Journal of Rajasthan Academy of Physical Sciences
ISSN : 0972-6306; URL : http://raops.org.in
Vol.17, No.3&4, July-December, 2018, 173-180

HOMOTHETIC β-CHANGE OF FINSLER METRIC

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Abstract: We have considered the conformal β-change of the Finsler metric given by L^*(x,y) $= e^{\sigma(x)}$ f(L(x y), β (x, y)), where σ (x) is a function of x, β (x, y) = $b_i(x)y^i$is a 1-form on the underlying manifold M^n, and f(L(x y), β (x, y)) is a homogeneous function of degree one in L and β. Let $F^n$and F^{*n} denote Finsler spaces with metric functions L and L^*respectively. When $F^n$is transformed to $F^{*n}$by Homothetic β-change, we obtained conditions for F^{*n} to be a Landberg space, a Berwald space or a locally Minkowskian space when F^n is a space of the same kind under the assumption that $b_i(x)$ is Cartan-parallel.

Keywords: Finslerspace, Conformal β-change, Homothetic β-change.

AMS (2010) Mathematics Subject Classification: 52B40, 53C60.

1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space on the differentiable manifold M^n, equipped with the fundamental function L(x,y). Prasad and BinduKumari [10] and Shibata [11] have studied the general case of β- change, i.e.,

$L^*(x, y)$= f(L, β),

where f is positively homogeneous function of degree one in L and β, and β given by β(x,y) = b_i(x)$y^i$is a one-form on M^n . The β-change of special Finsler spaces has been studied by Shukla, et al. [15].

The conformal theory of Finsler space was initiated by Knebelman [7] in 1929 and has been investigated in detail by many authors (Hashiguchi[3], Izumi[4,5], Kitayama[6], Abed [1,2]). The conformal change is defined as L^*(x,y)→ $e^{\sigma(x)}$L(x,y) , where σ (x) is a function of position only and known as conformal factor. In 2009 and 2010, Youssef, et al. [16,17] introduced the transformation $\mathring{L}$(x; y) = f ($e^{\sigma(x)}L, \beta$)) , which is general β-change of conformally changed Finsler metric L.

We have changed the order of combination of theabove two changes in our paper [12], i.e.,

L^*(x,y) =e^{σ} f(L, β) , (1)

where $\sigma(x)$ is a function of x,β(x,y) = b_i(x)y^i is a 1-form on M^n. We have called this change as conformal β-change of Finslermetric. In this paper we have investigate the condition under which a conformal β-change of Finsler metric leads a Douglas space into a Douglas space. We have also found the necessary and sufficient conditions for this change to be a projective change.

We have studied quasi-C-reducibility, C-reducibility, semi-C-reducibility and calculated T-tensor of $F^{*n} = [M^{*n}, L^*]$in [13]. Further, we have studied S3-likeness, S4-likeness and Killing correspondence between the Finsler spaces $F^n$and F^{*n}in [14].

When $\sigma = 0$, the change (1) reduces to a β–change. When $\sigma = contant$, it becomes a homothetic β–change. When $f(L,\beta)$ has special forms as $L+\beta\,, \frac{L^2}{L-\beta}\,, \frac{L^2}{\beta}\,, \frac{L^{m+1}}{\beta^m}$ (m≠ $0,-1$), we get conformal Randers change, conformal Matsumoto change, conformal Kropina change, conformal generalized Kropina change of Finsler metric[8,9] respectively. The Finsler space equipped with the metric L^*given by (1) will be denoted byF^{*n}. Throughout the paper the quantities corresponding to F^{*n} has been denoted by putting star on the top of them.

Homogeneity of f gives

$Lf_1 + \beta f_2 = f$

where subscripts "1" and "2" denote the partial derivatives with respect to$L\ and\ \beta$ respectively.

