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SOME NEW EXPANSION FORMULAE INVOLVING A BASIC ANALOGUE OF GENERALIZED H-FUNCTION

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Abstract: The aim of this paper is to establish certain new expansion formulae involving a basic analogue of generalized H-function by using the application of fractional q-Leibnitz rule for q-integrals of a product of two functions. In this paper, some new expansion formulae for Meijer's G-function and Mac Robert's E-function are also derived which follow as special cases of our main results.

Keywords: Fractional q-integrals, Fractional q-Leibnitz rule, Fox's H-function

Subject Classification: 33D60, 26A33

1. Introduction

The subject of fractional calculus (that is calculus of integrals and derivatives of any arbitrary real or complex order) has gained noticeable importance and popularity during the past three decades or so, due mainly to its demonstrated applications in many seemingly diverse fields of science and engineering. Much of the theory of fractional calculus is based upon the familiar Riemann-Liouville fractional integral (or derivative).

In the theory of q-calculus [4], the q-shifted factorial is defined as:

$$(a;q)_0 = 1, \; (a;q)_n = \prod_{i=0}^{n-1}\left(1 - aq^i\right), \qquad (n \in N) \tag{1}$$

or equivalently

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad \left(n \in N \cup \{\infty\}\right) \tag{2}$$

The definition (1) of $(a\,;q)_n$ is also extend to for any complex number α as

$$(a\,;q)_\alpha = \frac{(a\,;q)_\infty}{(aq^\alpha\,;q)_\infty}, \quad |q| < 1 \tag{3}$$

Where the principle value of q^α is (usually) taken when $q \neq 0$.

The q-binomial coefficient is defined for positive integer n, k as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \tag{4}$$

Now on using the following results

$$(a;q)_{n-k} = (a;q)_n \left(a q^n ;q\right)_{-k} \tag{5}$$

and

$$(a;q)_{-n} = \frac{(a;q)_\infty}{\left(aq^{-n};q\right)_\infty} = \frac{(-q/a)^n}{(q/a;q)_n} q^{\binom{n}{2}} \tag{6}$$

in equation (4), The definition (4) can be generalized as

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{\left(q^{-\alpha};q\right)_k}{(q;q)_k} \left(-q^{\alpha}\right)^k q^{-k(k-1)/2} \tag{7}$$

where the q-gamma function is given by

$$\Gamma_q(a) = \frac{(q;q)_\infty}{\left(q^a;q\right)_\infty (1-q)^{a-1}}, \quad a \neq 0,-1,-2.... \tag{8}$$

The q-derivative of a function f (z) is defined as [4]

$$\left(D_q f\right)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad z \neq 0 \tag{9}$$

and $\left(D_q f\right)(0) = \lim_{z\to 0} \left(D_q f\right)(z)$ (10)

where $D_q \to \frac{d}{dz}$, as $q \to 1$

Agarwal [2] introduced a q-analogue of the Riemann-Liouville fractional integral operator as follows:

$$I_q^{\mu} f(z) = \frac{1}{\Gamma_q(\mu)} \int_0^z (z - tq)_{\mu-1} f(t) d_q t, \ \left(\mathrm{Re}(\mu) > 0; |q| < 1\right) \tag{11}$$

The q-integral of a function involved in right hand side of (11) is defined as

$$\int_0^x f(t) d_q t = x(1-q) \sum_{k=0}^{\infty} q^k f\left(xq^k\right) \tag{12}$$

In view of the equation (12), (11) can be expressed in its series equivalent form as

$$I_q^{\mu} f(z) = \frac{z^{\mu}}{\Gamma_q(\mu)}(1-q)\sum_{k=0}^{\infty} q^k \left(1-q^{k+1}\right)_{\mu-1} f\left(zq^k\right) \tag{13}$$

In 1976, Agarwal [3] introduced a new q-extension of the Leibnitz rule for fractional q-integrals of a product of two functions in terms of a series involving fractional q-integrals of the functions in the following manner:

$$I_q^{\mu}\{f(z)g(z)\} = \sum_{k=0}^{\infty}\begin{bmatrix}-\mu\\ k\end{bmatrix}_q D_q^k f(z)\, I_q^{\mu+k} g\left(zq^k\right) \tag{14}$$

where f (z) and g (z) are two regular functions.

