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ON GENERALIZED FRACTIONAL DERIVATIVE INVOLVING PRODUCT OF TWO *H*-FUNCTIONS AND A GENERAL CLASS OF POLYNOMIALS

Rajeev Kumar Gupta, Bhupender Singh Shaktawat and Dinesh Kumar
Department of Mathematics & Statistics, Jai Narain Vyas University,
Jodhpur - 342005, India
E-mail: drrkbgupta@yahoo.co.in, b.s.shaktawatmath@gmail.com,
dinesh_dino03@yahoo.com

Abstract: The aim of this paper is to establish unified fractional derivative involving the product of two H-functions and the general class of polynomials. On account of the general nature of our main results a large number of new and known finite derivatives involving simpler special functions and polynomials, follow as its special cases. For the sake of illustration, we record here some special cases of our main derivatives which are also new and of interest by themselves.

Keywords: Generalized fractional calculus operators, H-function of several variables, general class of polynomials, special function.

Mathematics Subject Classification: 26A33, 33C45, 33C60, 33C70.

1. Introduction and Preliminaries

The Fox's H-function occurring in the present paper is defined and represented in the following manner [26, p.10, eq. (2.1.1)]:

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \theta(s)\,z^s\,ds \tag{1}$$

where $i=\sqrt{-1}$ and

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j-\beta_j s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\alpha_j s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\beta_j s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-\alpha_j s\right)} \tag{2}$$

The nature of the contour L of the integral (1), the conditions of existence of the H-function defined by (1) and other details can be found in the books Mathai et al. [11] and Kilbas and Saigo [7] (for reference see also, [14, 17, 24, 25]).

The H-function of several variables is defined and represented as follows (Srivastava et al. [26]) :

$$H\left[z_1,...,z_r\right]$$

$$= H^{0,n:m_1,n_1;...;m_r,n_r}_{p,q:p_1,q_1;...;p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\left| \begin{array}{l} \left(a_j;\alpha_j',...,\alpha_j^{(r)}\right)_{1,p} : \left(c_j',\gamma_j'\right)_{1,p_1};...;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j',...,\beta_j^{(r)}\right)_{1,q} : \left(d_j',\delta_j'\right)_{1,q_1};...;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \end{array}\right.\right]$$

$$= \frac{1}{(2\pi i)^r}\int_{L_1}...\int_{L_r}\phi(\xi_1,...,\xi_r)\theta_1(\xi_1)...\theta_r(\xi_r)z_1^{\xi_1}...z_r^{\xi_r}\,d\xi_1...d\xi_r, \tag{3}$$

where

$$\phi(\xi_1,...,\xi_r) = \frac{\prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i\right)\prod_{j=1}^{q}\Gamma\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\xi_i\right)}, \tag{4}$$

$$\theta_i(\xi_i) = \frac{\prod_{j=1}^{n_i}\Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}\xi_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma\left(c_j^{(i)}-\gamma_j^{(i)}\xi_i\right)\prod_{j=m_i+1}^{q_i}\Gamma\left(1-d_j^{(i)}+\delta_j^{(i)}\xi_i\right)}. \tag{5}$$

It is assumed that the various H-functions of several variables occurring in the paper always satisfy the appropriate existence and convergence conditions corresponding appropriately to those recorded in the book Srivastava et al. [26]. In case $r = 2$, (3) reduces to the H-function of two variables.

The general class of polynomials $S_n^m\left[x\right]$ is defined and represented as follows ([25, p.1, eq.1])

$$S_n^m\left[x\right] = \sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,x^k\ (n = 0,1,2,\cdots), \tag{6}$$

where m is an arbitrary positive integers and the coefficient $A_{n,k}\left(n,k\geq 0\right)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,k}$, the polynomial family $S_n^m\left[x\right]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel's polynomials and several others (see [29, 30]).

2. Generalized fractional derivative operators

The fractional calculus operators involving various special functions have been found significant importance and applications in various subfield of Mathematical analysis.

Saigo and Maeda [18] introduced the seven-parameter generalized left-sided and right-sided fractional derivative operators $D_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}$ and $D_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}$ with the Appell F_3 function as a kernel, respectively, as follows (see also, [3, 5, 10, 12, 14, 15, 16, 20, 21, 22, 23]):

For $\alpha,\alpha',\beta,\beta',\gamma\in\mathrm{C}$, and $x\in\mathrm{R}^+$,

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)\left(x\right)=\left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f\right)\left(x\right), \tag{7}$$

$$=\frac{d^n}{dx^n}\left(I_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n}f\right)\left(x\right)=\frac{1}{\Gamma\left(n-\gamma\right)}\frac{d^n}{dx^n}\left(x^{\alpha'}\right)\int_0^x\left(x-t\right)^{n-\gamma-1}t^{\alpha}$$

$$\times F_3\left(-\alpha',-\alpha,n-\beta',-\beta,n-\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)f\left(t\right)dt \qquad (\Re(\gamma)>0,\, n=[\Re(\gamma)]+1); \tag{8}$$

and

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)\left(x\right)=\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f\right)\left(x\right), \tag{9}$$

$$=\left(-1\right)^n\frac{d^n}{dx^n}\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+n,-\gamma+n}f\right)\left(x\right)=\frac{1}{\Gamma\left(n-\gamma\right)}\left(-1\right)^n\frac{d^n}{dx^n}\left(x^{\alpha}\right)\int_x^\infty\left(t-x\right)^{n-\gamma-1}t^{\alpha'}$$

$$\times F_3\left(-\alpha',-\alpha,-\beta',n-\beta,n-\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)f\left(t\right)dt \qquad (\Re(\gamma)>0, n=[\Re(\gamma)]+1), \tag{10}$$

where $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ and $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ are the Saigo-Maeda fractional integral operators. Here $F_3\left(\alpha,\alpha',\beta,\beta';\gamma;z,\xi\right)$ is the familiar Appell hypergeometric function of two variables defined by

$$F_3(\alpha,\alpha',\beta,\beta';\gamma;z,\xi)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{z^m}{m!}\frac{\xi^n}{n!}\quad(|z|<1 \text{ and } |\xi|<1), \quad (11)$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(\lambda)_n=\begin{cases}\lambda(\lambda+1)\ldots(\lambda+n-1) & (n\in \mathrm{N})\\ 1 & (n=0).\end{cases}$$

It is noted that the series in (11) is absolutely convergent for all $z,\xi\in \mathrm{C}$ with $|z|<1$ and $|\xi|<1$, and for all $z,\xi\in \mathrm{C}\setminus\{1\}$ with $|z|=1$ and $|\xi|=1$.