Differentiating above equations with respect to $L\ and\ \beta$ respectively, we get

$Lf_{12} + \beta f_{22} = 0$ and $Lf_{11} + \beta f_{21} = 0$

Hence, we have $\frac{f_{11}}{\beta^2} = \frac{-f_{12}}{L\beta} = \frac{f_{22}}{L^2}$,

which gives

$f_{11} = \beta^2\omega,\ f_{12} = -L\beta\omega,\ f_{22} = L^2\omega,$

where Weierstrass function ω is positively homogeneous of degree-3 in $L\ and\ \beta$. Therefore

$L\omega_1 + \beta\omega_2 + 3\omega = 0$

whereω_1 and ω_2 are positively homogeneous of degree – 4 in $L\ and\ \beta$.

Throughout the paper we frequently use the above equations without quoting them. Also we have assumed that f is not linear function of $L\ and\ \beta$ so that $\omega \neq 0$.

2. Fundamental quantities of F^n

Differentiating equation (1) with respect to y^iwe have

$$l_i^* = e^{\sigma}(f_1 l_i + f_2 b_i) \tag{2}$$

Differentiating equation (2) with respect to y^j we have

$$h_{ij}^* = e^{2\sigma}\left(ph_{ij} + q_0 m_i m_j \right) \tag{3}$$

where$p = \frac{ff_1}{L}$, $m_i = b_i - \frac{\beta}{L} L_i$, $q_0 = fL^2\omega$.

From (2) and (3) we get the following relation between metric tensors of F^n $andF^{*n}$:

$$g_{ij}^* = e^{2\sigma}\left(pg_{ij} + p_2 l_i l_j + p_1\left(l_i b_j + l_j b_i\right) + p_0 b_i b_j\right), \tag{4}$$

Where $p_0 = q_0 + f_2^2$, $p_1 = ff_1 - fL\beta\omega$, $p_2 = \frac{\beta(ff_1 - fL\beta\omega)}{L}$. (5)

$$l^i = g^{ij} l_j \ , b^i = g^{ij} b_j \ , b^2 = g^{ij} b_i b_j \ , \ \Delta = \frac{\beta^2}{L^2} - b^2 ,$$

Under the conformal change (1) we get the following relation between Cartan's C-tensors of F^n and F^{*n}:

$$C_{ijk}^* = e^{2\sigma}[\mathrm{p}C_{ijk} + \frac{p}{2L}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \frac{qL^2}{2} m_i m_j m_k] \tag{6}$$

$$C_{jk}^{*i} = C_{jk}^i + \frac{p}{2ff_1}\left(h_{jk} m^i + h_j^i m_k + h_k^i m_j\right) - \frac{pL\Delta}{2f^2 f_1 t} h_{jk} n^i + \frac{(2pL + qL^4\Delta)}{2f^2 f_1 t} m_j m_k m^i - \frac{L}{ft} C_{.\,jk} n^i - \frac{qL^3}{2ff_1} m_j m_k m^i , \tag{7}$$

where$C_{.jk} = C_{ijk} b^i$

The spray coefficient of is given by [14] :

$$G^{*i} = G^i + D^i , \tag{8}$$

Where the vector $D^i$is given by

$$D^i = \frac{f_2 L}{f_1} s_0^i - \frac{L}{ff_1 t}(f_1 r_{00} - 2Lf_2 s_{r0} b^r)\left(py^i - L^2 wfb^i\right) + \sigma_0 y^i - \frac{1}{2} f^2 \sigma^i), \tag{9}$$

$$2r_{ij} = b_{i|j} + b_{j|i} \quad , \ 2s_{ij} = b_{i|j} - b_{j|i} \tag{10}$$

The Cartan's non-linear connection of F^{*n} is given by [14]:

$$N_j^{*i} = N_j^i + D_j^i \tag{11}$$

Where the tensor $D_j^i = \dot{\partial}_j D^i$ is given by

$$D_j^i = \frac{Le^{2\sigma}}{f f_1} A_j^i - Q^i A_{rj} b^r + \frac{pLf_2}{f^2 f_1^2 t} b_{0j}(-Lf_1 b^i + (f\beta - \Delta L^2 f_2) y^i) + \sigma_j y^i - f\,\sigma^i (f_1 l_i + f_2 b_i) \tag{12}$$

in which

$A_{ij} = \frac{1}{2} r_{00} B_{ij} + e^{2\sigma} f f_2 s_{ij} + s_{i0} Q_j - (\frac{e^{2\sigma} f f_1}{L} C_{imj} + V_{ijm}) D^m$,

$$A_j^i = g^{ir} A_{rj} , V_{ijm} = g_{sj} V_{im}^s , Q_i = e^{2\sigma}(p y_i + f L^2 \omega y_i + f_2^2 b_i),$$

$B_{jk} = \frac{1}{2} e^{2\sigma}\left(p h_{jk} + q L^2 m_j m_k\right), \dot{\partial}_k Q_j = \frac{1}{2} B_{jk}$.

The Berwaldconnection coefficient of F^{*n} is given by [14]:

$$G_{jk}^{*i} = G_{jk}^i + B_{jk}^i \quad , \; B_{jk}^i = \dot{\partial}_k D_j^i \tag{13}$$

The Cartan's connection coefficient of F^{*n} is given by [14]:

$$F_{jk}^{*i} = F_{jk}^i + D_{jk}^i \ , \tag{14}$$

where we put

$$D_{jk}^i = \left[\frac{e^{-2\sigma} L}{f f_1} g^{is} - Q^i b^s + y^s \frac{e^{-2\sigma} p L}{f^3 f_1 t}\left(-L f b^i + (f\beta - \Delta L^2 f_2) y^i\right)\right]$$

$$(B_{sj} b_{0\,|k} + B_{sk} b_{0\,|j} - B_{kj} b_{0\,|s} + s_{sj} Q_k + s_{sk} Q_j + r_{kj} Q_s + \frac{e^{2\sigma} f f_1}{L} C_{jkr} D_s^r + V_{kjr} D_s^r - \frac{e^{2\sigma} f f_1}{L} C_{skm} D_j^m - V_{sjm} D_k^m - \frac{e^{2\sigma} f f_1}{L} C_{sjm} D_k^m - V_{skm} D_j^m) - e^{-2\sigma} \sigma^i g_{jk}^* \ . \tag{15}$$

The tensor D_{jk}^i called the difference tensor, has the following properties:

$$D_{j0}^i = B_{j0}^i = D_j^i , D_{00}^i = 2 D^i. \tag{16}$$

The (v)h-torsion tensor of F^n is defined as :

$$R_{jk}^i = \delta_k N_j^i - \delta_j N_k^i, \qquad \delta_k = \partial_k - N_k^r \dot{\partial}_r , \tag{17}$$

and the h-curvature tensor of F^n is defined as :

$$R_{hjk}^i = \vartheta_{(j,k)}[\delta_k F_{hj}^i - F_{hj}^m F_{mk}^i] + C_{hm}^i R_{jk}^m \ . \tag{18}$$

The (v)hv-torsion tensor of F^n is defined as :

$$P_{jk}^i = \dot{\partial}_k N_j^i - F_{jk}^i \ . \tag{19}$$

3. Homothetic β-Change with b_i(x) as cartan-parallel

We prove the following theorem :

Theorem 4.1. Under the conformal β-change (1), consider the following two assertions:

(1) The covariant vector b_i(x)is Cartan-parallel.

(2) The difference tensor D_{jk}^i vanishes identically.

Then we have:

(a) If (1) and (2) hold, then σ is constant.

(b) If σ is contant then (1) and (2) are equivalent.

