_s{}^{1/2}[(M_1' - M_4')D\Phi_2 + (1 - M_1' + M_4')\phi], \tag{10}$$

$$(D^2 - a^2 - \sigma P_r)\theta = -\,(1 - M_2)aR_a{}^{1/2}w - P_r'\,M_2\sigma D\Phi_1, \tag{11}$$

$$(D^2 - a^2 - \sigma P_s)\phi = -\,(1 - M_2')aR_s{}^{1/2}w - P_s' M_2'\sigma D\Phi_2, \tag{12}$$

$$(D^2 - a^2 M_3)\Phi_1 = D\theta, \tag{13}$$

$$(D^2 - a^2 M_3)\Phi_2 = D\phi, \tag{14}$$

where z is the real independent variable such that $0\ \leq z\ \leq 1$, D is differentiation with respect to z, a^2 is square of the wave number, $P_r = \frac{\nu\rho C_1}{K_1} > 0$ is the thermal Prandtl number,$P_s = \frac{\nu\rho C_1'}{K_1'} > 0$ is the solutal Prandtl number,, $\sigma = \frac{nd^2}{\nu}$ is the complex growth rate, $R_a = \frac{g\alpha\beta d^4\rho C_2}{K_1\nu} > 0$ is the thermal Rayleigh number, $R_s = \frac{g\alpha'\beta' d^4\rho C_2'}{K_1'\nu} > 0$ is the concentration Rayleigh number,$M_1 = \frac{\mu_0 K_2^2\,\beta}{(1+\chi)\alpha\rho_0 g} > 0$ is the ratio of magnetic force due to temperature fluctuation to the gravitational force, $M_2 = \frac{\mu_0 T_0 K_2^2}{(1+\chi)\rho C_2} > 0$ is the ratio of thermal flux due to magnetization to magnetic flux, $M_1' = \frac{\mu_0 K_3^2\beta'}{(1+\chi)\alpha'\rho_0 g} > 0$ is the ratio of magnetic flux due to concentration fluctuation to the gravitational force, $M_2' = \frac{\mu_0 S_0 K_3^2}{(1+\chi)\rho C_2'} > 0$ is the ratio of mass flux due to magnetization to magnetic flux, $M_4 = \frac{\mu_0 K_2 K_3\beta'}{(1+\chi)\alpha\,\rho_0 g} > 0$ and $M_4' = \frac{\mu_0 K_2 K_3\,\beta}{(1+\chi)\alpha'\rho_0 g} > 0$ are nondimentional parameters, ,$M_5 = \frac{M_4}{M_1} = \frac{M_1'}{M_4'} = \frac{K_3\beta'}{K_2\beta} > 0$ is the ratio of concentration effect on magnetic field to pyromagnetic coefficient, $M_3 = \frac{1+\frac{M_0}{H_0}}{(1+\chi)} > 0$ is the measure of the nonlinearity of magnetization, $\sigma = \sigma_r + i\sigma_i$ is a complex constant in general such that σ_r and σ_i are real constants and as a consequence the dependent variables $w(z) = w_r(z) + i\,w_i(z)$, $\theta(z) = \theta_r(z) + i\,\theta_i(z)$, $\phi(z) = \phi_r(z) + i\,\phi_i(z), \Phi_1(z) = \Phi_{1r}(z) + i\,\Phi_{1i}(z)$, $\Phi_2(z) = \Phi_{2r}(z) + i\,\Phi_{2i}(z)$ and are complex valued functions of the real variable z such that $w_r(z)$, $w_i(z)$, $\theta_r(z)$, $\theta_i(z)$, $\phi_r(z), \phi_i(z), \Phi_{1r}(z)$, $\Phi_{1i}(z), \Phi_{2r}(z)and\Phi_{2i}(z)$ are real valued functions of the real variable z.

Since, M_2 and M_2' are of very small order (Finlayson [4]), they are neglected in the subsequent analysis and thus Eqs. (11) and (12) takes the forms

$$(D^2 - a^2 - \sigma P_r)\theta = -aR_a{}^{1/2}w, \tag{15}$$

$$(D^2 - a^2 - \sigma P_s)\phi = -aR_s{}^{1/2}w, \tag{16}$$

respectively.