Saxena and Kumar [6] introduced a q-analogue of generalized H-function in terms of the Mellin Barnes type contour integral in the following manner :

$$I_{A_i,B_i}^{m,n}\left[z;q\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m} G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n} G\left(q^{1-a_j+\alpha_j s}\right)\pi z^s}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{B_i} G\left(q^{1-b_{ji}+\beta_{ji}s}\right)\prod_{j=n+1}^{A_i} G\left(q^{a_{ji}-\alpha_{ji}s}\right)\right\} G\left(q^{1-s}\right)\sin \pi s} ds \tag{15}$$

where $0 \le m \le B_i,\ 0 \le n \le A_i$; i = 1,2,…,r ; r is finite and

$$G\left(q^{\alpha}\right) = \prod_{n=0}^{\infty}\left\{\left(1-q^{\alpha+n}\right)\right\}^{-1} = \frac{1}{\left(q^{\alpha};q\right)_{\infty}} \tag{16}$$

Also, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}$ are real and positive and a_j,b_j,a_{ji},b_{ji} are complex numbers. The contour C runs from $-i\infty$ to $+i\infty$ in such a manner that all the poles of $G\left(q^{b_j-\beta_j s}\right);1 \le j \le m$, are to the right, and those of $G\left(q^{1-a_j+\alpha_j s}\right);1 \le j \le n$, to the left of C. The integral converges if $\mathrm{Re}\left[s\log(z)-\log\sin\pi s\right] < 0$, for large values of $|s|$ on the contour C, that is, if $|\arg z| < \pi$. For r = 1, $A_1 = A; B_1 = B$, equation (15) reduces to the q-analogue of the H-function defined by Saxena et al. [7].

2. Main Results

Purohit et al. [5] derived certain expansion formulae involving a basic analogue of Fox's H-function by the application of the q-Leibnitz rule for the Weyl type q-derivatives of a product of two functions. In this connection, some new results associated with the basic analogue of generalized H-function by assigning suitable values to the functions f (z), g (z) and μ in the q-Leibnitz rule defined by (14) are established.

Result 1

$$I^{m,n+1}_{A_i+1,B_i+1}\left[\rho z^{\sigma};q\left|\begin{matrix}(1-\lambda,\sigma),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i},(1-\lambda-\mu,\sigma)\end{matrix}\right.\right]=$$

$$\sum_{k=0}^{\infty}\frac{(q^{\mu};q)_k\,(q^{1-\lambda};q)_k}{(q;q)_k}$$

$$\times q^{\lambda k}\, I^{m,n+1}_{A_i+1,B_i+1}\left[\rho\left(zq^{k}\right)^{\sigma};q\left|\begin{matrix}(0,\sigma),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i},(-\mu-k,\sigma)\end{matrix}\right.\right] \quad (17)$$

Provided that

$0\le m\le B_i$, $0\le n\le A_i$; i = 1,2,…,r ; r is finite, $\mathrm{Re}\left[s\log(z)-\log\sin\pi s\right]<0$, $\sigma\ge 0$ and ρ being any complex quantity.

Result 2

$$I^{m+1,n}_{A_i+1,B_i+1}\left[\rho z^{\sigma};q\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i},(\lambda+\mu,-\sigma)\\(\lambda,-\sigma),(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right.\right]=$$

$$\sum_{k=0}^{\infty}\frac{(q^{\mu};q)_k\,(q^{1-\lambda};q)_k}{(q;q)_k}$$

$$\times q^{\lambda k}\, I^{m+1,n}_{A_i+1,B_i+1}\left[\rho\left(zq^{k}\right)^{\sigma};q\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i},(1+\mu+k,-\sigma)\\(1,-\sigma),(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right.\right] \quad (18)$$

where $0 \le m \le B_i$, $0 \le n \le A_i$;i = 1,2,…,r ; r is finite, $\operatorname{Re}\left[s\log(z) - \log\sin\pi s\right] < 0$, $\sigma < 0$ and ρ being any complex quantity.

Proof of Result 1: In order to prove result (17), we take $f(z) = z^{\lambda-1}$ and

$$g(z) = I_{A_i,B_i}^{m,n}\left[\rho z^{\sigma}; q\middle|\begin{matrix}(a_j,\alpha_j)_{1,n}, & (a_{ji},\alpha_{ji})_{n+1,A_i}\\ (b_j,\beta_j)_{1,m}, & (b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right]$$

in the q-Leibnitz rule defined by (14). We get

$$I_q^{\mu}\left\{z^{\lambda-1} I_{A_i,B_i}^{m,n}\left[\rho z^{\sigma}; q\middle|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right]\right\} = \sum_{k=0}^{\infty}(-1)^k \frac{(q^{\mu};q)_k}{(q;q)_k} q^{-\mu k - k(k-1)/2} D_q^k\left\{z^{\lambda-1}\right\}$$

$$I_q^{\mu+k}\left\{I_{A_i,B_i}^{m,n}\left[\rho\left(zq^k\right)^{\sigma}; q\middle|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right]\right\} \tag{19}$$

On making use of the definition (15), the left-hand side of equation (19) becomes

$$I_q^{\mu}\left\{z^{\lambda-1} I_{A_i,B_i}^{m,n}\left[\rho z^{\sigma}; q\middle|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right]\right\}$$

$$= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m} G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n} G\left(q^{1-a_j+\alpha_j s}\right)\pi\rho^s}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{B_i} G\left(q^{1-b_{ji}+\beta_{ji}s}\right)\prod_{j=n+1}^{A_i} G\left(q^{a_{ji}-\alpha_{ji}s}\right)\right\} G\left(q^{1-s}\right)\sin\pi s} I_q^{\mu}\left(z^{\lambda+\sigma s-1}\right) ds \tag{20}$$

