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ANALYTICAL SOLUTION OF THE TIME-FRACTIONAL FISHER EQUATION BY USING ITERATIVE LAPLACE TRANSFORM METHOD

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Abstract: The main object of the present paper is to derive analytical solutions of the non-linear time-fractional fisher's equations (TFFE) with initial conditions by using iterative Laplace transform method (ILTM). In this process the time-fractional derivatives are considered in the Caputo sense and solutions are found in the form of a series with easily computable components.

Keyword: Laplace transform, Iterative Laplace transform method, Fisher equation, fractional differential equations, Caputo fractional derivatives.

2010 AMS Classification: 26A33, 35R11, 35A20.

1. Introduction

The fractional calculus has become a strong tool for finding the solutions of many problems pertaining to control engineering, physics, signal processing, mathematical biology, viscoelasticity, electromagnetism, and mathematical physics and other areas of sciences as well as technology. Several methods can be found in the literature to find the solution of differential equation of fractional order such as differential transform method (DTM) [11], Homotopy analysis method (HAM) [8], Homotopy perturbation method (HPM) [18], Homotopy perturbation transform method (HPTM) [9] and fractional Laplace Adomian decomposition method (FLADM) [6] and so on. The above mentioned methods provide immediate and visible symbolic terms of numerical approximate solutions as well as of analytical solutions to both linear and nonlinear fractional differential equations.

In 2006, Daftardar-Gejji and Jafari introduced the iterative method for solving numerically non-linear functional equations [4, 5]. Since then the iterative method is being used to find the solution of several non-linear differential equations of integer and fractional order [1] and viewing fractional boundary value problem [3]. Recently, Jafari *et al*. has made elegant use of Laplace transform in this iterative method and it became a popular method known as iterative Laplace transform method (ILTM) [7] for having numerical solutions of a system of fractional partial differential equations, Fokker–Plank equation [17] as well. Recently, fractional Telegraph equations [13], time-fractional Schrödinger equations [14] and fractional heat and wave-like equation [15] are solved successfully by the use of iterative Laplace transform method.

The classical Fisher's equation is a partial differential equation with constant coefficients given by Fisher as a model for the propagation of a mutant gene as

$$u_t(x,t) = u_{xx}(x,t) + u(x,t)\left(1 - u(x,t)\right). \tag{1}$$

In this model, $u(x,t)$ is the population density and $u(u-1)$ denotes the logistic form. This equation is encountered in chemical kinetics and population dynamics which includes problems such as nonlinear evolution of a population in a one-dimensional habitat, neutron population in a nuclear reaction. Moreover, the same equation occurs in logistic population growth models, flame propagation, neurophysiology, autocatalytic chemical reactions and branching Brownian motion processes.

In the present study, the time-fractional model for fisher's equation can be written in following operator form as

$$D_t^{\alpha} u(x,t) = u_{xx}(x,t) + \lambda u(x,t)\left(1 - u(x,t)\right), 0 < \alpha \le 1, \tag{2}$$

where $D_t^{\alpha} u(x,t)$ denotes the Caputo fractional derivative of order α and λ is a real parameter. It can be obtained from the fisher's equation (1) by replacing the first- time derivative by a fractional derivative of order α, $0 < \alpha \le 1$.

2. Some Basic Definitions of Fractional Calculus and Laplace Transform theory

In this section, we list certain basic definitions of fractional calculus alongwith elegant properties of Laplace transform theory.

Definition 1. The Caputo fractional derivative [2] of function $u(x,t)$ is defined as

$$\begin{aligned} D_t^{\alpha} u(x,t) &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\eta)^{m-\alpha-1} u^{(m)}(x,\eta) d\eta, \quad m-1 < \alpha \le m, m \in N, \\ &= J_t^{m-\alpha} D^m u(x,t). \end{aligned} \tag{3}$$

Here $D^m \equiv \dfrac{d^m}{dt^m}$ and J_t^{α} stands for the Riemann-Liouville fractional integral operator of order $\alpha > 0$, defined as [10]

$$J_t^{\alpha} u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1} u(x,\eta) d\eta, \ \eta > 0, (m-1 < \alpha \le m), m \in N. \tag{4}$$

Definition 2. The Laplace transform of a function $f(t)$, $t > 0$ is defined as [10, 12]

$$L[f(t)] = F(t) = \int_0^\infty e^{-st} f(t)dt. \tag{5}$$

Definition 3. The Laplace transform of $D_t^\alpha u(x,t)$ is given as [10, 12]

$$L[D_t^\alpha u\,(x,t)] = L[u(x,t)] - \sum_{k=o}^{m-1} u^k(x,0)\, s^{\alpha-k-1}, m-1 < \alpha \le m, m \in N, \tag{6}$$

where $u^k(x,0)$ is the k-order derivative of $u(x,t)$ w.r.t. t at $t = 0$.