The bounding surfaces are considered to be either free or rigid. Hence the boundary conditions are given by

$$w = 0 = \theta = \phi = D\Phi_1 = D\Phi_2 \text{ at } z = 0 \text{ and } z = 1, \tag{17}$$

(both the boundaries are free)

$$\text{or } w = 0 = \theta = \phi = Dw = \Phi_1 = \Phi_2 \text{ at } z = 0 \text{ and } z = 1. \tag{18}$$

(both the boundaries are rigid)

It may further be noted that Eqs. (10) and (13) - (18) describe an eigenvalue problem for σ and govern ferrothermosolutal convection in densely packed porous medium heated and salted from below.

3. Mathematical Analysis

We now derive upper bounds for the complex growth rate of the arbitrary oscillatory perturbations of neutral or growing amplitude for the present configuration. The cases of free and rigid boundaries are studied separately. We prove the following theorems:

Theorem1: If $R_a > 0, R_s > 0, \quad M_1' > 0, 1 - M_5 > 0, P_s > 0, \sigma_r \geq 0, \delta > 0$ and $\sigma_i \ (\neq 0)$, then a necessary condition for the existence of a nontrivial solution $(w, \theta, \phi, \Phi_1, \Phi_2, \sigma)$ of the equations (10) and (13) - (18) together with the boundary conditions (17) is that $|\sigma| < \sqrt{\frac{\varepsilon R_s\left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]}{P_s}}$.

Proof: Multiplying Eq. (10) by w^* (the superscript * henceforth represents the complex conjugation) throughout and integrating the resulting equation over the interval [0,1] and using the boundary conditions (17), we obtain

$$\left(\frac{\sigma}{\varepsilon}+\frac{1}{k_1}\right)\int_0^1 w^*(D^2-a^2)w dz = aR_a^{1/2}(M_1-M_4)\int_0^1 w^* D\Phi_1\, dz - aR_a^{1/2}(1+M_1-M_4)\int_0^1 w^*\theta dz + aR_s^{1/2}(M_1'-M_4')\int_0^1 w^* D\Phi_2 dz + aR_s^{1/2}(1-M_1'+M_4')\int_0^1 w^*\phi\, dz. \quad (19)$$

Using Eqs. (13) - (16) and boundary conditions Eq. (17), the various terms of right hand side of Eq. (19) can respectively be written as

$$aR_a^{1/2}(M_1-M_4)\int_0^1 w^* D\Phi_1\, dz = -M_1(1-M_5)\int_0^1 D\Phi_1(D^2-a^2-\sigma^* P_r)\theta^*\, dz$$

$$= -M_1(1-M_5)\int_0^1 D\Phi_1 D^2\theta^* dz + M_1(1-M_5)(a^2+\sigma^* P_r)\int_0^1 \theta^* D\Phi_1 dz$$

$$= M_1(1-M_5)\int_0^1 D^2\Phi_1 D\theta^* dz - M_1(1-M_5)(a^2+\sigma^* P_r)\int_0^1 \Phi_1 D\theta^* dz$$

$= M_1(1-M_5)\int_0^1 D^2\Phi_1(D^2-a^2M_3)\Phi_1^* dz - M_1(1-M_5)(a^2+\sigma^* P_r)\int_0^1 \Phi_1(D^2-a^2M_3)\Phi_1^* dz$, (utilizing Eq. (13)) (20)

$$aR_a^{1/2}(1+M_1-M_4)\int_0^1 w^*\theta dz = -[1+M_1(1-M_5)]\int_0^1 \theta(D^2-a^2-\sigma^* P_r)\theta^* dz. \quad (21)$$

Following the same procedure as is used to obtain Eq. (20), we have on utilizing Eq. (14) that

$$aR_s^{1/2}(M_1'-M_4')\int_0^1 w^* D\Phi_2 dz =$$

$- M_4'(1-M_5)\int_0^1 D^2\Phi_2(D^2-a^2 M_3)\Phi_2^* dz + M_4'(1-M_5)(a^2+\sigma^* P_s)\int_0^1 \Phi_2(D^2-a^2 M_3)\Phi_2^* dz.$ (22)

Further, on using (16), we have

$$aR_s^{1/2}(1-M_1'+M_4')\int_0^1 w^*\phi\,dz=-\left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]\int_0^1 \phi\,(D^2-a^2-\sigma^*P_s)\phi^*dz. \tag{23}$$

