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ON η-EINSTEIN P-SASAKIAN MANIFOLD

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Abstract: In this paper, we have studied, if a p-Sasakian manifold is η-Einstein then the relation R(X,ξ)C = C(X,ξ)R = 0 holds. If R(X, Y)S = 0 holds on η-Einstein p-Sasakian manifold then it is an Einstein manifold, where R, C and S are the Curvature tensor, Weyl conformal curvature tensor and Ricci tensor respectively. Moreover we have also studied ξ-conformally flat contact metric manifold.

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1. Introduction

Let M^n be n-dimensional C^∞-manifold. If there exist a tensor field F of type (1, 1), a vector field ξ and a 1-form η in M^n satisfying.

$$\bar{\bar{X}} = X - \eta(X)\,\xi,\ \bar{X} = F(X),\ \eta(\xi) = 1 \tag{1}$$

Then M^n is called an almost para contact manifold[2].

Let g be the Riemannian metric satisfying

$$g(X,\xi) = \eta(X) \tag{2}$$

$$\eta(F\,X) = 0,\ F\xi = 0,\ \text{rank } F = (n-1). \tag{3}$$

$$g(FX,FY) = g(X,Y) - \eta(X)\eta(Y) \tag{4}$$

Then the set (F, ξ, η, g) satisfying (1), (2), (3) and (4) is called an almost para-contact Riemannian structure. The manifold with such structure is called an almost para-contact Riemannian manifold [6].

If we define $'F(X, Y) = g(\bar{X}, Y)$, then in addition to the above the following relations

$$'F(X,Y) = {}'F(Y,X), \tag{5}$$

$$'F(\bar{X}, \bar{Y}) = 'F(X, Y), \tag{6}$$

$$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0, \tag{7}$$

$$(\nabla_X \nabla_Y \eta)(Z) = -g(X, Z)\eta(Y) - g(X, Y)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z) \tag{8}$$

And $(\nabla_X \xi) = \bar{X}$ (9)

also holds, where 'F is 2-forms.

Let us consider an n-dimensional differentiable manifold M with connection ∇ where, ∇ is the covariant differentiation with respect to g.

Then it can be easily verified that the manifold in consideration becomes an almost para-contact Riemannian manifold. Such a manifold is called p-Saskian manifolds [1][4].

For a p-Saskian manifold the following relations hold:

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X. \tag{10}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \tag{11}$$

$$S(X, \xi) = -(n-1)\eta(X) \tag{12}$$

$$Q\xi = -(n-1)\xi \tag{13}$$

Where R and S are the curvature tensor and Ricci tensor respectively.

An almost Para contact Riemannian manifold M is said to be η-Einstein [5] if its Ricci tensor S is of the form

$$S = a\, g + b\eta \otimes \eta$$

Where a, b are smooth functions on M.

In this case we have

$$S(X, Y) = a\, g(X, Y) + b\eta(X)\eta(Y) \tag{14}$$

Putting $X = Y = \xi$ in (14), we get

$$a + b = -(n-1) \tag{15}$$

Also (12) implies that

$$r = n + b. \tag{16}$$

From (15) and (16), we have

$$b = -\left(n + \frac{r}{n-1}\right), \quad a = 1 + \frac{r}{n-1} \tag{17}$$

Again from (14), we have

$$QX = \left(1 + \frac{r}{n-1}\right)X - \left(n + \frac{r}{n-1}\right)\eta(X)\xi \tag{18}$$

Where Q denotes the Ricci operator, i.e. $S(X, Y) = g(QX, Y)$.

2. Curvature Tensors on η-Einstein p-Sasakian Manifold

Weyl conformal curvature tensor C is defined as

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}\{g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y\} + \frac{r}{(n-1)(n-2)}\{g(Y, Z)X - g(X, Z)Y\} \tag{19}$$

Using (14) and (18) in (19), we get

$$C(X, Y)Z = R(X, Y)Z - \frac{1+a}{n-2}\{g(Y, Z)X - g(X, Z)Y\}$$

$$- \frac{b}{n-2}\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}$$

$$C(X, Y)Z = R(X, Y)Z - \left(\frac{2}{n-2} + \frac{r}{(n-1)(n-2)}\right)\{g(Y, Z)X - g(X, Z)Y\}$$

$$+ \left(\frac{n}{n-2} + \frac{r}{(n-1)(n-2)}\right)\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + (\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y)\} \tag{20}$$

