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COEFFICIENT INEQUALITY OF A SIGNIFICANT CLASS OF ANALYTIC FUNCTION

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Abstract: In the present investigation, we have introduced and studied a new significant class of analytic function defined on the unit disk for which $(\alpha+\beta-2\alpha\beta)\frac{zf'(z)}{f(z)}+(1-\alpha-\beta+2\alpha\beta)\frac{\{zf'(z)\}'}{f'(z)} \prec (-1 \leq \mathbf{B} \leq \mathbf{A} \leq 1, \delta > 0)$. We have also investigated the coefficient estimates under Fekete–Szego problem and some other corollaries by using principle of subordination.

Keywords: Starlike function, Convex function, Extremal function and Fekete-Szego inequality.

Subject Classification: 30C45, 30C50.

1. Introduction

Let the class $\mathcal{A}$ consists of all analytic functions $f(z)$ in the open disk $D = \{z: |z| < 1\}$, normalized by $f(0) = f^{I}(0) - 1 = 0$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. (1)

Let us consider two analytic functions j and k defined in the domain **D**, then the function j is called subordinate to k, if there exists an analytic function w, satisfing the condition $|w(z)| < 1$ such that $j(z) = k(w(z))$, and we write it as $j(z) \prec k(z)$. Let **S** denotes the class of analytic functions in $\mathcal{A}$.

In 1916, Bieberbach [2, 3] formulated, Bieberbach conjecture regarding the coefficients of the univalent analytic functions of the unit disk. He proved that $|a_2| \leq 2$ for the regular function $f(z) \in S$. In 1923, Lowner [7] proved that $|a_3| \leq 3$ for the function. Fekete and Szego [5] investigated that the intricate inequality $|a_3 - \mu a_2^2| \leq 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right)$, holds good for each $0 \leq \mu \leq 1.$

Concise history of the Fekete and Szego inequality for the class of starlike and convex functions, can be seen by the research work done by Srivastava et al. [9], Orhan and Gunes [8], Denize and Orhan [4], Arıkan et al. [1].

We define subclasses $\boldsymbol{S}^*$ and $\boldsymbol{K}$ of $\boldsymbol{S}$ in the following manner:

Let $\boldsymbol{S}^*$ denotes the class of univalent starlike functions $\boldsymbol{g(z) = z + \sum_{n=2}^{\infty} b_n z^n} \in \boldsymbol{\mathcal{A}}$, satisfying the condition $\boldsymbol{Re}\left\{\frac{\boldsymbol{zg'(z)}}{\boldsymbol{g(z)}}\right\} > 0, z \in D$, and (2)

$\boldsymbol{K}$ denotes the class of univalent convex functions $\boldsymbol{h(z) = z + \sum_{n=2}^{\infty} c_n z^n} \in \boldsymbol{\mathcal{A}}$, satisfying the condition $\boldsymbol{Re}\frac{\{\boldsymbol{zh'(z)}\}'}{\boldsymbol{h'(z)}} > 0, z \in D$ (3)

Here we introduce the class $\boldsymbol{\mathcal{A}(\alpha, \beta; A, B; \delta)}$ of the functions $\boldsymbol{f(z) = z + \sum_{n=2}^{\infty} a_n z^n} \in \boldsymbol{\mathcal{A}}$ satisfying the condition

$$\boldsymbol{Re}\left[(\boldsymbol{\alpha+\beta-2\alpha\beta})\frac{\boldsymbol{zf'(z)}}{\boldsymbol{f(z)}} + (\boldsymbol{1-\alpha-\beta+2\alpha\beta})\frac{\{\boldsymbol{zf'(z)}\}'}{\boldsymbol{f'(z)}}\right] > 0, z \in D$$

i.e $(\boldsymbol{\alpha+\beta-2\alpha\beta})\frac{\boldsymbol{zf'(z)}}{\boldsymbol{f(z)}} + (\boldsymbol{1-\alpha-\beta+2\alpha\beta})\frac{\{\boldsymbol{zf'(z)}\}'}{\boldsymbol{f'(z)}} \prec \left(\frac{\mathbf{1+Az}}{\mathbf{1+Bz}}\right)^{\boldsymbol{\delta}}; \boldsymbol{-1 \le \mathbf{B} \le \mathbf{A} \le 1, \delta} > 0$ (4)

Here it is to be noted:

[1] $\boldsymbol{\mathcal{A}(\alpha, \beta; A, B; 1)} = \boldsymbol{\mathcal{A}(\alpha, \beta; A, B)}$

[2] $\boldsymbol{\mathcal{A}(\alpha, \beta; 1, -1; \delta)} = \boldsymbol{\mathcal{A}(\alpha, \beta; \delta)}$

[3] $\boldsymbol{\mathcal{A}(\alpha, \beta; 1, -1; 1)} = \boldsymbol{\mathcal{A}(\alpha, \beta)}$

[4] $\boldsymbol{\mathcal{A}(1, 0)} = \boldsymbol{S}^*$

[5] $\boldsymbol{\mathcal{A}(0, 1)} = \boldsymbol{S}^*$

[6] $\boldsymbol{\mathcal{A}(0, 0)} = \boldsymbol{K}$

[7] $\boldsymbol{\mathcal{A}(1, 1)} = \boldsymbol{K}$

Fekete- Szego Inequality

THEROEM 2 :- If f(z) $\in \boldsymbol{\mathcal{A}}(\alpha, \beta, A, B, \delta), then$

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{(A-B)\delta}{(2-\alpha-\beta+2\alpha\beta)^2}\left\{\frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)} - \mu(A-B)\delta\right\}.if \\ \mu \le \frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B-1\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)(A-B)\delta} & (5) \\ \frac{(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)},\quad if\frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B-1\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)(A-B)\delta} \le \mu \le \\ \frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B+1\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)(A-B)\delta} & (6) \\ \frac{(A-B)\delta}{(2-\alpha-\beta+2\alpha\beta)^2}\left\{\mu(A-B)\delta - \frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)}\right\}.if \\ \mu \le \frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B+1\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)(A-B)\delta} & (7) \end{cases}$$

And the results are sharp.

Proof: by definition of $\mathcal{A}(\alpha,\beta,A,B,\delta)$, we have

$$(\boldsymbol{\alpha}+\boldsymbol{\beta}-\mathbf{2}\boldsymbol{\alpha\beta})\frac{\boldsymbol{zf'(z)}}{\boldsymbol{f(z)}}+(\mathbf{1}-\boldsymbol{\alpha}-\boldsymbol{\beta}+\mathbf{2}\boldsymbol{\alpha\beta})\frac{\{\boldsymbol{zf'(z)}\}'}{\boldsymbol{f'(z)}}\left(\frac{\mathbf{1+Az}}{\mathbf{1+Bz}}\right)^{\boldsymbol{\delta}} \tag{8}$$

Expanding (8), we have

$$1+(2-\alpha-\beta+2\alpha\beta)a_2z+\{(6-4\alpha-4\beta+8\alpha\beta)a_3-(4-3\alpha-3\beta+6\alpha\beta)a_2^2\}z^2\ldots\ldots\ldots\ldots$$

$$=$$

$$1+(A-B)\delta c_1z+(A-B)\delta\left[C_2+\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B\right\}C_1^2\right]z^2\ldots\ldots\ldots\ldots\ldots \tag{9}$$

From the equation (9), we have

$$\begin{cases} a_2=\frac{(A-B)\delta}{(2-\alpha-\beta+2\alpha\beta)}c_1 \\ a_3=\frac{(A-B)\delta}{(6-4\alpha-4\beta+8\alpha\beta)}\left[c_2+\left\{\frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{(2-\alpha-\beta+2\alpha\beta)^2}\right\}c_1^2\right] \end{cases} \tag{10}$$

Using (10) we have

$$a_3-\mu a_2^2$$
$$=\frac{(A-B)\delta}{(6-4\alpha-4\beta+8\alpha\beta)}\left[c_2+\left\{\frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{(2-\alpha-\beta+2\alpha\beta)^2}\right\}c_1^2\right]$$
$$-\mu\frac{(A-B)^2\delta^2}{(2-\alpha-\beta+2\alpha\beta)^2}c_1^2$$

This leads to

$$\left|a_3-\mu a_2^2\right|\le\frac{(A-B)\delta}{(6-4\alpha-4\beta+8\alpha\beta)}+\left[\left|\frac{(A-B)\delta}{(2-\alpha-\beta+2\alpha\beta)^2}\left\{\frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)}-\mu(A-B)\delta\right\}\right|-\frac{(A-B)\delta}{(6-4\alpha-4\beta+8\alpha\beta)}\right]\left|c_1^2\right| \tag{11}$$