Combining Eqs. (19) - (23), we obtain

$$\left(\frac{\sigma}{\varepsilon}+\frac{1}{k_1}\right)\int_0^1 w^*(D^2-a^2)wdz=M_1(1-M_5)\int_0^1 D^2\Phi_1(D^2-a^2M_3)\Phi_1^*dz-M_1(1-M_5)(a^2+\sigma^*P_r)\int_0^1 \Phi_1(D^2-a^2M_3)\Phi_1^*dz+[1+M_1(1-M_5)]\int_0^1 \theta(D^2-a^2-\sigma^*P_r)\theta^*dz-M_4'(1-M_5)\int_0^1 D^2\Phi_2(D^2-a^2M_3)\Phi_2^*dz+M_4'(1-M_5)(a^2+\sigma^*P_s)\int_0^1 \Phi_2(D^2-a^2M_3)\Phi_2^*dz-\left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]\int_0^1 \phi\,(D^2-a^2-\sigma^*P_s)\phi^*dz. \tag{24}$$

Integrating the various terms of Eq. (24), by parts, for a suitable number of times with the help of the boundary conditions (17) and the equality

$$\int_0^1 \psi^* D^{2n}\psi dz=(-1)^n\int_0^1 |D^n\psi|^2dz, \tag{25}$$

where $\psi=w,\theta,\phi,\Phi_1,\Phi_2 (n=1)$,

we get

$$\left(\frac{\sigma}{\varepsilon}+\frac{1}{k_1}\right)\int_0^1(|Dw|^2+a^2|w|^2)\,dz=-M_1(1-M_5)\int_0^1(|D^2\Phi_1|^2+a^2M_3|D\Phi_1|^2)dz-M_1(1-M_5)(a^2+\sigma^*P_r)\int_0^1(|D\Phi_1|^2+a^2M_3|\Phi_1|^2)\,dz+[1+M_1(1-M_5)]\int_0^1(|D\theta|^2+(a^2+\sigma^*P_r)|\theta|^2)\,dz+M_4'(1-M_5)\int_0^1(|D^2\Phi_2|^2+a^2M_3|D\Phi_2|^2)dz+M_4'(1-M_5)(a^2+\sigma^*P_s)\int_0^1(|D\Phi_2|^2+a^2M_3|\Phi_2|^2)\,dz-\left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]\int_0^1(|D\phi|^2+(a^2+\sigma^*P_s)|\phi|^2)dz. \tag{26}$$

Equating imaginary parts of both sides of Eq. (26) and cancelling $\sigma_i(\neq 0)$, we obtain

$$\frac{1}{\varepsilon}\int_0^1(|Dw|^2+a^2|w|^2)\,dz=M_1(1-M_5)P_r\int_0^1(|D\Phi_1|^2+a^2M_3|\Phi_1|^2)\,dz-[1+M_1(1-M_5)]P_r\int_0^1|\theta|^2dz-M_4'(1-M_5)P_s\int_0^1(|D\Phi_2|^2+a^2M_3|\Phi_2|^2)\,dz+\left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]P_s\int_0^1|\phi|^2dz. \tag{27}$$

Now multiplying Eq. (13) by Φ_1^* and integrating over the interval [0,1], we get

$$\int_0^1(|D\Phi_1|^2+a^2M_3|\Phi_1|^2)\,dz=\int_0^1(D\Phi_1^*)\theta dz$$

$$\leq\left|\int_0^1(D\Phi_1^*)\theta dz\right|\leq\int_0^1|(D\Phi_1^*)\theta|dz\leq\int_0^1|D\Phi_1^*|\,|\theta|dz\leq\int_0^1|D\Phi_1||\theta|dz$$

$$\leq \left(\int_0^1 |D\Phi_1|^2 dz\right)^{1/2} \left(\int_0^1 |\theta|^2 dz\right)^{1/2}, \tag{28}$$

(using the Schwartz's inequality)

which gives

$$\int_0^1 |D\Phi_1|^2 dz \leq \left(\int_0^1 |D\Phi_1|^2 dz\right)^{1/2} \left(\int_0^1 |\theta|^2 dz\right)^{1/2},$$

and hence

$$\left(\int_0^1 |D\Phi_1|^2 dz\right)^{1/2} \leq \left(\int_0^1 |\theta|^2 dz\right)^{1/2}. \tag{29}$$

On the similar lines Eq. (14) yieldsthe inequality

$$\left(\int_0^1 |D\Phi_2|^2 dz\right)^{1/2} \leq \left(\int_0^1 |\phi|^2 dz\right)^{1/2}. \tag{30}$$

Combining inequality (28) in inequality Eq. (29), we get

$$\int_0^1 (|D\phi_1|^2 + a^2 M_3 |\Phi_1|^2)\, dz \leq \int_0^1 |\theta|^2 dz. \tag{31}$$