The Saigo-Maeda fractional derivative operators reduce to Saigo derivative operators [17, 18, 24] by virtue of the following identities:

$$\left(D_{0+}^{0,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma}f\right)(x)\quad(\Re(\gamma)>0) \quad (12)$$

and

$$\left(D_{-}^{0,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma}f\right)(x)\quad(\Re(\gamma)>0). \quad (13)$$

Our results in the next section are based on the following composition formula of Saigo-Maeda fractional integrals with a power function (see Saigo and Maeda [18]).

Lemma 1 *Let* $\alpha,\alpha',\beta,\beta',\gamma\in C$*; if* $Re(\gamma)>0$ *and*

$Re(\rho)>\max\left[0,Re(\alpha+\alpha'+\beta-\gamma),Re(\alpha'-\beta')\right]$, then

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}=x^{\rho-\alpha-\alpha'+\gamma-1}\frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)\Gamma(\rho+\beta')}. \quad (14)$$

Lemma 2 *Let* $\alpha,\alpha',\beta,\beta',\gamma\in C$*; if* $Re(\gamma)>0$ *and*

$Re(\rho)<1+\min\left[Re(-\beta),Re(\alpha+\alpha'-\gamma),Re(\alpha+\beta'-\gamma)\right]$, then

$$I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}=x^{\rho-\alpha-\alpha'+\gamma-1}\frac{\Gamma(1+\alpha+\alpha'-\gamma-\rho)\Gamma(1+\alpha+\beta'-\gamma-\rho)\Gamma(1-\beta-\rho)}{\Gamma(1-\rho)\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\Gamma(1+\alpha-\beta-\rho)}. \quad (15)$$

3. Main results

In this section, we study the left-sided $D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ and right-sided $D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ generalized fractional derivative defined by (7) and (9). We establish two theorems of fractional derivative involving products of two H-functions and a general class of polynomials.

Theorem 1 *Let* $\alpha,\alpha',\beta,\beta',\gamma,\mu,\eta,\delta,\upsilon_1,\upsilon_2,z_1,z_2,a,b\in \mathbb{C},\lambda,\sigma_1,\sigma_2>0$ *and* $Re(\gamma)>0$ *, then we have*

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}(b-at)^{-\eta}S_n^m\left[t^{\lambda}(b-at)^{-\delta}\right]\right.\right.$$

$$\left.\left.\times H_{p_1,q_1}^{m_1,n_1}\left[z_1t^{\sigma_1}(b-at)^{-\upsilon_1}\left|\begin{matrix}(a_j,A_j)_{1,p_1}\\(b_j,B_j)_{1,q_1}\end{matrix}\right.\right]H_{p_2,q_2}^{m_2,n_2}\left[z_2t^{\sigma_2}(b-at)^{-\upsilon_2}\left|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right.\right]\right)\right\}(x)$$

$$=b^{-\eta}x^{\mu+\alpha+\alpha'-\gamma-1}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}b^{-\delta k}x^{\lambda k}$$

$$\times H_{4,4:p_1,q_1;p_2,q_2;0,1}^{0,4:m_1,n_1;m_2,n_2;1,0}\left[\begin{matrix}z_1x^{\sigma_1}b^{-\upsilon_1}\\z_2x^{\sigma_2}b^{-\upsilon_2}\\-\frac{a}{b}x\end{matrix}\left|\begin{matrix}E_1:(a_j,A_j)_{1,p_1};(c_j,C_j)_{1,p_2};-\\E_2:(b_j,B_j)_{1,q_1};(d_j,D_j)_{1,q_2};(0,1)\end{matrix}\right.\right], \tag{16}$$

where $E_1=(1-\eta-\delta k;\upsilon_1,\upsilon_2,1),(1-\mu-\lambda k;\sigma_1,\sigma_2,1),$
$(1-\mu+\gamma-\alpha-\alpha'-\beta'-\lambda k;\sigma_1,\sigma_2,1)$,
$(1-\mu-\alpha+\beta-\lambda k;\sigma_1,\sigma_2,1)$;and
$E_2=(1-\eta-\delta k;\upsilon_1,\upsilon_2,0),(1-\mu+\gamma-\alpha-\alpha'-\lambda k;\sigma_1,\sigma_2,1),$
$(1-\mu+\gamma-\alpha-\beta'-\lambda k;\sigma_1,\sigma_2,1),(1-\mu+\beta-\lambda k;\sigma_1,\sigma_2,1).$

Also, satisfy the following conditions:

$$(i)\left|\arg z_1\right|<\frac{1}{2}\Omega_1\pi,\Omega_1>0,\ where\ \Omega_1=\sum_{j=1}^{m_1}B_j+\sum_{j=1}^{n_1}A_j-\sum_{j=m_1+1}^{q_1}B_j-\sum_{j=n_1+1}^{p_1}A_j. \tag{17}$$

$$(ii)\left|\arg z_2\right|<\frac{1}{2}\Omega_2\pi,\Omega_2>0,\ where\ \Omega_2=\sum_{j=1}^{m_2}D_j+\sum_{j=1}^{n_2}C_j-\sum_{j=m_2+1}^{q_2}D_j-\sum_{j=n_2+1}^{p_2}C_j. \tag{18}$$

