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ON K-CONTACT η-EINSTEIN MANIFOLDS

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Abstract: In this paper, we study conformal curvature tensor, projective curvature tensor and Q curvature tensor on K-contact η-Einstein manifold. Here we study ξ-projectively flat and ξQ flat manifold on K-contact manifold. Finally, we consider K-contact η-Einstein manifold satisfying the curvature conditions $R.C = Q(S,C)$ and $R.P = Q(S,P)$, where C and P are the conformal and projective curvature tensors respectively.

Key words: Conformal curvature tensor, Projective curvature tensor, Q curvature tensor and K-contact η-Einstein manifold.

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1. Introduction

Let (M^n, g) be a Riemannian manifold with contact form η, associated vector field ξ, (1,1) tensor field ϕ and associated Riemannian metric g. If ξ is a killing vector field then M^n is called a K-contact Riemannian manifold or simply a K-contact manifold [1], [8]. An almost contact manifold is Sasakian [1] if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{1}$$

where ∇ is the Levi-Civita connection. It is well known that K-contact manifold is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2}$$

for any vector field X, Y on (M^n, g), where $R$is the Riemannian curvature tensor of type (1,3) defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{3}$$

Every Sasakian manifold is a K-contact manifold, but the converse need not be always true, except in dimension three [4].

A K-contact manifold M^n is said to be η- Einstein if it's Ricci tensor S is of the form

$$S = ag + b\eta \otimes \eta, \tag{4}$$

where a and b are certain functions, η is called the associated 1-form and vector field ξ defined by

$$g(X, \xi) = \eta(X), \tag{5}$$

is called the generator and a, b are called associated scalars. It is known that in[2][11], a K-contact η- Einstein manifold of dimension n $(n > 3)$, a and b are constants. If $b = 0$, the manifold reduces to an Einstein manifold. Recently Sheikh, De, Binh [6], Yildiz and Murathan [12] studied K-contact η-Einstein manifold satisfying certain curvature conditions.

A Riemannian manifold (M^n, g) is called locally symmetric if it's curvature tensor R is parallel, i.e. $\nabla R = 0$. The notation of semi symmetric, a proper generalization of locally symmetric manifold, is defined by $R(X,Y).R = 0$, where $R(X,Y)$ acts on R as derivation. A complete intrinsic classification of these manifolds was given by Szabo in [7]. In [3], Chaki and Tarafdar studied a Sasakian manifold satisfying the condition $R(X,Y).C = 0$, where C denotes the Weyl conformal curvature tensor defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]$$
$$+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \tag{6}$$

where S is the Ricci tensor of type (0,2), Q is the Ricci operator defined by

$$S(X,Y) = g(QX,Y) \tag{7}$$

and r is the scalar curvature of M^n.

Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. M^n is said to be locally projective flat for $n \geq 1$ if and only if the projective curvature tensor P vanishes. The projective curvature tensor P is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y] \tag{8}$$

for all $X, Y, Z \in \chi(M)$, where R is the curvature tensor and S is the Ricci tensor.

In a recent paper Mantica and Suh [5] introduced a new curvature tensor of type (1,3) in an n-dimensional Riemannian manifold $(M^n, g), n > 2$, denoted by Q and defined by

$$Q(X,Y)Z = R(X,Y)Z - \frac{\Psi}{n-1}[g(Y,Z)X - g(X,Z)Y] \tag{9}$$

where Ψ is an arbitrary scalar function. Such a tensor Q is known as Q-curvature tensor. The notation of Q tensor is also suitable to interpret again some differential structure on a Riemannian manifold. If $\Psi = \frac{r}{n}$, then Q curvature tensor reduces to concircular curvature tensor.

For a $(0,k)$ tensor field T, $k \geq 1$, on (M^n, g) we define the tensor $R.T$ and $Q(S,T)$ by

$$\begin{aligned}(R(X,Y).T)(X_1, X_2, \dots\dots\dots X_k) = & -T(R(X,Y)X_1, X_2, \dots\dots\dots, X_k) \\ & - T(X_1, R(X,Y)X_2, \dots\dots\dots, X_K) \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & -T(X_1, X_2, \dots\dots\dots, R(X,Y)X_k) \end{aligned} \tag{10}$$

and

$$\begin{aligned} Q(S,T)(X_1, X_2, \dots\dots\dots X_k) = & -T\big((X \wedge_S Y)X_1, X_2, \dots\dots\dots, X_k\big) \\ & - T(X_1, (X \wedge_S Y)X_2, \dots\dots\dots, X_K) \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & -T(X_1, X_2, \dots\dots\dots, (X \wedge_S Y)X_k) \end{aligned} \tag{11}$$

respectively [10].