Now multiplying Eq. (16) by its complex conjugate and integratingover the interval [0,1]for suitable of times by making use of boundary conditions (17).The real part of this resulting equation gives

$$\int_0^1 (|D^2\phi|^2 + 2a^2|D\phi|^2 + a^4|\phi|^2) dz +$$
$$2\sigma_r P_s \int_0^1 (|D\phi|^2 + a^2|\phi|^2) dz + |\sigma|^2 {P_s}^2 \int_0^1 |\phi|^2 dz = a^2 R_s \int_0^1 |w|^2 dz. \tag{32}$$

Since $P_s \geq 0$, therefore from Eq. (32), we obtain

$$\int_0^1 |\phi|^2 dz < \frac{R_s a^2}{{P_s}^2 |\sigma|^2} \int_0^1 |w|^2 dz\,. \tag{33}$$

Using inequalities Eqs. (31) and (33) in Eq. (27), we obtain

$$\frac{1}{\varepsilon}\int_0^1 |Dw|^2 dz + a^2 \left[\frac{1}{\varepsilon} - \frac{\left[1 - M_1'\left(1 - \frac{1}{M_5}\right)\right] R_s}{P_s |\sigma|^2}\right] \int_0^1 |w|^2 dz + P_r \int_0^1 |\theta|^2 dz + M_4'(1 -$$
$$M_5) P_s \int_0^1 (|D\Phi_2|^2 + a^2 M_3 |\Phi_2|^2) dz < 0, \tag{34}$$

which clearly implies that

$$|\sigma| < \sqrt{\frac{\varepsilon R_s \left[1 - M_1'\left(1 - \frac{1}{M_5}\right)\right]}{P_s}}.$$

This establishes the theorem.

The theorem proved above may be stated in an equivalent form, from the physical point of view, as: the complex growth rate of an arbitrary oscillatory motion of neutral or growing amplitude in ferrothermosolutal convection in a densely packed porous medium, for the case of free boundaries, lies inside a semi-circle in the right half of the $\sigma_r\sigma_i$- plane whose centre is at the origin and radius equals to $\sqrt{\frac{\varepsilon\, R_s\left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]}{P_s}}$.

Theorem2: If $R_a > 0$, $R_s > 0, M_1 > 0, M_1' > 0, 1 - M_5 > 0, P_r > 0$, $P_s > 0$, $\sigma_r \geq 0$, $\delta > 0$ and $\sigma_i \neq 0$, then a necessary condition for the existence of a nontrivial solution $(w, \theta, \phi, \Phi_1, \Phi_2\, \sigma,)$ of the equations (10) and (13) - (16) together with the boundaryconditions (18) is that

$$|\sigma|^2\sigma_i^2 < \varepsilon^2\left\{\frac{R_a\, M_1(1-M_5)}{P_r} + \frac{R_s}{P_s}\left(1 + M_1'\left|1-\frac{1}{M_5}\right| - M_1'\left(1-\frac{1}{M_5}\right)\right)\right\}^2.$$

Proof: Multiplying equation (10) by w^* throughout and integrating the resulting equation over the interval [0,1], we get

$$\left(\frac{\sigma}{\varepsilon}+\frac{1}{k_1}\right)\int_0^1 w^*(D^2-a^2)\,wdz = a\,{R_a}^{1/2}(M_1 - M_4)\int_0^1 w^*\,D\Phi_1 dz - a\,{R_a}^{\frac{1}{2}}[1+M_1 - M_4]\int_0^1 w^*\,\theta\, dz + a\,{R_s}^{\frac{1}{2}}(M_1' - M_4')\int_0^1 w^*\,D\Phi_2 dz + a\,{R_s}^{1/2}[1-M_1'+M_4']\int_0^1 w^*\,\phi dz. \quad (35)$$

Eqs. (15) and (16) respectively yields

$$-a\,{R_a}^{1/2}[1+M_1-M_4]\int_0^1 w^*\theta\, dz = [1+M_1(1-M_5)]\int_0^1 \theta\,(D^2-a^2-P_r\sigma^*)\theta^* dz, \quad (36)$$

and

$$a\,{R_s}^{1/2}[1-M_1'+M_4']\int_0^1 w^*\,\phi dz = -\left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]\int_0^1 \phi\,(D^2-a^2-P_s\sigma^*)\phi^* dz. \quad (37)$$