(iii) $\left|\frac{a}{b}x\right|<1$, also we have

$$Re(\mu)+\sigma_1 \min_{1\le j\le m_1} Re\left(\frac{b_j}{B_j}\right)+\sigma_2 \min_{1\le j\le m_2} Re\left(\frac{d_j}{D_j}\right)+\max\left[0, Re(\alpha-\beta), Re(\alpha+\alpha'+\beta'-\gamma)\right]>0,$$

$$Re(\eta)+\upsilon_1 \min_{1\le j\le m_1} Re\left(\frac{b_j}{B_j}\right)+\upsilon_2 \min_{1\le j\le m_2} Re\left(\frac{d_j}{D_j}\right)+\max\left[0, Re(\alpha-\beta), Re(\alpha+\alpha'+\beta'-\gamma)\right]>0.$$

Proof To prove the fractional derivative formula (FDF) (16), we first express the general class of polynomials occurring on its left-hand side in the series from given by (6), replace the both H-functions occurring therein by its well-known Mellin-Barnes contour integral given by (1), interchange the order of summations, (ξ_1,ξ_2)- integrals and taking generalized fractional derivative operator inside (which is permissible under the conditions stated with (16)) and make a little simplification.

Next, we express the following binomial expansion for $(b-ax)^{-\gamma}$ as

$$(b-ax)^{-\gamma}=b^{-\gamma}\sum_{s=0}^{\infty}\frac{(\gamma)_s}{s!}\left(\frac{ax}{b}\right)^s,\quad \left|\frac{ax}{b}\right|<1, \tag{19}$$

obtained in terms of Mellin-Barnes contour integral (see Srivastava *et al.* [26]).

Now, interchanging the order of integrals, we obtain the following form after a little simplification (say D_1):

$$\mathrm{D}_1=b^{-\eta}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}b^{-\delta k}\left(\frac{1}{2\pi i}\right)^3\int_{L_1}\theta(\xi_1)z_1^{\xi_1}(b)^{-\upsilon_1\xi_1}d\xi_1\int_{L_2}\theta(\xi_2)z_2^{\xi_2}(b)^{-\upsilon_2\xi_2}d\xi_2$$

$$\times\int_{L_3}\frac{\Gamma(\eta+\delta k+\upsilon_1\xi_1+\upsilon_2\xi_2+\xi_3)}{\Gamma(\eta+\delta k+\upsilon_1\xi_1+\upsilon_2\xi_2)\Gamma(1+\xi_3)}\left(-\frac{a}{b}\right)^{\xi_3}d\xi_3\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\mu+\lambda k+\sigma_1\xi_1+\sigma_2\xi_2+\xi_3-1}\right)(x). \tag{20}$$

$$\mathrm{D}_1=b^{-\eta}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}b^{-\delta k}\left(\frac{1}{2\pi i}\right)^3\int_{L_1}\theta(\xi_1)z_1^{\xi_1}(b)^{-\upsilon_1\xi_1}d\xi_1\int_{L_2}\theta(\xi_2)z_2^{\xi_2}(b)^{-\upsilon_2\xi_2}d\xi_2$$

$$\times\int_{L_3}\frac{\Gamma(\eta+\delta k+\upsilon_1\xi_1+\upsilon_2\xi_2+\xi_3)}{\Gamma(\eta+\delta k+\upsilon_1\xi_1+\upsilon_2\xi_2)\Gamma(1+\xi_3)}\left(-\frac{a}{b}\right)^{\xi_3}d\xi_3$$

$$\times\frac{\Gamma(\mu+\lambda k+\sigma_1\xi_1+\sigma_2\xi_2+\xi_3)\Gamma(\mu+\lambda k+\sigma_1\xi_1+\sigma_2\xi_2+\xi_3-\gamma+\alpha+\alpha'+\beta')}{\Gamma(\mu+\lambda k+\sigma_1\xi_1+\sigma_2\xi_2+\xi_3-\gamma+r+\alpha+\alpha')\Gamma(\mu+\lambda k+\sigma_1\xi_1+\sigma_2\xi_2+\xi_3-\gamma+\alpha+\beta')}$$

$$\times\frac{\Gamma(\mu+\lambda k+\sigma_1\xi_1+\sigma_2\xi_2+\xi_3+\alpha-\beta)}{\Gamma(\mu+\lambda k+\sigma_1\xi_1+\sigma_2\xi_2+\xi_3-\beta)}\frac{d^r}{dx^r}x^{\mu+\lambda k+\sigma_1\xi_1+\sigma_2\xi_2+\xi_3+\alpha+\alpha'-\gamma+r-1}, \tag{21}$$

where $r = \left[\Re(\gamma)\right]+1$. By using $\frac{d^n}{dx^n} x^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$ $(m \ge n)$, and re-interpreting the Mellin-Barnes counter integral in terms of the H-function of three variables defined by (3), we obtain the right-hand side of (16) after little simplifications. This completes proof of Theorem 1.

In view of the relation (12), then we get the following corollary concerning left-sided Saigo fractional derivative operator [16, 17].