3. Basic Idea of Iterative Laplace Transform Method

To explain the basic idea of iterative Laplace transform method [7], we take the following fractional non-linear non-homogeneous partial differential equation having the prescribed initial conditions written in the form of an operator as

$$D_t^\alpha u(x,t) + Ru(x,t) + N u(x,t) = g(x,t), \;\; m-1 < \alpha \le m, \;\; m \in N, \tag{7}$$

$$u^k(x,0) = h_k(x), \quad k = 0,1,2,...,m-1, \tag{8}$$

where $D_t^\alpha u(x,t)$ is the Caputo fractional derivative of order α, $m-1 < \alpha \le m$, defined by equation (3), R is a linear operator and may include other fractional derivatives of order less than α, N is a non-linear operator which may include other fractional derivatives of order less than α and $g(x,t)$ is a known analytic function.

Applying the Laplace transform w.r.t. on both sides of equation (7), we have

$$L[D_t^\alpha u(x,t)] + L[Ru\,(x,t) + Nu(x,t)] = L[g(x,t)]. \tag{9}$$

Using the differentiation property (6) of the Laplace transform, we find

$$L[u(x,t)] = \frac{1}{s^\alpha} \sum_{k=0}^{m-1} s^{\alpha-1-k} u^k(x,0) + \frac{1}{s^\alpha} L[g(x,t)] - \frac{1}{s^\alpha} L[Ru(x,t) + N\,u(x,t)]. \tag{10}$$

On taking inverse Laplace transform of equation (10), we obtain

$$\begin{aligned} u(r,t) = L^{-1}&\left[\frac{1}{s^\alpha}\left(\sum_{k=0}^{m-1} s^{\alpha-1-k} u^k(x,0) + L[g(x,t)]\right)\right] \\ &- L^{-1}\left[\frac{1}{s^\alpha} L[Ru(x,t) + Nu(x,t)]\right]. \end{aligned} \tag{11}$$

Now, applying the iterative method,

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t). \tag{12}$$

Since R is a linear operator

$$R\left(\sum_{i=0}^{\infty}u_i(x,t)\right)=\sum_{i=0}^{\infty}R\left[u_i(x,t)\right], \tag{13}$$

and the non-linear operator N is splitted as

$$N\left(\sum_{i=0}^{\infty}u_i(x,t)\right)=N\left[u_0(x,t)\right]$$
$$+\sum_{i=1}^{\infty}\left\{N\left(\sum_{k=0}^{i}u_k(x,t)\right)-N\left(\sum_{k=0}^{i-1}u_k(x,t)\right)\right\}. \tag{14}$$

Putting the results given by equations from (12) to (14) in the equation (11), we obtain

$$\sum_{i=0}^{\infty}u_i(x,t)=L^{-1}\left[\frac{1}{s^{\alpha}}\left(\sum_{k=0}^{m-1}s^{\alpha-1-k}u^k(x,0)+L[g(x,t)]\right)\right]$$
$$-L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\sum_{i=0}^{\infty}R\left[u_i(x,t)\right]+N\left[u_0(x,t)\right]\right.\right. \tag{15}$$
$$\left.\left.+\sum_{i=1}^{\infty}\left\{N\left(\sum_{k=0}^{i}u_k(x,t)\right)-N\left(\sum_{k=0}^{i-1}u_k(x,t)\right)\right\}\right]\right].$$

We have defined the recurrence relations as

$$u_0(x,t)=L^{-1}\left[\frac{1}{s^{\alpha}}\left(\sum_{k=0}^{m-1}s^{\alpha-1-k}u^k(x,0)+L\big(g(x,t)\big)\right)\right] \tag{16}$$

$$u_1(x,t)=-L^{-1}\left[\frac{1}{s^{\alpha}}L\left[R\big(u_0(x,t)\big)+N\big(u_0(x,t)\big)\right]\right], \tag{17}$$

$$u_{m+1}(x,t)=-L^{-1}\left[\frac{1}{s^{\alpha}}L\left[R\big(u_m(x,t)\big)-\left\{N\left(\sum_{k=0}^{m}u_k(x,t)\right)-N\left(\sum_{k=0}^{m-1}u_k(x,t)\right)\right\}\right]\right],$$
$$m\geq 1 \tag{18}$$

Therefore the m-term approximate solution of equations (7) and (8) in series form is given by

$$u(x,t)\cong u_0(x,t)+u_1(x,t)+u_2(x,t)+,...,+u_m(x,t),\ \ m=1,2,.... \tag{19}$$

4. Solution of the time-fractional fisher equations

In this section, we apply iterative Laplace transform Method (ILTM) for solving the non-linear time-fractional Fisher's equations with initial conditions.