Now, two cases arise:

Case I: when $\mu\le\frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)(A-B)\delta}$, we have

$$\left|a_3-\mu a_2^2\right| \leq \frac{(A-B)\delta}{(6-4\alpha-4\beta+8\alpha\beta)}+\left[\left|\frac{(A-B)\delta}{(2-\alpha-\beta+2\alpha\beta)^2}\left\{\frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B-1\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)}-\mu(A-B)\delta\right\}\right|\right]\left|c_1^2\right| \quad (12)$$

Under this case, two sub cases arise:

Sub case I (a): when $\mu \leq \frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B-1\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)(A-B)\delta}$, we have

$$\left|a_3-\mu a_2^2\right| \leq \frac{(A-B)\delta}{(2-\alpha-\beta+2\alpha\beta)^2}\left\{\frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)}-\mu(A-B)\delta\right\} \quad (13)$$

Sub case I (b): when $\mu \geq \frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B-1\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)(A-B)\delta}$, we have

$$\left|a_3-\mu a_2^2\right| \leq \frac{(A-B)\delta}{(6-4\alpha-4\beta+8\alpha\beta)} \quad (14)$$

Case II: when $\mu \geq \frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)(A-B)\delta}$, we have

$$\left|a_3-\mu a_2^2\right| \leq \frac{2\delta}{(6-4\alpha-4\beta+8\alpha\beta)}+\left[\left|\frac{2\delta}{(2-\alpha-\beta+2\alpha\beta)^2}\left\{(A-B)\delta\mu-\frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B+1\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)}\right\}\right|\right]\left|c_1^2\right| \quad (15)$$

Now, again two sub cases arise under this case:

Sub case II (a): if $\mu \leq \frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B+1\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)(A-B)\delta}$, we have

$$\left|a_3-\mu a_2^2\right| \leq \frac{(A-B)\delta}{(6-4\alpha-4\beta+8\alpha\beta)} \quad (16)$$

Combining sub cases I (b) and II (a), we get

$$\left|a_3-\mu a_2^2\right| \leq \frac{(A-B)\delta}{(6-4\alpha-4\beta+8\alpha\beta)}\ if\ \frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B-1\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)(A-B)\delta} \leq \mu \leq \frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B+1\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)(A-B)\delta}$$

Sub case II (b): if $\mu \geq \frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B+1\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)(A-B)\delta}$, we have

$$|a_3-\mu a_2^2| \leq \frac{(A-B)\delta}{(2-\alpha-\beta+2\alpha\beta)^2}\left\{\mu(A-B)\delta-\frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B\right\}(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)}\right\} \quad (17)$$

By combining (13), (14), (16) and (17), we get our desired result

Extremal function for the first inequality (5) and third inequality (7) is given by

$$f_1(z)=z\left(1+\frac{w-2v^2}{\sqrt{w}}z\right)^{\frac{w}{w-2v^2}}$$

Where v = $\frac{(A-B)\delta}{(2-\alpha-\beta+2\alpha\beta)^2}\left\{\frac{\left\{\left(\frac{\delta-1}{2}\right)(A-B)-B\right\}(2-\alpha-\beta+4\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)}\right\}$ and

$$w=\left(\frac{(A-B)\delta}{2-\alpha-\beta+2\alpha\beta}\right)^2 \quad (18)$$

Extremal function for the second inequality (6) is given by

$$f_2(z)=(1+z)^{\frac{(A-B)\delta}{2(3-2\alpha-2\beta+4\alpha\beta)}} \quad (19)$$

Corollary: 2.1 For A=1, B = -1, δ=1 in the inequality (8) provides

$$(\boldsymbol{\alpha}+\boldsymbol{\beta}-\mathbf{2}\boldsymbol{\alpha}\boldsymbol{\beta})\mathbf{z}\frac{\mathbf{f}'(\mathbf{z})}{\mathbf{f}(\mathbf{z})}+(\mathbf{1}-\boldsymbol{\alpha}-\boldsymbol{\beta}+\mathbf{2}\boldsymbol{\alpha}\boldsymbol{\beta})\frac{\{\mathbf{z}\mathbf{f}'(\mathbf{z})\}'}{\mathbf{f}'(\mathbf{z})}\prec\frac{\mathbf{1}+\mathbf{z}}{\mathbf{1}-\mathbf{z}}$$