Combining Eqs. (35) – (37), we obtain

$$\left(\frac{\sigma}{\varepsilon}+\frac{1}{k_1}\right)\int_0^1 w^*(D^2-a^2)\,wdz = a\,{R_a}^{\frac{1}{2}}M_1(1-M_5)\int_0^1 w^*\,D\Phi_1 dz + [1+M_1(1-M_5)]\int_0^1 \theta\,(D^2-a^2-P_r\sigma^*)\theta^* dz + a\,{R_s}^{\frac{1}{2}}M_1'\left(1-\frac{1}{M_5}\right)\int_0^1 w^*\,D\Phi_2 dz - \left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]\int_0^1 \phi\,(D^2-a^2-P_s\sigma^*)\phi^* dz. \quad (38)$$

Integrating the various terms of Eq. (38), by parts, for a suitable number of times and utilizing the boundary conditions (18) and equality (25) we get

$$\left(\frac{\sigma}{\varepsilon}+\frac{1}{k_1}\right)\int_0^1(|Dw|^2 + a^2|w|^2)dz = -aR_a^{1/2}M_1(1-M_5)\int_0^1 w^* D\Phi_1 dz + [1+M_1(1-M_5)]\int_0^1(|D\theta|^2 + a^2|\theta|^2 + P_r\sigma^*|\theta|^2)dz - aR_s^{1/2}M_1'\left(1-\frac{1}{M_5}\right)\int_0^1 w^* D\Phi_2 dz - \left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]\int_0^1(|D\phi|^2 + a^2|\phi|^2 + P_s\sigma^*|\phi|^2)\,dz. \quad (39)$$

Equating imaginary parts of both sides of Eq. (39) and dividing the resulting equation by $\sigma_i(\neq 0)$ throughout, we have

$$\frac{1}{\varepsilon}\int_0^1(|Dw|^2 + a^2|w|^2)dz = -\frac{aR_a^{1/2}M_1(1-M_5)}{\sigma_i}\,imaginary\ part\ \text{of}\int_0^1 w^* D\Phi_1 dz - [1+M_1(1-M_5)]P_r\int_0^1|\theta|^2\,dz - \frac{aR_s^{1/2}M_1'\left(1-\frac{1}{M_5}\right)}{\sigma_i}\,imaginary\ part\ \text{of}\int_0^1 w^* D\Phi_2 dz + \left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]P_s\int_0^1|\phi|^2\,dz. \quad (40)$$

Now multiplying Eq. (15) by its complex conjugate and integrating over the interval [0,1], by parts, for an appropriate number of times, using the boundary conditions (18) and then equating the real parts of both sides, we obtain

$$\int_0^1(|D^2\theta|^2 + 2a^2|D\theta|^2 + a^4|\theta|^2)dz + 2\sigma_r P_r\int_0^1(|D\theta|^2 + a^2|\theta|^2)dz + |\sigma|^2 {P_r}^2\int_0^1|\theta|^2 dz = a^2R_a\int_0^1|w|^2 dz, \quad (41)$$

since $\sigma_r \geq 0$, we have from Eq. (41) that

$$\int_0^1|\theta|^2 dz \leq \frac{R_a a^2}{P_r^2|\sigma|^2}\int_0^1|w|^2 dz\,. \quad (42)$$

Combining inequalities (29) and (42), we obtain

$$\left(\int_0^1|D\Phi_1|^2 dz\right)^{1/2} \leq \frac{aR_a^{1/2}}{P_r|\sigma|}\left(\int_0^1|w|^2 dz\right)^{1/2}. \quad (43)$$

Similarly combining inequalities (30) and (33) to obtain

$$\left(\int_0^1|D\Phi_2|^2 dz\right)^{1/2} \leq \frac{aR_s^{1/2}}{P_s|\sigma|}\left(\int_0^1|w|^2 dz\right)^{1/2}. \quad (44)$$

Now, $-\frac{aR_a^{1/2}M_1(1-M_5)}{\sigma_i}\,imaginary\ part$ of $\int_0^1 w^* D\Phi_1 dz$

$$\leq aR_a^{1/2}M_1(1-M_5)\left|\frac{1}{\sigma_i}\int_0^1 w^* D\Phi_1 dz\right|$$

$$\leq \frac{aR_a^{1/2}\,M_1(1-M_5)}{|\sigma_i|}\int_0^1|w||D\Phi_1|\,dz$$

$\leq \frac{aR_a^{1/2} M_1(1-M_5)}{|\sigma_i|}\left(\int_0^1 |w|^2 dz\right)^{1/2}\left(\int_0^1 |D\Phi_1|^2 dz\right)^{1/2}$ (using Schwartz inequality)