Corollary 1.1 *Let* $\alpha,\beta,\gamma,\mu,\eta,\delta,\upsilon_1,\upsilon_2,z_1,z_2,a,b \in \mathrm{C}$; $\lambda,\sigma_1,\sigma_2 > 0$ *and* $Re(\alpha) > 0$. *Then the following result holds true:*

$$\Bigg\{D_{0+}^{\alpha,\beta,\gamma}\Bigg(t^{\mu-1}(b-at)^{-\eta}\, S_n^m\left[t^{\lambda}(b-at)^{-\delta}\right]$$

$$\times H_{p_1,q_1}^{m_1,n_1}\left[z_1 t^{\sigma_1}(b-at)^{-\upsilon_1}\left|\begin{matrix}(a_j,A_j)_{1,p_1}\\(b_j,B_j)_{1,q_1}\end{matrix}\right.\right] H_{p_2,q_2}^{m_2,n_2}\left[z_2 t^{\sigma_2}(b-at)^{-\upsilon_2}\left|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right.\right]\Bigg)\Bigg\}(x)$$

$$= b^{-\eta} x^{\mu+\beta-1}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}\, A_{n,k}\, b^{-\delta k} x^{\lambda k}$$

$$\times H_{3,3:p_1,q_1;p_2,q_2;0,1}^{0,3:m_1,n_1;m_2,n_2;1,0}\left[\begin{matrix} z_1 x^{\sigma_1} b^{-\upsilon_1}\\ z_2 x^{\sigma_2} b^{-\upsilon_2}\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix} E_1' : (a_j,A_j)_{1,p_1};(c_j,C_j)_{1,p_2};-\\ E_2' : (b_j,B_j)_{1,q_1};(d_j,D_j)_{1,q_2};(0,1)\end{matrix}\right.\right], \tag{22}$$

where

$E_1' = (1-\eta-\delta k;\upsilon_1,\upsilon_2,1),(1-\mu-\lambda k;\sigma_1,\sigma_2,1),(1-\mu-\alpha-\beta-\gamma-\lambda k;\sigma_1,\sigma_2,1)$;

and $E_2' = (1-\eta-\delta k;\upsilon_1,\upsilon_2,0),(1-\mu-\beta-\lambda k;\sigma_1,\sigma_2,1),(1-\mu-\gamma-\lambda k;\sigma_1,\sigma_2,1)$.

Also, satisfy the following conditions:

$$Re(\mu)+\sigma_1 \min_{1\le j\le m_1} Re\left(\frac{b_j}{B_j}\right)+\sigma_2 \min_{1\le j\le m_2} Re\left(\frac{d_j}{D_j}\right)+\max\left[0,Re(\beta),Re(\alpha+\beta+\gamma)\right] > 0,$$

$$Re(\eta)+\upsilon_1 \min_{1\le j\le m_1} Re\left(\frac{b_j}{B_j}\right)+\upsilon_2 \min_{1\le j\le m_2} Re\left(\frac{d_j}{D_j}\right)+\max\left[0,Re(\beta),Re(\alpha+\beta+\gamma)\right] > 0.$$

Next, if we set $\beta=-\alpha$ in (22), we obtain the following result concerning left-sided Riemann-Liouville fractional derivative operator [12, 16]:

Corollary 1.2 *Let* $\alpha,\mu,\eta,\delta,\upsilon_1,\upsilon_2,z_1,z_2,a,b\in\mathrm{C};\ \lambda,\sigma_1,\sigma_2>0$ *and* $Re(\alpha)>0$, *then*

$$\Big\{D_{0+}^{\alpha}\Big(t^{\mu-1}(b-at)^{-\eta}\,S_n^m\Big[t^{\lambda}(b-at)^{-\delta}\Big]$$

$$\times H_{p_1,q_1}^{m_1,n_1}\left[z_1t^{\sigma_1}(b-at)^{-\upsilon_1}\left|\begin{matrix}(a_j,A_j)_{1,p_1}\\(b_j,B_j)_{1,q_1}\end{matrix}\right.\right]H_{p_2,q_2}^{m_2,n_2}\left[z_2t^{\sigma_2}(b-at)^{-\upsilon_2}\left|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right.\right]\Big)\Big\}(x)$$

$$=b^{-\eta}x^{\mu-\alpha-1}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,b^{-\delta k}x^{\lambda k}$$

$$\times H_{2,2:p_1,q_1;p_2,q_2;0,1}^{0,2:m_1,n_1;m_2,n_2;1,0}\left[\begin{matrix}z_1x^{\sigma_1}b^{-\upsilon_1}\\ z_2x^{\sigma_2}b^{-\upsilon_2}\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix}E_1'':(a_j,A_j)_{1,p_1};(c_j,C_j)_{1,p_2};-\\ E_2'':(b_j,B_j)_{1,q_1};(d_j,D_j)_{1,q_2};(0,1)\end{matrix}\right.\right],\qquad(23)$$

where $E_1''=(1-\eta-\delta k;\upsilon_1,\upsilon_2,1),(1-\mu-\lambda k;\sigma_1,\sigma_2,1)$; and

$E_2''=(1-\eta-\delta k;\upsilon_1,\upsilon_2,0),(1-\mu+\alpha-\lambda k;\sigma_1,\sigma_2,1)$,

and the existence conditions of the above corollary easily follows with the help of (16).

Further, if we set $\beta=0$ in Corollary 1.1, one can easily obtain similar type of result concerning left-sided Erdélyi-Kober fractional derivative operator.

Theorem 2 *Suppose that* $\alpha,\alpha',\beta,\beta',\gamma,\mu,\eta,\delta,\upsilon_1,\upsilon_2,z_1,z_2,a,b\in\mathrm{C},\ \lambda,\sigma_1,\sigma_2>0$, *and* $Re(\gamma)>0$. Then we have the following result:

$$\Big\{D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\Big(t^{\mu-1}(b-at)^{-\eta}\,S_n^m\Big[t^{\lambda}(b-at)^{-\delta}\Big]$$

$$\times H_{p_1,q_1}^{m_1,n_1}\left[z_1t^{\sigma_1}(b-at)^{-\upsilon_1}\left|\begin{matrix}(a_j,A_j)_{1,p_1}\\(b_j,B_j)_{1,q_1}\end{matrix}\right.\right]H_{p_2,q_2}^{m_2,n_2}\left[z_2t^{\sigma_2}(b-at)^{-\upsilon_2}\left|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right.\right]\Big)\Big\}(x)$$

$$= b^{-\eta} x^{\mu+\alpha+\alpha'-\gamma-1} \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} b^{-\delta k} x^{\lambda k}$$

$$\times\ H^{0,4:m_1,n_1;m_2,n_2;1,0}_{4,4:p_1,q_1;p_2,q_2;0,1} \left[\begin{matrix} z_1 x^{\sigma_1} b^{-\upsilon_1} \\ z_2 x^{\sigma_2} b^{-\upsilon_2} \\ -\frac{a}{b} x \end{matrix} \left| \begin{matrix} F_1 : (a_j, A_j)_{1,p_1} ; (c_j, C_j)_{1,p_2} ; - \\ F_2 : (b_j, B_j)_{1,q_1} ; (d_j, D_j)_{1,q_2} ; (0,1) \end{matrix} \right. \right], \tag{24}$$

where

$F_1 = (1-\eta-\delta k; \upsilon_1, \upsilon_2, 1), (\mu+\alpha+\alpha'-\gamma+\lambda k; -\sigma_1, -\sigma_2, -1)$,