Example 1. Consider the following non-linear fisher's equation concerning to time–fractional, defined by

$$D_t^{\alpha} u = \frac{\partial^2 u}{\partial x^2} + 6u(1-u) \quad , \; 0 < \alpha \leq 1 , \tag{20}$$

with the initial condition

$$u(x,0) = \frac{1}{\left(1+e^x\right)^2}, \tag{21}$$

where $D_t^{\alpha} u(x,t)$ is the Caputo fractional derivative of order α given by (3).

Taking the Laplace transform of the above equation, and making use of the result given by (21), we have,

$$L[u(x,t)] = \frac{1}{s}\,\frac{1}{\left(1+e^x\right)^2} + \frac{1}{s^{\alpha}}\left[L\left(\frac{\partial^2 u}{\partial x^2} + 6u(1-u)\right)\right]. \tag{22}$$

Applying inverse Laplace transform to the equation (4.3), we obtain

$$u(x,t) = \frac{1}{\left(1+e^x\right)^2} + L^{-1}\left[\frac{1}{s^{\alpha}} L\left(\frac{\partial^2 u}{\partial x^2} + 6u(1-u)\right)\right]. \tag{23}$$

Now, making use of the iterative method, substituting the results of the equations from (12) to (14) in the equation (23) and making use of the results given by the equations (16) to (18), we determine the components of the solution as follows

$$u_0(x,t) = \frac{1}{\left(1+e^x\right)^2}. \tag{24}$$

$$u_1(x,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L\left(\frac{\partial^2 u_0}{\partial x^2} + 6u_0(1-u_0)\right)\right].$$

$$= 10\,\frac{e^x}{\left(1+e^x\right)^3}\,\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{25}$$

$$u_2(x,t) = L^{-1}\left[\frac{1}{s^\alpha}L\left(\frac{\partial^2 u_1}{\partial x^2}+6u_1(1-u_1)\right)\right]. = 50\,\frac{e^x(-1+2e^x)}{\left(1+e^x\right)^4}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{26}$$

$$u_3(x,t) = L^{-1}\left[\frac{1}{s^\alpha}L\left(\frac{\partial^2 u_2}{\partial x^2}+6u_2(1-u_2)\right)\right].$$

$$= 50e^x\left(5-6e^x-15e^{2x}+20e^{3x}-12e^x\,\frac{\Gamma(2\alpha+1)}{\left(\Gamma(\alpha+1)\right)^2}\right)\frac{t^{3\alpha}}{\left(1+e^x\right)^6\Gamma(3\alpha+1)} \tag{27}$$

and so on. The other components may be obtained accordingly.

Thus, the analytical solution in the series form is given by

$$u\left(x,t\right) = u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+,...,$$

$$\begin{aligned} &= \frac{1}{\left(1+e^x\right)^2}+10\,\frac{e^x}{\left(1+e^x\right)^3}\,\frac{t^\alpha}{\Gamma(\alpha+1)}+50\,\frac{e^x(-1+2e^x)}{\left(1+e^x\right)^4}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad +50e^x\left(5-6e^x-15e^{2x}+20e^{3x}-12e^x\,\frac{\Gamma(2\alpha+1)}{\left(\Gamma(\alpha+1)\right)^2}\right)\frac{t^{3\alpha}}{\left(1+e^x\right)^6\Gamma(3\alpha+1)}+,..., \end{aligned} \tag{28}$$

Special Cases

(i) The result in (28) was derived by Zhang and Liu [17] using the different method that is HPM.

(ii) The result in (28) deduced by Khan *et al*. [8] by the application of HAM.

(iii) For $\alpha = 1,$ the result in (28) reduces to the following exact solution

$$u(x,t) = \frac{1}{\left(1+e^{x-5t}\right)^2},. \tag{29}$$

This result was obtained earlier by Wazwaz and Gorguis [16] by using the method of ADM.