For which, we have theorem 2, as follows

$$|a_3-\mu a_2^2| \leq \begin{cases} \left(\frac{1}{2-\alpha-\beta+2\alpha\beta}\right)^2\left\{\frac{(2-\alpha-\beta+2\alpha\beta)^2+2(4-3\alpha-3\beta+6\alpha\beta)}{(3-2\alpha-2\beta+4\alpha\beta)}-4\mu\right\}, \text{if}\mu \leq \frac{(4-3\alpha-3\beta+6\alpha\beta)}{2(3-2\alpha-2\beta+4\alpha\beta)} \\ \frac{1}{(3-2\alpha-2\beta+4\alpha\beta)}, \text{if } \frac{(4-3\alpha-3\beta+6\alpha\beta)}{2(3-2\alpha-2\beta+4\alpha\beta)} \leq \mu \leq \frac{(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)}{2(3-2\alpha-2\beta+4\alpha\beta)} \\ \left(\frac{1}{2-\alpha-\beta+2\alpha\beta}\right)^2\left\{4\mu-\frac{(2-\alpha-\beta+2\alpha\beta)^2+2(4-3\alpha-3\beta+6\alpha\beta)}{(3-2\alpha-2\beta+4\alpha\beta)}\right\}, \text{if } \mu \geq \frac{(2-\alpha-\beta+2\alpha\beta)^2+(4-3\alpha-3\beta+6\alpha\beta)}{2(3-2\alpha-2\beta+4\alpha\beta)} \end{cases} \quad (20)$$

Corollary: 2.2 For δ=1 in the inequality (8) provides

$$(\boldsymbol{\alpha}+\boldsymbol{\beta}-\mathbf{2}\boldsymbol{\alpha}\boldsymbol{\beta})\frac{\boldsymbol{z}\boldsymbol{f}'(\boldsymbol{z})}{\boldsymbol{f}(\boldsymbol{z})}+(\mathbf{1}-\boldsymbol{\alpha}-\boldsymbol{\beta}+\mathbf{2}\boldsymbol{\alpha}\boldsymbol{\beta})\frac{\{\boldsymbol{z}\boldsymbol{f}'(\boldsymbol{z})\}'}{\boldsymbol{f}'(\boldsymbol{z})}\prec\frac{\mathbf{1}+\mathbf{Az}}{\mathbf{1}-\mathbf{Bz}}$$

For which, we have theorem 2, as follows

$$|a_3-\mu a_2^2| \leq$$
$$\begin{cases} \frac{1}{(2-\alpha-\beta+2\alpha\beta)^2}\left\{\frac{(4-3\alpha-3\beta+6\alpha\beta)(A-B)^2+(B^2-AB)(2-\alpha-\beta+2\alpha\beta)^2}{(6-4\alpha-4\beta+8\alpha\beta)}-(A-B)^2\mu\right\} & \mu \leq \frac{(4-3\alpha-3\beta+6\alpha\beta)(A-B)^2+(B^2-AB-A+B)(2-\alpha-\beta+2\alpha\beta)^2}{(A-B)^2(6-4\alpha-4\beta+8\alpha\beta)} \\ \frac{A-B}{(6-4\alpha-4\beta+8\alpha\beta)} \quad \frac{(4-3\alpha-3\beta+6\alpha\beta)(A-B)^2+(B^2-AB-A+B)(2-\alpha-\beta+2\alpha\beta)^2}{(A-B)^2(6-4\alpha-4\beta+8\alpha\beta)} \leq \mu \leq & \frac{(4-3\alpha-3\beta+6\alpha\beta)(A-B)^2+(B^2-AB+A-B)(2-\alpha-\beta+2\alpha\beta)^2}{(A-B)^2(6-4\alpha-4\beta+8\alpha\beta)} \\ \frac{1}{(2-\alpha-\beta+2\alpha\beta)^2}\left\{(A-B)^2\mu-\frac{(4-3\alpha-3\beta+6\alpha\beta)(A-B)^2+(B^2-AB)(2-\alpha-\beta+2\alpha\beta)^2}{(6-4\alpha-4\beta+8\alpha\beta)}\right\} & \mu \leq \frac{(4-3\alpha-3\beta+6\alpha\beta)(A-B)^2+(B^2-AB+A-B)(2-\alpha-\beta+2\alpha\beta)^2}{(A-B)^2(6-4\alpha-4\beta+8\alpha\beta)}) \end{cases}$$