The endomorphism $X\wedge Y$ and $X\wedge_S Y$ and the homeomorphism $R(X,\xi)C$ and $C(X,\xi)R$ are defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y \tag{21}$$

$$(X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y \tag{22}$$

$$(R(X,\xi)C)(U, V)W = R(X,\xi)C(U, V)W - C(R(X,\xi)U, V)W - C(U, R(X,\xi)V)W - C(U, V)R(X,\xi)W \tag{23}$$

$$(C(X,\xi)R)(U, V)W = C(X, \xi)R(U, V)W - R(C(X, \xi)U, V)W - R(U, C(X, \xi)V)W - R(U, V)C(X, \xi)W \tag{24}$$

Respectively, where X, Y, Z are vector fields. Putting $U = W = \xi$ in (23) yields

$$(R(X, \xi)C)(\xi, V)\xi = R(X, \xi)C(\xi, V)\xi - C(R(X, \xi)\xi, V)\xi - C(\xi, R(X, \xi)V)\xi - C(\xi, V)R(X, \xi)\xi$$

From (20) we have

$$C(\xi, Y)\xi = R(\xi, Y)\xi - \left(\frac{2}{n-2} + \frac{r}{(n-1)(n-2)}\right)\{\eta(Y)\xi - Y\} + \left(\frac{n}{n-2} + \frac{r}{(n-1)(n-2)}\right)\{(\eta(Y) - \eta(Y))\xi + \eta(Y)\xi - Y\}$$

$$C(\xi, Y)\xi = 0 \text{ for all vector field } Y \tag{25}$$

And similarly

$$C(Y, \xi)\xi = 0 \tag{26}$$

Thus, we have

$(R(X,\xi)C)(\xi, V)\, \xi = -\, C(R(X,\xi)\xi, V)\xi - C(\xi, V)R(X,\xi)\, \xi$

Since

$C(R(X,\xi)\xi, V)\, \xi = C(\eta(X)\xi - X, V)\xi$

$= \eta(X)C(\xi, V)\xi - C(X, V)\xi = -\, C(X, V)\xi$

$C(\xi, V)R(X, \xi)\xi = C(\xi, V)(\eta(X)\xi - X) = -C(\xi, V)X$

Therefore,

$$(R(X,\xi)\, C)(\xi, V)\, \xi = C(X, V)\xi + C(\xi, V)X \quad (27)$$

On the other hand

$$(C(X, \xi)\, R)(\xi, V)\xi = C(X, \xi)R(\xi, V)\xi - R(C(X, \xi)\xi, V)\xi - R(\xi, C(X, \xi)V)\xi - R(\xi, V)C(X, \xi)\xi. \quad (28)$$

Using (26) and $R(X, \xi)\xi = \eta(X)\xi - X$, we have the following

$$C(X, \xi)R(\xi, V)\xi = C(X, \xi)(-\eta(V)\xi + V) = C(X, \xi)V. \quad (29)$$

$$R(C(X, \xi)\xi, V)\xi = 0 \quad (30)$$

$$R(\xi, C(X, \xi)V)\xi = C(X, \xi)V \quad (31)$$

$$R(\xi, V)C(X, \xi)\xi = 0 \quad (32)$$

Using in (28)

$$(C(X, \xi)R)\, (\xi, V)\xi = 0 \quad (33)$$

If $R(X, \xi)C = C\, (X, \xi)R = 0$, then

$C(X, V)\xi + C\, (\xi, V)X = 0.$

Which gives

$R(X, V)\xi + \eta(V)X - \eta(X)V + R(\xi, V)\, X + g(V, X)\xi - \eta(X)V = 0.$

This implies

$R(X, V)\xi = -\eta(V)X + \eta(X)V$

Therefore

$R(\xi, V)X = -g\, (V, X)\xi + \eta(X)V$

Which is true in p-Sasakian manifold.

Therefore, we have the following theorem:

Theorem 1. If a p-Sasakian manifold (M^n, g) is η-Einstein manifold then the relation $R(X, \xi)C = C(X, \xi)R = 0$ holds.

3. On η-Einstein p-Sasakian Manifold Satisfying R(X, Y) S = 0

Let R(X, Y)S = 0

S(R(X, Y)Z, U) + S (Z, R(X, Y)U) = 0

if Z = ξ, in the above, we get

S(R(X, Y)ξ, U) + S(ξ, R(X, Y)U) = 0, which yields

S (η(X)Y – η(Y)X, U) + S (ξ, R(X, Y)U) = 0

Using (12), we get

S (η(X)Y – η(Y)X, U) – (n–1)ηR(X, Y)U = 0

(n–1)η(R(X, Y)U) = η(X) S(Y, U) – η(Y)S(X, U).

$R(X, Y)U = \frac{1}{n-1}\{S(Y, U)X - S(X, U)Y\}$.