$(\mu+\alpha'+\beta-\gamma+\lambda k; -\sigma_1, -\sigma_2, -1)$, $(\mu-\beta'+\lambda k; -\sigma_1, -\sigma_2, -1)$; and

$F_2 = (1-\eta-\delta k; \upsilon_1, \upsilon_2, 0), (\mu+\lambda k; -\sigma_1, -\sigma_2, -1)$

$(\mu+\alpha+\alpha'+\beta-\gamma+\lambda k; -\sigma_1, -\sigma_2, -1), (\mu+\alpha'-\beta'+\lambda k; -\sigma_1, -\sigma_2, -1)$.

Also, satisfy the following conditions:

$$Re(\mu) + \sigma_1 \max_{1\le j\le n_1} \left(\frac{Re(a_j)-1}{A_j} \right) + \sigma_2 \max_{1\le j\le n_2} \left(\frac{Re(c_j)-1}{C_j} \right)$$
$$< 1 + \min\left[Re(-\beta), Re(\gamma-\alpha-\alpha'-k), Re(-\alpha'-\beta+\gamma) \right],$$

$$Re(\eta) + \upsilon_1 \max_{1\le j\le n_1} \left(\frac{Re(a_j)-1}{A_j} \right) + \upsilon_2 \max_{1\le j\le n_2} \left(\frac{Re(c_j)-1}{C_j} \right)$$
$$< 1 + \min\left[Re(-\beta), Re(\gamma-\alpha-\alpha'-k), Re(-\alpha'-\beta+\gamma) \right],$$

here $k = [Re(\gamma)] + 1$, and the conditions (i)-(iii) given in Theorem 1 are also satisfied.

Proof In order to prove (24), first write the general class of polynomials in the series from given by (6), replace the both H-functions by its well-known Mellin-Barnes contour integral representation using (1), interchange the order of summations, take generalized fractional derivative inside (which is valid under conditions stated with Theorem 2) and make a little simplification.

By using the Mellin-Barnes contour integral representation for the binomial expansion $\left(b-ax\right)^{-\gamma}$ stated by (19), and interchanging the order of integrals, we obtain the following form after a little simplification (say D_2):

$$\mathrm{D}_2=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}(b)^{-\eta-\delta k}\left(\frac{1}{2\pi i}\right)^3\int_{L_1}\theta(\xi_1)z_1^{\xi_1}(b)^{-\upsilon_1\xi_1}\,d\xi_1\int_{L_2}\theta(\xi_2)z_2^{\xi_2}(b)^{-\upsilon_2\xi_2}\,d\xi_2$$

$$\times\int_{L_3}\frac{\Gamma\left(\eta+\delta k+\upsilon_1\xi_1+\upsilon_2\xi_2+\xi_3\right)}{\Gamma\left(\eta+\delta k+\upsilon_1\xi_1+\upsilon_2\xi_2\right)\Gamma\left(1+\xi_3\right)}\left(-\frac{a}{b}\right)^{\xi_3}d\xi_3\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\mu+\lambda k+\sigma_1\xi_1+\sigma_2\xi_2+\xi_3-1}\right)(x). \quad (25)$$

$$=(-1)^r b^{-\eta}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}b^{-\delta k}\left(\frac{1}{2\pi i}\right)^3\int_{L_1}\theta(\xi_1)z_1^{\xi_1}(b)^{-\upsilon_1\xi_1}\,d\xi_1$$

$$\times\int_{L_2}\theta(\xi_2)z_2^{\xi_2}(b)^{-\upsilon_2\xi_2}\,d\xi_2\int_{L_3}\frac{\Gamma\left(\eta+\delta k+\upsilon_1\xi_1+\upsilon_2\xi_2+\xi_3\right)}{\Gamma\left(\eta+\delta k+\upsilon_1\xi_1+\upsilon_2\xi_2\right)\Gamma\left(1+\xi_3\right)}\left(-\frac{a}{b}\right)^{\xi_3}d\xi_3$$

$$\times\frac{\Gamma\left(1+\beta'-\mu-\lambda k-\sigma_1\xi_1-\sigma_2\xi_2-\xi_3\right)\Gamma\left(1-\alpha'-\beta+\gamma-\mu-\lambda k-\sigma_1\xi_1-\sigma_2\xi_2-\xi_3\right)}{\Gamma\left(1-\mu-\lambda k-\sigma_1\xi_1-\sigma_2\xi_2-\xi_3\right)\Gamma\left(1-\alpha'+\beta'-\mu-\lambda k-\sigma_1\xi_1-\sigma_2\xi_2-\xi_3\right)}$$

$$\times\frac{\Gamma\left(1-\alpha-\alpha'+\gamma-r-\mu-\lambda k-\sigma_1\xi_1-\sigma_2\xi_2-\xi_3\right)}{\Gamma\left(1-\alpha-\alpha'-\beta+\gamma-\mu-\lambda k-\sigma_1\xi_1-\sigma_2\xi_2-\xi_3\right)}$$