Corollary: 2.3 For A=1, B = -1 in the inequality (8) provides

$$(\boldsymbol{\alpha}+\boldsymbol{\beta}-\mathbf{2}\boldsymbol{\alpha}\boldsymbol{\beta})\frac{\boldsymbol{z}\boldsymbol{f}'(\boldsymbol{z})}{\boldsymbol{f}(\boldsymbol{z})}+(\mathbf{1}-\boldsymbol{\alpha}-\boldsymbol{\beta}+\mathbf{2}\boldsymbol{\alpha}\boldsymbol{\beta})\frac{\{\boldsymbol{z}\boldsymbol{f}'(\boldsymbol{z})\}'}{\boldsymbol{f}'(\boldsymbol{z})} \prec \left(\frac{\mathbf{1}+\mathbf{z}}{\mathbf{1}-\mathbf{z}}\right)^{\boldsymbol{\delta}}$$

For which, we have theorem 2, as follows

$$|a_3-\mu a_2^2| \leq$$
$$\begin{cases} \frac{\delta^2}{(2-\alpha-\beta+2\alpha\beta)^2}\left(\frac{2(4-3\alpha-3\beta+6\alpha\beta)+(2-\alpha-\beta+2\alpha\beta)^2}{(3-2\alpha-2\beta+4\alpha\beta)}-4\mu\right) .\, if \\ \qquad \mu \leq \frac{(\delta-1)(2-\alpha-\beta+2\alpha\beta)^2+2\delta(4-3\alpha-3\beta+6\alpha\beta)}{4\delta(3-2\alpha-2\beta+4\alpha\beta)} \\ \frac{\delta}{(3-2\alpha-2\beta+4\alpha\beta)} \;, if\; \frac{(\delta-1)(2-\alpha-\beta+2\alpha\beta)^2+2\delta(4-3\alpha-3\beta+6\alpha\beta)}{4\delta(3-2\alpha-2\beta+4\alpha\beta)} \leq \mu \leq \\ \qquad \frac{(\delta+1)(2-\alpha-\beta+2\alpha\beta)^2+2\delta(4-3\alpha-3\beta+6\alpha\beta)}{4\delta(3-2\alpha-2\beta+4\alpha\beta)} \\ \frac{\delta^2}{(2-\alpha-\beta+2\alpha\beta)^2}\left(4\mu-\frac{2(4-3\alpha-3\beta+6\alpha\beta)+(2-\alpha-\beta+2\alpha\beta)^2}{(3-2\alpha-2\beta+4\alpha\beta)}\right) .\, if \\ \qquad \mu \geq \frac{(\delta+1)(2-\alpha-\beta+2\alpha\beta)^2+2\delta(4-3\alpha-3\beta+6\alpha\beta)}{4\delta(3-2\alpha-2\beta+4\alpha\beta)} \end{cases}$$

(21)

Corollary: 2.4 For α=0, β=1, δ=1, A=1, B=-1 and α=1, β=0, δ=1,A=1, B=-1 in the theorem 2, we get

$$|a_3-\mu a_2^2| \leq \begin{cases} \mathbf{3-4\mu}, & \mathbf{if \mu \leq \frac{1}{2}} \\ \mathbf{1}\,, & \mathbf{if\ \frac{1}{2} \leq \mu \leq 1} \\ \mathbf{4\mu-3} & \mathbf{if \mu \geq 1} \end{cases} \tag{22}$$

Corollary: 2.5 For α=1, β=1, δ=1, A=1, B=-1 and α=0, β=0, δ=1,A=1, B=-1 in the theorem 2, we get

$$|a_3-\mu a_2^2| \leq \begin{cases} \mathbf{1-\mu}, & \mathbf{if\ \mu \leq \frac{2}{3}} \\ \mathbf{\frac{1}{3}}\,, & \mathbf{if\ \frac{2}{3} \leq \mu \leq \frac{4}{3}} \\ \mathbf{\mu-1}\,, & \mathbf{if\ \mu \geq \frac{4}{3}} \end{cases} \tag{23}$$

The estimates obtained under corollaries (2.4) and (2.5), were derived by Keogh and Merkes [6] and therefore the results are sharp for the class of univalent star like function and univalent convex function respectively.

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