Example 2. Consider the following Fisher equation with respect to time–fractional, given by

$$D_t^\alpha u = \frac{\partial^2 u}{\partial x^2}+u(1-u) \quad , \quad 0<\alpha\le 1\,, \tag{30}$$

with the initial condition

$$u(x,0) = \beta\,, \tag{31}$$

where $D_t^{\alpha}u(x,t)$ is the Caputo fractional derivative of order α given by (3).

Taking the Laplace transform of the above equation, and making use of the result given by (31), we have,

$$L[u(x,t)]=\frac{\beta}{s} + \frac{1}{s^{\alpha}}\left[L\left(\frac{\partial^2 u}{\partial x^2}+u(1-u)\right)\right]. \tag{32}$$

Applying inverse Laplace transform to the equation (32), we obtain

$$u(x,t)=\beta+L^{-1}\left[\frac{1}{s^{\alpha}}L\left(\frac{\partial^2 u}{\partial x^2}+u(1-u)\right)\right]. \tag{33}$$

Now, making use of the iterative method, substituting the results of the equations from (12) to (14) in the equation (33) and making use of the results given by the equations (16) to (18), we determine the components of the solution as follows

$$u_0(r,t)= \beta, \tag{34}$$

$$u_1(x,t)= L^{-1}\left[\frac{1}{s^{\alpha}}L\left(\frac{\partial^2 u_0}{\partial x^2}+u_0(1-u_0)\right)\right].$$

$$=\beta(1-\beta)\frac{t^{\alpha}}{\Gamma(\alpha+1)} \tag{35}$$

$$u_2(x,t)= L^{-1}\left[\frac{1}{s^{\alpha}}L\left(\frac{\partial^2 u_1}{\partial x^2}+u_1(1-u_1)\right)\right].$$

$$=\beta(1-\beta)(1-2\beta)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{36}$$

$$u_3(x,t)= L^{-1}\left[\frac{1}{s^{\alpha}}L\left(\frac{\partial^2 u_2}{\partial x^2}+u_2(1-u_2)\right)\right].$$

$$=\left(\beta-5\beta^2+8\beta^3-4\beta^4\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}-\left(\beta^2-2\beta^3+\beta^4\right)\frac{\Gamma(2\alpha+1)}{\left[\Gamma(\alpha+1)\right]^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \tag{37}$$

$$u_4(x,t)= L^{-1}\left[\frac{1}{s^{\alpha}}L\left(\frac{\partial^2 u_3}{\partial x^2}+u_3(1-u_3)\right)\right].$$

$$= (1-2\beta)\left(\beta-5\beta^2+8\beta^3-4\beta^4\right)-\left(\beta^2-2\beta^3+\beta^4\right)\frac{\Gamma(2\alpha+1)}{\left[\Gamma(\alpha+1)\right]^2}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$
$$-2\left(\beta-\beta^2\right)\left(\beta-3\beta^3+2\beta^3\right)\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} \quad (38)$$

and so on. The other components may be obtained accordingly.

Thus, the analytical solution in the series form can be obtained as

$$u(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+u_4(x,t)+,...,$$
$$= \beta+\beta(1-\beta)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\beta(1-\beta)(1-2\beta)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+\left(\beta-5\beta^2+8\beta^3-4\beta^4\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}-\left(\beta^2-2\beta^3+\beta^4\right)\frac{\Gamma(2\alpha+1)}{\left[\Gamma(\alpha+1)\right]^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \quad (39)$$
$$+(1-2\beta)\left(\beta-5\beta^2+8\beta^3-4\beta^4\right)-\left(\beta^2-2\beta^3+\beta^4\right)\frac{\Gamma(2\alpha+1)}{\left[\Gamma(\alpha+1)\right]^2}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$
$$-2\left(\beta-\beta^2\right)\left(\beta-3\beta^3+2\beta^3\right)\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+,...,$$

Special Cases

(i) The result in (39) was derived by Zhang and Liu [17] using the different method that is HPM.

(ii) The result in (39) deduced by Mirzazadeh [11] by the application of DTM.

(iii) For $\alpha = 1,$ the result in (39) reduces to the following exact solution

$$u(x,t) = \frac{\beta e^x}{1-\beta+\beta e^t} \quad . \quad (40)$$

This result was obtained earlier by Wazwaz and Gorguis [16] by using the method of ADM.

5. Conclusion

The analytical solutions of the non-linear time-fractional fisher's equations with initial conditions by the use of iterative Laplace transform Method (ILTM) were derived. The outcomes of study reveal that the proposed approach performs extremely well in terms of efficiency and simplicity and it can be utilized to investigate more problems in the field of non-linear fractional differential equations. Our findings provide interesting unifications and extensions of many results, hither to scattered in the literature.

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