$$\times\frac{d^r}{dx^r}\left(x^{\mu+\lambda k+\sigma_1\xi_1+\sigma_2\xi_2+\xi_3+\alpha+\alpha'-\gamma+r-1}\right), \quad (26)$$

where $r=\left[\Re\left(\gamma\right)\right]+1$. By using $\frac{d^n}{dx^n}x^m=\frac{\Gamma\left(m+1\right)}{\Gamma\left(m-n+1\right)}x^{m-n}\quad\left(m\geq n\right)$, we arrive at

$$\mathrm{D}_2=b^{-\eta}x^{\mu+\alpha+\alpha'-\gamma-1}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}b^{-\delta k}x^{\lambda k}\left(\frac{1}{2\pi i}\right)^3\int_{L_1}\theta(\xi_1)\left(z_1x^{\sigma_1}b^{-\upsilon_1}\right)^{\xi_1}d\xi_1$$

$$\times\int_{L_2}\theta(\xi_2)\left(z_2x^{\sigma_2}b^{-\upsilon_2}\right)^{\xi_2}d\xi_2\int_{L_3}\frac{\Gamma\left(1-\left(1-\eta-\delta k\right)+\upsilon_1\xi_1+\upsilon_2\xi_2+\xi_3\right)}{\Gamma\left(1-\left(1-\eta-\delta k\right)+\upsilon_1\xi_1+\upsilon_2\xi_2\right)\Gamma\left(1+\xi_3\right)}\left(-\frac{a}{b}x\right)^{\xi_3}d\xi_3$$

$$\times\frac{\Gamma\left(1-\left(\alpha+\alpha'-\gamma+\mu+\lambda k\right)-\sigma_1\xi_1-\sigma_2\xi_2-\xi_3\right)\Gamma\left(1-\left(\mu-\beta'+\lambda k\right)-\sigma_1\xi_1-\sigma_2\xi_2-\xi_3\right)}{\Gamma\left(1-\left(\mu+\lambda k\right)-\sigma_1\xi_1-\sigma_2\xi_2-\xi_3\right)\Gamma\left(1-\left(\mu+\alpha'-\beta'+\lambda k\right)-\sigma_1\xi_1-\sigma_2\xi_2-\xi_3\right)}$$

$$\times\frac{\Gamma\left(1-\left(\alpha'+\beta-\gamma+\mu+\lambda k\right)-\sigma_1\xi_1-\sigma_2\xi_2-\xi_3\right)}{\Gamma\left(1-\left(\mu+\alpha+\alpha'+\beta-\gamma+\lambda k\right)-\sigma_1\xi_1-\sigma_2\xi_2-\xi_3\right)}. \tag{27}$$

Re-interpreting the Mellin-Barnes counter integral in terms of the H-function of three variables given by (3), we obtain the right-hand side of (24) after little simplifications. This completes proof of Theorem 2.

In view of the relation (13), we obtain the following corollary concerning right-sided Saigo fractional derivative operator:

Corollary 2.1 *Let* $\alpha,\beta,\gamma,\mu,\eta,\delta,\upsilon_1,\upsilon_2,z_1,z_2,a,b\in\mathbb{C},\ \lambda,\sigma_1,\sigma_2>0,$ *and* $Re\left(\alpha\right)>0$. *Then we have the following result:*

$$\Big\{D_{-}^{\alpha,\beta,\gamma}\Big(t^{\mu-1}\left(b-at\right)^{-\eta}S_n^m\left[t^{\lambda}\left(b-at\right)^{-\delta}\right]$$

$$\times H_{p_1,q_1}^{m_1,n_1}\left[z_1t^{\sigma_1}\left(b-at\right)^{-\upsilon_1}\left|\begin{matrix}\left(a_j,A_j\right)_{1,p_1}\\ \left(b_j,B_j\right)_{1,q_1}\end{matrix}\right.\right]H_{p_2,q_2}^{m_2,n_2}\left[z_2t^{\sigma_2}\left(b-at\right)^{-\upsilon_2}\left|\begin{matrix}\left(c_j,C_j\right)_{1,p_2}\\ \left(d_j,D_j\right)_{1,q_2}\end{matrix}\right.\right]\Big)\Big\}\left(x\right)$$

$$=b^{-\eta}x^{\mu+\beta-1}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}b^{-\delta k}x^{\lambda k}$$

$$\times H_{3,3:p_1,q_1;p_2,q_2;0,1}^{0,3:m_1,n_1;m_2,n_2;1,0}\left[\begin{matrix}z_1x^{\sigma_1}b^{-\upsilon_1}\\ z_2x^{\sigma_2}b^{-\upsilon_2}\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix}F_1':\left(a_j,A_j\right)_{1,p_1};\left(c_j,C_j\right)_{1,p_2};-\\ F_2':\left(b_j,B_j\right)_{1,q_1};\left(d_j,D_j\right)_{1,q_2};\left(0,1\right)\end{matrix}\right.\right], \tag{28}$$

where

$F_1'=\left(1-\eta-\delta k;\upsilon_1,\upsilon_2,1\right),\left(\mu+\beta+\lambda k;-\sigma_1,-\sigma_2,-1\right),$
$\left(\mu-\alpha-\gamma+\lambda k;-\sigma_1,-\sigma_2,-1\right);$

and

$F_2'=\left(1-\eta-\delta k;\upsilon_1,\upsilon_2,0\right),\left(\mu+\beta-\gamma+\lambda k;-\sigma_1,-\sigma_2,-1\right),\ \left(\mu+\lambda k;-\sigma_1,-\sigma_2,-1\right).$