$\leq \frac{a^2 R_a M_1(1-M_5)}{P_r|\sigma||\sigma_i|}\int_0^1 |w|^2 dz.$ (using inequality (43)) (45)

Similarly, $-\frac{aR_s^{1/2} M_1'\left(1-\frac{1}{M_5}\right)}{\sigma_i}$ *imaginary part* of $\int_0^1 w^* D\Phi_2 dz$

$$\leq \frac{aR_s^{1/2} M_1'\left|1-\frac{1}{M_5}\right|}{|\sigma_i|}\int_0^1 |w||D\Phi_2|\, dz$$

$$\leq \frac{aR_s^{1/2} M_1'\left|1-\frac{1}{M_5}\right|}{|\sigma_i|}\left(\int_0^1 |w|^2 dz\right)^{1/2}\left(\int_0^1 |D\Phi_2|^2 dz\right)^{1/2}$$

$\leq \frac{a^2 R_s M_1'\left|1-\frac{1}{M_5}\right|}{P_s|\sigma||\sigma_i|}\int_0^1 |w|^2 dz.$ (utilizing inequality (44)) (46)

Now, multiplying Eq. (16) by ϕ^* and integrating the resulting equation, by parts, for an appropriate number of times over the vertical range of z; and then from imaginary part of the final equation, we obtain

$$\int_0^1 |\phi|^2 dz = \frac{1}{\sigma_i}\, imaginary\ part\ of\ \frac{aR_s^{1/2}}{P_s}\int_0^1 \phi^* w dz$$

$$\leq \frac{aR_s^{1/2}}{|\sigma_i|P_s}\left|\int_0^1 \phi^* w dz\right|$$

$\leq \frac{aR_s^{1/2}}{|\sigma_i|P_s}\left(\int_0^1 |\phi|^2 dz\right)^{\frac{1}{2}}\left(\int_0^1 |w|^2 dz\right)^{\frac{1}{2}}$ (using Schwartz inequality)

$\leq \frac{a^2 R_s}{|\sigma||\sigma_i|P_s^2}\int_0^1 |w|^2 dz.$ (utilizing inequality (33)) (47)

Thus using inequalities (45), (46) and (47) in Eq. (40), we finally, get

$\frac{1}{\varepsilon}\int_0^1 |Dw|^2 dz + a^2\left(\frac{1}{\varepsilon} - \frac{R_a M_1(1-M_5)}{P_r|\sigma||\sigma_i|} - \frac{R_s M_1'\left|1-\frac{1}{M_5}\right|}{P_s|\sigma||\sigma_i|} - \frac{R_s\left[1-M_1'\left(1-\frac{1}{M_5}\right)\right]}{|\sigma||\sigma_i|P_s}\right)\int_0^1 |w|^2 dz +$
$[1+M_1(1-M_5)]P_r\int_0^1 |\theta|^2\, dz \leq 0,$

which implies that

$|\sigma|^2\sigma_i^2 < \varepsilon^2\left\{\frac{R_a M_1(1-M_5)}{P_r} + \frac{R_s}{P_s}\left(1 + M_1'\left|1-\frac{1}{M_5}\right| - M_1'\left(1-\frac{1}{M_5}\right)\right)\right\}^2,$ (48)

Above theorem may be stated, from the physical point of view, as: the complex growth rate of an arbitrary oscillatory disturbance of neutral or growing amplitude in ferrothermosolutal convection in a densely packed porous medium must lie inside the region represented by inequality (61). This result is valid when both the boundaries are rigid.

Note: It is worth to point out here that for most of the ferrofluids, which are formed by changing ferric oxides and carrier organic fluids (e. g. kerosene, alcohol and hydrocarbon etc.), the parametric value M_5 varies between 0.1 and 0.5 (Finlayson [4] and Gupta and Gupta [5]), so that the condition $1 - M_5 > 0$ and hence $1 - \frac{1}{M_5} < 0$ remains valid.

4. Conclusion

The present problem under consideration has been investigated by using the linear stability theory. The upper limits for the complex growth rates in ferrothermosolutal convection in a densely packed porous medium heated and salted from below has been obtained by using Darcy model. These results are important especially when at least one of the boundaries is rigid so that exact solutions in the closed form are not obtainable. The present investigations can present theoretical support to experimentalists.

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