Further, we define the endomorphism $X \wedge_A Y$ by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y \tag{12}$$

where $X, Y, Z \in \chi(M)$, $\chi(M)$ is the Lie algebra of vector fields on M and A is a symmetric tensor of type (0,2).

A Riemannian manifold or semi–Riemannian manifold $(M^n, g), n > 3$, is said to be Ricci generalized pseudo symmetric [13] if and only if

$$R.R = fQ(S,R) \tag{13}$$

holds on the set $U_R = \{x \in M : R \neq 0 \; at \; x\}$, where f is some function on U_R. A very important subclass of this class of manifolds realizing the condition is

$$R.R = Q(S,R).$$

After the preliminaries, in section 3 we consider ξ-projectively flat in K-contact η-Einstein manifold and we have shown that the manifold becomes Sasakian manifold. In section 4, we prove that ξQ flat K-contact manifold is also a Sasakian manifold. Finally, in the section 5 and 6 we consider $R.C = Q(S,C)$ and $R.P = Q(S,P)$ on K-contact η-Einstein manifold respectively and shown that under these conditions manifold is Sasakian.

2. Preliminaries

In a K-contact manifold the following relations holds [1][8][9]:

$$(a)\ \phi\xi = 0 \qquad (b)\ \eta(\xi) = 1 \qquad (c)\ g(X,\xi) = \eta(X) \tag{14}$$

$$\phi^2 X = -X + \eta(X)\xi \tag{15}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{16}$$

$$\nabla_X \xi = -\phi X \tag{17}$$

$$g(R(\xi,X)Y,\xi) = \eta(R(\xi,X)Y) = g(X,Y) - \eta(X)\eta(Y) \tag{18}$$

$$R(\xi,X)\xi = -X + \eta(X)\xi \tag{19}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{20}$$

$$(\nabla_X \phi)Y = R(\xi,X)Y \tag{21}$$

In K-contact η-Einstein manifold, we have

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{22}$$

Putting $X = Y = \xi$ in (22) and then using (20) and $(14)(b)$, we get

$$a + b = (n-1) \tag{23}$$

Also (22) implies that

$$r = an + b \tag{24}$$

From (23) and (24), we have

$$a = \frac{r}{n-1} - 1, \quad b = n - \frac{r}{n-1} \tag{25}$$

Again from (22), we obtain

$$QX = \left(\frac{r}{n-1} - 1\right)X + \left(n - \frac{r}{n-1}\right)\eta(X)\xi \tag{26}$$

where Q denotes the Ricci operator, i.e. $g(QX,Y) = S(X,Y)$.

Using (22) and (26) in (6) ,we get

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}\Big[g(Y,Z)\Big\{\Big(\frac{r}{n-1} - 1\Big)X + \Big(n - \frac{r}{n-1}\Big)\eta(X)\xi\Big\}$$

$$-g(X,Z)\Big\{\Big(\frac{r}{n-1} - 1\Big)Y + \Big(n - \frac{r}{n-1}\Big)\eta(Y)\xi\Big\}$$

$$+\{ag(Y,Z)+b\eta(Y)\eta(Z)\}X-\{ag(X,Z)+b\eta(X)\eta(Z)\}Y]$$
$$+\frac{r}{(n-1)(n-2)}[g(Y,Z)X-g(X,Z)Y]$$

Using (25) in above, we obtain

$$C(X,Y)Z=R(X,Y)Z+\left(\frac{2}{n-2}-\frac{r}{(n-1)(n-2)}\right)[g(Y,Z)X-g(X,Z)Y]$$
$$-\left(\frac{n}{n-2}-\frac{r}{(n-1)(n-2)}\right)[g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi$$
$$+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y] \tag{27}$$

Putting $Z=\xi$ in (27), we get

$$C(X,Y)\xi=R(X,Y)\xi-[\eta(Y)X-\eta(X)Y] \tag{28}$$

Again putting $X=\xi$ in (27), we obtain

$$C(\xi,Y)Z=R(\xi,Y)Z-[g(Y,Z)\xi-\eta(Z)Y] \tag{29}$$

Putting $Z=\xi$ in (29) and using (19), we obtain

$$C(\xi,Y)\xi=0 \tag{30}$$

for all vector fields X,Y and Z on M.