Now applying the well-known fractional q-integral formula given by Agarwal [2]:

$$I_{\mu}^{q}\left(z^{k-1}\right) = \frac{\Gamma_q(k)}{\Gamma_q(k+\mu)} z^{k+\mu-1} \tag{21}$$

in the above equation (20) and using equation (8), we arrive at the following transformation after a little simplification:

$$I_q^{\mu}\left\{z^{\lambda-1} I_{A_i,B_i}^{m,n}\left[\rho z^{\sigma};q\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right.\right]\right\}$$

$$=(1-q)^{\mu} z^{\lambda+\mu-1} I_{A_i+1,B_i+1}^{m,n+1}\left[\rho z^{\sigma};q\left|\begin{matrix}(1-\lambda,\sigma),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i},(1-\lambda-\mu,\sigma)\end{matrix}\right.\right] \quad (22)$$

where $\sigma \geq 0$.

Further, on using the fractional q-derivative formula [3] given by

$$D_q^k\left\{z^{\lambda-1}\right\} = \frac{\left(q^{1-\lambda};q\right)_k}{(1-q)^k}(-1)^k q^{\lambda k-k(k+1)/2} z^{\lambda-k-1} \quad (23)$$

and using the result (22) by setting $\lambda=1$, μ by $\mu+k$ and z by zq^k in right hand side of equation (19), one can easily arrive at the desired result.

Remark: The proof of Result 2 follows similar to proof of Result 1, when $\sigma<0$

3. Special Cases

If we take r = 1, $A_1=A, B_1=B, \alpha_j=\beta_i=1, j=1,...,A; i=1,...,B$ and $\sigma=1$, in the main result (17), we obtain the following expansion formula involving Meijer's $G_q(.)$ function [7], namely

$$G_{A+1,B+1}^{m,n+1}\left[\rho z;q\left|\begin{matrix}1-\lambda,a_1,...,a_A\\b_1,...,b_B,1-\lambda-\mu\end{matrix}\right.\right] = \sum_{k=0}^{\infty}\frac{\left(q^{\mu};q\right)_k\left(q^{1-\lambda};q\right)_k}{(q;q)_k} q^{\lambda k}$$

$$G_{A+1,B+1}^{m,n+1}\left[\rho\left(zq^k\right);q\left|\begin{matrix}0,a_1,...,a_A\\b_1,...,b_B,-\mu-k\end{matrix}\right.\right] \quad (24)$$

where $0\leq m\leq B$, $0\leq n\leq A$, $\mathrm{Re}\left[s\log(z)-\log\sin\pi s\right]<0$ and ρ being any complex quantity.

Similarly, for r = 1, $A_1=A, B_1=B, \alpha_j=\beta_i=1, j=1,...,A; i=1,...,B$ and $\sigma=-1$, the equation (18) reduces to yet another expansion formula for Meijer's $G_q(.)$ function given by

$$G_{A+1,B+1}^{m+1,n}\left[\rho/z;q\left|\begin{matrix}a_1,\ldots,a_A,\lambda+\mu\\ \lambda,b_1,\ldots,b_B\end{matrix}\right.\right] = \sum_{k=0}^{\infty}\frac{\left(q^{\mu};q\right)_k\left(q^{1-\lambda};q\right)_k}{\left(q;q\right)_k}q^{\lambda k}$$

$$\times G_{A+1,B+1}^{m+1,n}\left[\rho/\left(zq^{k}\right);q\left|\begin{matrix}a_1,\ldots,a_A,1+\mu+k\\ 1,b_1,\ldots,b_B\end{matrix}\right.\right] \tag{25}$$

where $0\le m\le B,\ 0\le n\le A$, $\mathrm{Re}\left[s\log(z)-\log\sin\pi s\right]<0$ and ρ being any complex quantity.

Further, on taking n =0 and m = B in the equation (24), it yields to an expansion formula involving MacRobert's $E_q(.)$ [1] namely

$$E_q\left[B+1;b_j,1-\lambda-\mu:A+1;a_j,1-\lambda:\rho z\right] = \sum_{k=0}^{\infty}\frac{\left(q^{\mu};q\right)_k\left(q^{1-\lambda};q\right)_k}{\left(q;q\right)_k}q^{\lambda k}$$

$$\times E_q\left[B+1;b_j,-\mu-k:A+1;a_j,0:\rho\left(zq^{k}\right)\right] \tag{26}$$

where $\mathrm{Re}\left[s\log(z)-\log\sin\pi s\right]<0$ and ρ being any complex quantity.

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