It follows that the manifold is projectively flat. But it is known that projectively flat n-dimensional (n >2) manifold is of constant curvature is an Einstein manifold.

Hence we have the following theorem:

Theorem 2. If in an η-Einstein p-Sasakian manifold[3], the relation R(X, Y)S = 0 holds, then the manifold is an Einstein manifolds.

4. ξ-Conformally flat η-Einstein p-Sasakian manifold

Let M^n be ξ-conformally flat contact metric manifold then it follows that [8]

C(X,Y) ξ = 0, where X,Y ϵ TM

We have from (19)

$$R(X,Y)\xi = \frac{1}{(n-2)}\{g(Y,\xi)QX - g(X,\xi)QY + S(Y,\xi)X - S(X,\xi)Y\} - \frac{r}{(n-1)(n-2)}\{g(Y,\xi)X - g(X,\xi)Y\}. \quad (34)$$

Using (2), (10) and (11) we get

$$\{\eta(X)Y - \eta(Y)X\} = \frac{1}{(n-2)}\{\eta(Y)QX - \eta(X)QY - (n-1)\eta(Y)X + (n-1)\eta(X)Y\} - \frac{r}{(n-1)(n-2)}\{\eta(Y)X - \eta(X)Y\} \quad (35)$$

Putting Y = ξ and using (13), we get

$$QX = \left(1 + \frac{r}{n-1}\right) X - \left(n + \frac{r}{n-1}\right) \eta(X)\xi = a X + b\, \eta(X)\xi, \quad (36)$$

Where $a = \left(1 + \frac{r}{n-1}\right)$, $b = -\left(n + \frac{r}{n-1}\right)$ and a+b= – (n-1).

Hence the manifold is η-Einstein p-Sasakian manifold.

Conversely let us suppose that the manifold M^n is η-Einstein p-Sasakian manifold.

From (19), we get

$$C(X,Y)\xi = R(X,Y)\xi - \frac{1}{(n-2)}\{g(Y,\xi)QX - g(X,\xi)QY + S(Y,\xi)X - S(X,\xi)Y\}$$
$$+ \frac{r}{(n-1)(n-2)}\{g(Y,\xi)X - g(X,\xi)Y\} \quad (37)$$

From (2) and (12), we obtain

$$C(X,Y)\xi = R(X,Y)\xi - \frac{1}{(n-2)}\{\eta(Y)QX - \eta(X)QY - (n-1)\eta(Y)X + (n-1)\eta(X)Y\}$$
$$+ \frac{r}{(n-1)(n-2)}\{\eta(Y)X - \eta(X)Y\} \quad (38)$$

With the help of (10), (15), (16) and (18) the relation (35) reduces to

$C(X,Y)\xi = R(X,Y)\xi - \{\eta(X)Y - \eta(Y)X\} = 0$.

Thus the manifold is ξ- conformally flat.

Hence we have the following theorem:

Theorem 3. A contact metric manifold M^n is ξ-conformally flat if and only if it is an η-Einstein p-Sasakian manifold.

5. Contravariant Almost Analytic Vector Field

Let the vector field ξ on a p-Sasakian manifold be contravariant almost analytic, then we have

$£_\xi F = 0$ implies $[\xi,\overline{X}] = [\overline{\xi, X}]$. Which gives

$(\nabla_\xi F)(X) = 0$.

Thus, we have the following theorem:

Theorem 4. On a p-Sasakian manifold if the vector field ξ is contravariant almost analytic. Then F is parallel alongξ, i.e. $(\nabla_\xi F)(X) = 0$.

We know that on a p-Sasakian manifold if the vector field ξ is contravariant almost analytic. Then we have the following:

(a) $(\nabla_\xi {}'R)(X, Y, Z, \xi) = 0$

(b) $(\nabla_\xi S)(Y, \xi) = 0$

(c) $\xi r = 0$.

From (17), we have $\xi a = \xi b = 0$

Hence, we can state the following theorem:

Theorem 5. On an η-Einstein p-Sasakian manifold if the vector field ξ is contravariant almost analytic, then a and b are constant function.

Since $S(Y, Z) = a\, g(Y, Z) + b\eta(Y)\eta(Z)$.

Differentiating covariantly w.r.t. ξ, we get

$(\nabla_{\xi}S)(Y, Z) = (\xi a)g(Y, Z) + (\xi b)\eta(Y)\eta(Z) = 0$.

Thus we have the following theorem:

Theorem 6. An η-Einstein p-Sasakian manifold with contravariant almost analytic vector field ξ, is Ricci flat.

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