Further, putting $Z=\xi$ in (8) and using (20), we obtain

$$P(X,Y)\xi=R(X,Y)\xi-\eta(Y)X+\eta(X)Y \tag{31}$$

Putting $X=\xi$ in (8) and using (22), we get

$$P(\xi,Y)Z=R(\xi,Y)Z-\frac{1}{n-1}[ag(Y,Z)\xi+b\eta(Y)\eta(Z)\xi-(a+b)\eta(Z)Y] \tag{32}$$

Taking $Z=\xi$ in (32) and using (19), (23) we obtain

$$P(\xi,Y)\xi=0 \tag{33}$$

3. ξ - Projectively flat K-contact manifold

Definition 3.1. An almost contact manifold (M^n,g) is said to be ξ-projectively flat if it satisfies $P(X,Y)\xi=0$, where $X,Y\in\chi(M)$ and ξ is associated vector field.

A ξ–projectively flat manifold can be written as

$$R(X,Y)\xi=\frac{1}{n-1}[S(Y,\xi)X-S(X,\xi)Y] \tag{34}$$

Using (20) in (34), we get

$R(X,Y)\xi = \eta(Y)X - \eta(X)Y.$

Which implies that manifold is Sasakian.

This leads to the following result:

Theorem 3.1. A ξ-projectively flat K-contact manifold is a Sasakian manifold.

Again using (22) in (34), we get

$$R(X,Y)\xi = \frac{a+b}{n-1}\,[\eta(Y)X - \eta(X)Y]$$

Using (23) in above, we get

$R(X,Y)\xi = \eta(Y)X - \eta(X)Y.$

Which implies that manifold is Sasakian.

This leads to the following result:

Theorem 3.2. A ξ-projectively flat K-contact η- Einstein manifold is a Sasakian manifold.

4. ξQ flat K-contact manifold

Definition 4.1. An almost contact manifold (M^n, g) is said to be ξQ flat if $Q(X,Y)\xi = 0$, where $X, Y \in \chi(M)$ and ξ is associated vector field.

A ξQ flat manifold can be written as

$$R(X,Y)\xi = \frac{\Psi}{n-1}[\eta(Y)X - \eta(X)Y].$$

This leads to the following result:

Theorem 4.1. A ξQ flat manifold K-contact manifold is a Sasakian manifold if and only if $\Psi = (n-1)$.

For $\Psi = \frac{r}{n}$, Q curvature tensor reduces to concircular curvature tensor. Then in view of above theorem we obtain

Corollary 4.1. A ξ-concircularly flat K-contact manifold is a Sasakian manifold if and only if $r = n(n-1)$.

5. K-contact η-Einstein manifold satisfying $R(\xi, Y).C = Q(S, C)$

This section is devoted to study K-contact η-Einstein manifold satisfying the curvature condition

$R(\xi,Y).C=Q(S,C)$

i.e. $(R(\xi,Y).C)(U,V)W=Q(S,C)(U,V)W$,

for all Y,U,V and $W\in\chi(M)$.