Also, satisfy the following conditions:

$$Re(\mu)+\sigma_1\max_{1\le j\le n_1}\left(\frac{Re\left(a_j\right)-1}{A_j}\right)+\sigma_2\max_{1\le j\le n_2}\left(\frac{Re\left(c_j\right)-1}{C_j}\right)$$

$$< 1+\min\left[0,[Re(\alpha)]-Re(\beta)-1,Re(\alpha+\gamma)\right],$$

$$Re(\eta)+\upsilon_1 \max_{1\le j\le n_1}\left(\frac{Re(a_j)-1}{A_j}\right)+\upsilon_2 \max_{1\le j\le n_2}\left(\frac{Re(c_j)-1}{C_j}\right)$$

$$< 1+\min\left[0,[Re(\alpha)]-Re(\beta)-1,Re(\alpha+\gamma)\right].$$

Next, if we set $\beta = 0$ in above result, then we obtain the following corollary concerning right-sided Riemann-Liouville fractional derivative operator:

Corollary 2.2 *Let* $\alpha,\mu,\eta,\delta,\upsilon_1,\upsilon_2,z_1,z_2,a,b\in \mathrm{C};\ \lambda,\sigma_1,\sigma_2>0$ *and* $Re(\alpha)>0$*, then we have*

$$\Bigg\{D_-^{\alpha}\Bigg(t^{\mu-1}(b-at)^{-\eta}\, S_n^m\left[t^{\lambda}(b-at)^{-\delta}\right]$$

$$\times H_{p_1,q_1}^{m_1,n_1}\left[z_1 t^{\sigma_1}(b-at)^{-\upsilon_1}\left|\begin{matrix}(a_j,A_j)_{1,p_1}\\(b_j,B_j)_{1,q_1}\end{matrix}\right.\right]H_{p_2,q_2}^{m_2,n_2}\left[z_2 t^{\sigma_2}(b-at)^{-\upsilon_2}\left|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right.\right]\Bigg)\Bigg\}(x)$$

$$=b^{-\eta}x^{\mu-\alpha-1}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\, b^{-\delta k}x^{\lambda k}$$

$$\times H_{2,2:p_1,q_1;p_2,q_2;0,1}^{0,2:m_1,n_1;m_2,n_2;1,0}\left[\begin{matrix}z_1x^{\sigma_1}b^{-\upsilon_1}\\ z_2x^{\sigma_2}b^{-\upsilon_2}\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix}F_1'':(a_j,A_j)_{1,p_1};(c_j,C_j)_{1,p_2};-\\ F_2'':(b_j,B_j)_{1,q_1};(d_j,D_j)_{1,q_2};(0,1)\end{matrix}\right.\right], \tag{29}$$

where $F_1''=(1-\eta-\delta k;\upsilon_1,\upsilon_2,1),(\mu-\alpha+\lambda k;-\sigma_1,-\sigma_2,-1)$; and

$F_2''=(1-\eta-\delta k;\upsilon_1,\upsilon_2,0),(\mu+\lambda k;-\sigma_1,-\sigma_2,-1)$,

and the existence conditions of (29) easily follows from Theorem 2.

Again, if we set $\beta = 0$ in (28), then similar type of result concerning right-sided Erdélyi-Kober fractional derivative operator.

4. Special cases and concluding remarks

The general class of polynomials involved in (16) and (24) reduce to a large spectrum of polynomials listed by Srivastava and Singh [29] and so from Theorem 1 and 2, we can further obtain various fractional derivative results involving a number of simpler polynomials. Here, we provide a few special cases of our main findings (see also, [3, 5, 9, 10, 19]).

(i) If we reduce the first H-function in (16) to the exponential function by taking $\sigma_1 = \lambda = 1, \upsilon_1 = \delta = 0$ and S_n^m to the Hermite polynomial [29] by setting

$$S_n^2[x] = x^{n/2} H_n\left[\frac{1}{2\sqrt{x}}\right] (\text{for } m = 2, A_{n,k} = (-1)^k);$$

we obtain the following result of the (16) after a little simplification:

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}(b-at)^{-\eta}\, t^{n/2} H_n\left[\frac{1}{2\sqrt{t}}\right] e^{-z_1 t}\right.\right.$$

$$\left.\left. H_{p_2,q_2}^{m_2,n_2}\left[z_2 t^{\sigma_2}(b-at)^{-\upsilon_2}\left|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right.\right]\right)\right\}(x)$$

$$= b^{-\eta} x^{\mu+\alpha+\alpha'-\gamma-1} \sum_{k=0}^{[n/2]} \frac{(-n)_{2k}}{k!}(-x)^k$$

$$\times H_{4,3:0,1;p_2,q_2+1;0,1}^{0,4:1,0;m_2,n_2;1,0}\left[\begin{matrix} z_1 x\\ z_2 x^{\sigma_2} b^{-\upsilon_2}\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix} R_1 : -;(c_j,C_j)_{1,p_2};-\\ R_2 : (0,1);(d_j,D_j)_{1,q_2},(1-\eta,\upsilon_2);(0,1)\end{matrix}\right.\right], \tag{30}$$

where $R_1 = (1-\eta;0,\upsilon_2,1),(1-\mu-k;1,\sigma_2,1),(1-\mu+\gamma-\alpha-\alpha'-\beta'-k;1,\sigma_2,1),$

$(1-\mu-\alpha+\beta-k;1,\sigma_2,1)$; and

$R_2 = (1-\mu+\gamma-\alpha-\alpha'-k;1,\sigma_2,1),\ (1-\mu+\gamma-\alpha-\beta'-k;1,\sigma_2,1),$
$(1-\mu+\beta-k;1,\sigma_2,1).$

(ii) Next, if we take $z_1, \upsilon_2 = 0$ in (30), we obtain the following result:

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}(b-at)^{-\eta}\, t^{n/2} H_n\left[\frac{1}{2\sqrt{t}}\right] H_{p_2,q_2}^{m_2,n_2}\left[z_2 t^{\sigma_2}\left|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right.\right]\right)\right\}(x)$$

$$= b^{-\eta} x^{\mu+\alpha+\alpha'-\gamma-1}\sum_{k=0}^{[n/2]}\frac{(-n)_{2k}}{k!}(-x)^k$$

$$\times H_{4,3:0,1;p_2,q_2+1;0,1}^{0,4:1,0;m_2,n_2;1,0}\left[\begin{matrix}z_2 x^{\sigma_2}\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix}S_1:-;(c_j,C_j)_{1,p_2};-\\ S_2:(0,1);(d_j,D_j)_{1,q_2},(1-\eta,0);(0,1)\end{matrix}\right.\right]. \tag{31}$$

where $S_1 = (1-\eta;0,1),(1-\mu-k;\sigma_2,1),(1-\mu+\gamma-\alpha-\alpha'-\beta'-k;\sigma_2,1),$
$(1-\mu-\alpha+\beta-k;\sigma_2,1)$; and
$S_2 = (1-\mu+\gamma-\alpha-\alpha'-k;\sigma_2,1),(1-\mu+\gamma-\alpha-\beta'-k;\sigma_2,1),(1-\mu+\beta-k;\sigma_2,1).$

If we reduce the H-function of one variable to generalized Wright hypergeometric function ${}_p\psi_q$ [6, 29] in (31) by using the relation with H-function and Wright generalized hypergeometric function ${}_p\psi_q$ ([11], eq.(1.140), p.25), we can easily obtain result for generalized Wright hypergeometric function.

(iii) If we use the relation involving Mittag-Leffler function, obtained by Saxena et al. ([11], eq. (1.137), p. 25) given as $E_{\beta,\gamma}^{\delta}(z)=\frac{1}{\Gamma(\delta)}H_{1,2}^{1,1}\left[-z\left|\begin{matrix}(1-\delta,1)\\(0,1),(1-\gamma,\beta)\end{matrix}\right.\right]$ in (31) then we arrive at $\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}(b-at)^{-\eta}\, t^{n/2} H_n\left[\frac{1}{2\sqrt{t}}\right] E_{\tau_1,\tau_2}^{\rho}[t]\right)\right\}(x)$

$$=b^{-\eta} x^{\mu+\alpha+\alpha'-\gamma-1}\frac{1}{\Gamma(\rho)}\sum_{k=0}^{[n/2]}\frac{(-n)_{2k}}{k!}(-x)^k$$

$$\times H_{4,3:0,1;1,3;0,1}^{0,4:1,0;1,1;1,0}\left[\begin{matrix}-x\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix}T_1:-;(1-\rho,1);-\\ T_2:(0,1);(0,1),(1-\eta,0),(1-\tau_2,\tau_1);(0,1)\end{matrix}\right.\right], \tag{32}$$

where
$T_1 = (1-\eta;0,1),(1-\mu-k;1,1),(1-\mu+\gamma-\alpha-\alpha'-\beta'-k;1,1),$
$(1-\mu-\alpha+\beta-k;1,1);$

and $T_2 = (1-\mu+\gamma-\alpha-\alpha'-k;1,1),(1-\mu+\gamma-\alpha-\beta'-k;1,1),(1-\mu+\beta-k;1,1).$

(iv) If we reduce one H-function to the exponential function by taking $\sigma_1 = 1$, $\upsilon_1 \to 0$ in (16), we obtain the following result:

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}(b-at)^{-\eta} S_n^m\left[t^\lambda (b-at)^{-\delta}\right]e^{-z_1 t}\right.\right.$$

$$\left.\left. H_{p_2,q_2}^{m_2,n_2}\left[z_2 t^{\sigma_2}(b-at)^{-\upsilon_2}\left|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right.\right]\right)\right\}(x)$$

$$= b^{-\eta}x^{\mu+\alpha+\alpha'-\gamma-1}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}b^{-\delta k}\ x^{\lambda k}$$

$$\times H_{4,3:0,1;p_2,q_2+1;0,1}^{0,4:1,0;m_2,n_2;1,0}\left[\begin{matrix}z_1 x\\ z_2 x^{\sigma_2}b^{-\upsilon_2}\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix}U_1:-;(c_j,C_j)_{1,p_2};-\\ U_2:(0,1);(d_j,D_j)_{1,q_2},(1-\eta-\delta k,\upsilon_2);(0,1)\end{matrix}\right.\right], \quad (33)$$

where
$U_1 = (1-\eta-\delta k;0,\upsilon_2,1),(1-\mu-\lambda k;1,\sigma_2,1),(1-\mu+\gamma-\alpha-\alpha'-\beta'-\lambda k;1,\sigma_2,1),$
$(1-\mu-\alpha+\beta-\lambda k;1,\sigma_2,1)$; and $U_2 = (1-\mu+\gamma-\alpha-\alpha'-\lambda k;1,\sigma_2,1),$
$(1-\mu+\gamma-\alpha-\beta'-\lambda k;1,\sigma_2,1),\ (1-\mu+\beta-\lambda k;1,\sigma_2,1).$

The conditions of validity of the above results (i)-(iv) can be easily follow directly from Theorem 1.

Remark 4.1 *We can also obtain results for ordinary Bessel function of the first kind* $J_\upsilon(z)$, *modified Bessel functions* $K_\upsilon(z)$ *(Macdonald function) and* $Y_\upsilon(z)$ *(Neumann function), generalized Bessel-Maitland function* $J_{\upsilon,\lambda}^{\mu}$, *Kummer's confluent hypergeometric function* $\phi(a;d;-z)$, *Gauss' hypergeometric function* ${}_2F_1(b,a;d;-zt^\mu)$, *MacRobert's* E*-function, Whittaker function by using the relation with* H*-function and these function.*

5. Conclusion

In the present paper, we have given the two theorems of generalized fractional derivative operators given by Saigo-Maeda. The theorems have been developed in terms of the product of two H -functions and a general class of polynomials in a compact and elegant form with the help of Saigo-Maeda power function formulas. The main fractional derivatives whose integrands being the products of two H -functions and a general class of polynomials, as shown in Section 4, can be specialized to yield a large number of simpler results. So the main results may find (potentially) useful applications in a variety of areas.

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