Proof: (a) If $D^i_{jk} = 0$, thenby second equation of (16), $D^i = 0$. Moreover, $b_{j\,|k} = 0$ implies that $r_{ij} = s_{ij} = 0$. Consequently, (9) reduces to

$$\sigma_0 y^i - \frac{1}{2} f^2 \sigma^i = 0 \tag{20}$$

Transvecting equation (20) by y_i, we get $\sigma_0 = 0$. Then by second equation of (16) gives $\sigma_i = 0$. Hence σ is constant.

(b) Let σ is constant and $b_{j\,|k} = 0$. Then $D^i = 0$, by virtue of (9). Consequently $D^i_{jk} = 0$ by virtue of (15).

On the other hand, let σ is constant $D^i_{jk} = 0$. Then by second equation of (16), $D_i = 0$. Hence, (9) reduces to

$$\frac{f_2 L}{f_1} s^i_0 - \frac{L}{f f_1 t}(f_1 r_{00} - 2L f_2 s_{r0} b^r)(p y^i - L^2 w f b^i) = 0 \tag{21}$$

Transvecting (21) by y_i and noting that $(L^2(p - \omega\beta f))$ is not equal tozero, we get

$$f_1 r_{00} - 2L f_2 s_{r0} b^r = 0 \tag{22}$$

This, together with (21), implies that $s^i_0 = 0$. Consequently $r_{00} = 0$ by virtue of (22). Since $s_{ij} = \dot{\partial}_k\, s_{i0}$ and $r_{j0} = \dot{\partial}_j\, r_{00}$, therefore $s_{ij} = 0$ and $r_{j0} = 0$, which leads to $b_{i|0} = b_{0|j} = 0$. Consequently, (15) gives

$$r_{kj}\, Q_s \left[\frac{e^{-2\sigma} L}{f f_1} g^{is} - Q^i b^s + y^s \frac{e^{-2\sigma} p L}{f^3 f_1 t}\left(-L f b^i + (f\beta - \Delta L^2 f_2) y^i\right)\right] = 0 \tag{23}$$

Hence $r_{kj}\, Q_s = 0$ .Trasvecting this with y^s we gets $r_{kj} = 0$.Thus b_i(x) is Cartan-parallel.

Corollary (3.1): (a) Let the conformal β-change(1)be a conformal β-change ($\sigma = 0$), then D^i_{jk} vanishesidentically iff $b_i(x)$ is Cartan-parallel.

If F^n is a Landsberg space, a Berwald space or a locally Minkowskian space, we want to see whether F^{*n} is a space of the same kind under some conditions.

When σ is constant and $b_i(x)$ is Cartan-parallel, then by virtue of (10) and (15), we have

$$R^{*i}_{jk} = R^i_{jk} \tag{24}$$

Under the aforesaid conditions we have

$$R^{*i}_{hjk} = R^i_{hjk} + E^i_{hm} R^m_{jk}\ ,\ E^i_{jk} = C^{*i}_{jk} - C^i_{jk} \tag{25}$$

by virtue of (18), (14) and (24). Again, when σ is constant and $b_i(x)$ is Cartan-parallel, thenby virtue of (19), (11) and (14), we have

$$P^{*i}_{jk} = P^i_{jk}$$

A Finsler space F^n is called a Landesberg space if P^i_{jk} = 0. From (26) we have

Theorem (3.2). A Landesberg space remains Landesberg space under homothetic β-change and $b_i(x)$ is Cartan-parallel.

A Finsler space F^n is called a Berwald space if $\dot{\partial}_h G^i_{jk} = 0$. When σ is constant and $b_i(x)$ is Cartan-parallel, then by virtue of (11), we have

$$G^{*i}_{jk} = G^i_{jk}\,. \tag{26}$$

Thus we have the theorem:

Theorem (3.3). A Berwald space remains Berwald space underhomothetic β-change and $b_i(x)$ is Cartan-parallel.

A Finsler space F^n is a locally Minkowskian space if F^n is a Berwald space and the h-curvature tensor R^i_{hjk} vanishes.