The above equation implies

$$R(\xi,Y)C(U,V)W-C(R(\xi,Y)U,V)W-C(U,R(\xi,Y)V)W-C(U,V)R(\xi,Y)W$$
$$=(\xi\wedge_S Y)C(U,V)W-C\big((\xi\wedge_S Y)U,V\big)W-C(U,(\xi\wedge_S Y)V)W$$
$$-C(U,V)(\xi\wedge_S Y)W$$

Using (12) and (22) in above, we obtain

$$\begin{aligned}R(\xi,Y)C(U,V)W-C(R(\xi,Y)U,V)W-C(U,R(\xi,Y)V)W-C(U,V)R(\xi,Y)W\\ =[ag(Y,C(U,V)W)+b\eta(Y)\eta(C(U,V)W)]\xi\\ -[ag(\xi,C(U,V)W)+b\eta(\xi)\eta(C(U,V)W)]Y\\ -[ag(Y,U)+b\eta(Y)\eta(U)]C(\xi,V)W\\ +[ag(\xi,U)+b\eta(\xi)\eta(U)]C(Y,V)W\\ -[ag(Y,V)+b\eta(Y)\eta(V)]C(U,\xi)W\\ +[ag(\xi,V)+b\eta(\xi)\eta(V)]C(U,Y)W\\ -[ag(Y,W)+b\eta(Y)\eta(W)]C(U,V)\xi\\ +[ag(\xi,W)+b\eta(\xi)\eta(W)]C(U,V)Y\end{aligned} \tag{35}$$

Substituting $U=W=\xi$ in (35) and using (30), we obtain

$$-C(R(\xi,Y)\xi,V)\xi-C(\xi,V)R(\xi,Y)\xi=(a+b)[C(Y,V)\xi+C(\xi,V)Y]$$

Using (19) and (23), we obtain

$$(n-2)[C(Y,V)\xi+C(\xi,V)Y]=0$$

Hence for $n>3$, it follows that

$$[C(Y,V)\xi+C(\xi,V)Y]=0 \tag{36}$$

Using (28) and (29) in (36), we obtain

$$R(Y,V)\xi+R(\xi,V)Y-g(V,Y)\xi+2\eta(Y)V-\eta(V)Y=0 \tag{37}$$

Interchanging Y and V in (37), we obtain

$$R(V,Y)\xi+R(\xi,Y)V-g(Y,V)\xi+2\eta(V)Y-\eta(Y)V=0 \tag{38}$$

Subtracting (38) from (37), we get

$R(Y,V)\xi + R(\xi,V)Y - R(V,Y)\xi - R(\xi,Y)V + 3[\eta(Y)V - \eta(V)Y] = 0$

Using the Bianchi's first identity, we have

$R(Y,V)\xi = \eta(V)Y - \eta(Y)V.$

Which implies that manifold is Sasakian.

This leads to the following result:

Theorem 5.1. Let (M^n, g) be an $n(n > 3)$ dimensional K-contact η-Einstein manifold satisfying the curvature condition $R(\xi,Y).C = Q(S,C)$ then the manifold is a Sasakian manifold.

6. K-contact η-Einstein manifold satisfying $R(\xi,Y).P = Q(S,P)$

This section is again devoted to study K-contact η-Einstein manifold satisfying the curvature condition

$R(\xi,Y).P = Q(S,P)$

i.e. $(R(\xi,Y).P)(U,V)W = Q(S,P)(U,V)W,$

for all Y, U, V and $W \in \chi$(M).

The above equation implies

$$R(\xi,Y)P(U,V)W - P(R(\xi,Y)U,V)W - P(U,R(\xi,Y)V)W - P(U,V)R(\xi,Y)W$$
$$= (\xi \wedge_S Y)P(U,V)W - P\big((\xi \wedge_S Y)U,V\big)W - P(U,(\xi \wedge_S Y)V)W - P(U,V)(\xi \wedge_S Y)W$$

Using (12) and (22) in above, we obtain

$$\begin{aligned}
R(\xi,Y)P(U,V)W - P(R(\xi,Y)U,V)W - P(U,R(\xi,Y)V)W - P(U,V)R(\xi,Y)W \\
= [ag(Y,P(U,V)W) + b\eta(Y)\eta(P(U,V)W)]\xi \\
- [ag(\xi,P(U,V)W) + b\eta(\xi)\eta(P(U,V)W)]Y \\
- [ag(Y,U) + b\eta(Y)\eta(U)]P(\xi,V)W \\
+ [ag(\xi,U) + b\eta(\xi)\eta(U)]P(Y,V)W \\
- [ag(Y,V) + b\eta(Y)\eta(V)]P(U,\xi)W \\
+ [ag(\xi,V) + b\eta(\xi)\eta(V)]P(U,Y)W \\
- [ag(Y,W) + b\eta(Y)\eta(W)]P(U,V)\xi \\
+ [ag(\xi,W) + b\eta(\xi)\eta(W)]P(U,V)Y
\end{aligned} \tag{39}$$