We prove the following theorem:

Theorem (3.3). A locally Minkowskian space remains a locally Minkowskian under homothetic β-change if $b_i(x)$ is Cartan-parallel.

Proof: First we prove that R^{*i}_{hjk} vanishes if fR^i_{hjk} vanishes. Let R^i_{hjk} = 0. Then R^i_{jk} = 0 and hence by virtue of (25) we have R^{*i}_{jk}=0. Conversely, let R^{*i}_{hjk} = 0. Then

$R^i_{hjk} + E^i_{hm} R^m_{jk}$= 0. Transvecting by y^h gives R^i_{jk}= 0 and therefore R^i_{hjk}= 0. The proof follows from theorem (3.3) and the above fact.

Acknowledgement: The authors are thankful to the Referee for valuable comments and suggestions. The work contained in this research paper is part of Major Research Project 'Certain Investigations in Finsler Geometry' financed by the U.G.C., New Delhi.

References

[1] Abed, S.H. (2008). Conformal β-changes in Finslerspaces, Proc. Math. Phys. Soc. Egypt, **86**, 79-89.ArXiv No.:math.DG/0602404.

[2] Abed, S.H. (2008). Cartan connections associated with a β-conformal change in Finsler geometry, Tensor, N.S., **70**, 146-158.ArXiv No.:math.DG/0701491.

[3] Hashiguchi, M. (1976). On conformal transformation of Finslermetric, J.Math.Kyoto Univ., **16**, 25-50.

[4] Izumi, H. (1977). Conformal transformations of Finsler spaces I, Tensor, N.S., **31**, 33-41.

[5] Izumi, H. (1980). Conformal transformations of Finsler spaces II. An h-conformally at Finsler space, Tensor, N.S., **33**, 337-369.

[6] Kitayama, M. (2000). Geometry of transformations of Finsler met-rics, Hokkaido University of Education, Kushiro Campus, Japan.

[7] Knebelman, M.S. (1929). Conformal geometry of generalized metric spaces, Proc. Nat. Acad. Sci., USA, **15**, 376-379.

[8] Mastsumoto, M. (1986). Foundations of Finsler geometry and special Finsler spaces, Kaiseisha Press, Saikawa, Otsu, 520 Japan.

[9] Mastsumoto, M. (1985). The induced and intrinsic connections of a hypersurface and Finslerian projective geometry, J. Math. Kyoto Univ., **25**, 107-144.

[10] Prasad, B.N and Kumari, Bindu (2013). The β-change of Finsler metric and imbedding classes of their tangent spaces, Tensor, N.S., **74**, 48-59.

[11] Shibata, C. (1984). On invariant tensors of β-change of Finslermetric, J. Math. Kyoto Univ., **24**, 163-188.

[12] Shukla, H.S. and Mishra, Neelam (2017). Conformal β-change of Finslermetric, J. Int. Acad. Phys. Sci., **21(1)**, 19-30.

[13] Shukla, H.S. and Mishra, Neelam (2018). Some intrinsic properties of con-formal β-change, J. Math. Combin., **2**, 13-23.

[14] Shukla, H.S. and Mishra, Neelam. On conformal β-change, Communicated for publication.

[15] Shukla, H.S., Pandey, O.P. and Mandal, Khageshwar (2017). The β-change of special Finslerspaces, International J. Math. Combin., **1**, 78-87.

[16] Youssef, N.L., Abed, S.H. and Elgendi, S.G. (2010). Generalized β-conformal change of Finsler metric, Int. J. Geom. Meth. Mod. Phys., **7**, 4. ArXiv No.:math.DG/ 0906.5369.

[17] Youssef, N.L., Abed, S.H. and Elgendi, S.G. (2010). Generalized β-conformal change and special Finsler spaces, Int. J. Geom. Math. Mod. Phys., **8**, 6. ArXivNo.:math. DG/1004. 5478v3.