Substituting $U = W = \xi$ in (39) and using (33), we obtain

$$-P(R(\xi,Y)\xi,V)\xi - P(\xi,V)R(\xi,Y)\xi = (a+b)[P(Y,V)\xi + P(\xi,V)Y]$$

Using (19) and (23), we obtain

$$(n-2)[P(Y,V)\xi + P(\xi,V)Y] = 0$$

Hence for $n > 3$, it follows that

$$[P(Y,V)\xi + P(\xi,V)Y] = 0 \tag{40}$$

Using (31) and (32) in (40), we obtain

$$R(Y,V)\xi + R(\xi,V)Y - \eta(V)Y + \eta(Y)V - \frac{1}{n-1}[ag(V,Y)\xi$$
$$+b\eta(V)\eta(Y)\xi - (a+b)\eta(Y)V] = 0 \tag{41}$$

Interchanging Y and V in (41), we obtain

$$R(V,Y)\xi + R(\xi,Y)V - \eta(Y)V + \eta(V)Y - \frac{1}{n-1}[ag(Y,V)\xi$$
$$+b\eta(Y)\eta(V)\xi - (a+b)\eta(V)Y] = 0 \tag{42}$$

Subtracting (42) from (41) and using (23), we get

$$R(Y,V)\xi + R(\xi,V)Y - R(V,Y)\xi - R(\xi,Y)V + 3[\eta(Y)V - \eta(V)Y] = 0$$

Using the Bianchi's first identity, we have

$$R(Y,V)\xi = \eta(V)Y - \eta(Y)V.$$

Which implies that manifold is Sasakian.

This leads to the following result:

Theorem 6.1. Let (M^n, g) be an $n(n > 3)$ dimensional K-contact η-Einstein manifold satisfying the curvature condition $R(\xi, Y).P = Q(S, P)$ then the manifold is a Sasakian manifold.

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References

[1] Balair, D.E. (1976). Contact manifolds in Riemannian Geometry, Lecture notes in Math. 509, Springer – Verlag, Berlin.

[2] Balair, D.E., Koufogurgos, T. and Sharma, R. (1990). A classification of 3-dimensional contact metric manifolds with $Q\phi = \phi Q$, Kodai Math. J. **13**, 391- 401.

[3] Chaki, M.C. and Tarafdar, M. (1990). On a type of Sasakian Manifolds, Soochow J. Math. **16**, 23-28.

[4] Jun, J-B, Kim, U.K. (1994). On 3-dimensional almost contact metric manifolds, Kyungpook Math. J. **34**, 293-301.

[5] Mantica, C.A. and Suh, Y.J. (2013). Pseudo Q Symmetric Riemannian manifolds, Int. J. Geom. Methods Mod. Phys. **10(5)**, 25 pages.

[6] Shaikh, A.A., De, U.C. and Binh, T.Q. (2000). On K-contact η-Einstein manifolds, Steps in Differential Geometry, Debrecen, 311-315.

[7] Szabo, Z.I. (1982). Structure theorem on Riemannian spaces Satisfying $R(X,Y).R = 0$, I the local version. J. Diff. Geom. **17**, 531-582.

[8] Sasaki, S. (1965). Lecture note on almost contact manifold, Part-I, Tohoku University.

[9] Sasaki, S. (1967). Lecture note on almost contact manifold, Part-II, Tohoku University.

[10] Verstraelen, L. (1994). Comments on pseudo-symmetry in sence of R. Deszcz, in: Geometry and Topology of submanifolds, World Sci. Publishing, **6**, 199-209.

[11] Yano, K. and Kon, M. (1984). Structure on Manifolds, World Scientific, 285.

[12] Yildiz, A. and Murathan, C. (2018). On K-contact η-Einstein Manifolds, to appear in Radovi Mathematiski.

[13] Yildiz, A. and Murathan, C. (2008). Ricci generalized pseudo-parallel Kaehlerian submanifolds in complex space forms, Bull. Malays. Math. Sci. Soc. **31**, 153-163.