CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH QUASI-SUBORDINATION

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Abstract: In this paper, we introduce and investigate two new subclasses of the function class $\Sigma$ of bi-univalent function defined in the open unit disk which are associated with the quasi-subordination. We find estimates on the Taylor-Maclaurin coefficient $|a_2|$ and $|a_3|$ for functions in these subclasses. Several known and new consequences of these results are also pointed out.

Key words: Bi-univalent functions, quasi-subordination, coefficient estimates.

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1. Introduction

Let $A$ denote the class of analytic functions in the unit disk $U = \{ z \in \mathbb{C} \mid |z| < 1 \}$ that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and let $S$ be the class of all functions from $A$ which are univalent in $U$.

The Koebe one quarter theorem [5] states that the image of $U$ under every function $f$ from $S$ contains a disk of radius $\frac{1}{4}$. Thus such univalent function has an inverse $f^{-1}$ which satisfies

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_o(f), \ r_o(f) \geq \frac{1}{4}).$$

In fact, the inverse function $f^{-1}$ is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2a_3 + a_4) w^4 + ... = g(w).$$

A function $f \in A$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$.

Let $\Sigma$ denotes the class of bi-univalent functions defined in the unit disk $U$. 

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2a_3 + a_4) w^4 + ... = g(w).$$
Robertson [12] introduced the concept of quasi-subordination which is generalization of subordination and majorization. For two analytic functions $f$ and $h$, the function $f$ is quasi subordination to $h$ written as

$$f(z) \prec_q h(z) \quad (z \in U)$$  (3)

If there exist analytic functions $\phi$ and $\omega$, with $|\phi(z)| \leq 1, \omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = \phi(z)h(\omega(z)).$ Observe that if $\phi(z) = 1$, then $f(z) = h(\omega(z))$, so that $f(z) \prec h(z)$ in $U$. also if $\omega(z) = z$, then $f(z) = \phi(z)h(z)$ and it is said that $f(z)$ is majorized by $h(z)$ and written as $f(z) \ll h(z)$ in $U$.

In 1967, Lewin [7] investigated the class $\Sigma$ of bi-univalent functions and obtained the bound for the second coefficient $|a_2|$. Brannan and Taha [3] considered certain subclasses of bi-univalent functions similar to the familiar subclasses of univalent functions consisting of starlike, strongly starlike and convex functions. They introduced the bi-starlike function, bi-convex function classes and obtained non sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

Recently Ali et al. [1], Deniz [4], Tang et al. [14], Peng et al.[9], Ramchandran et al. [11], Peng et al. [8] etc. have introduced and investigated Ma-Minda type subclasses of bi-univalent functions class $\Sigma$.

Motivated by the work of Goyal [6] we define subclasses of $\Sigma$ by means of quasi-subordination.

Throughout this paper it is assumed that $h$ is analytic in $U$ with $h(0) = 1$ and let

$$\phi(z) = A_0 + A_1z + A_2z^2 + ... \quad (|\phi(z)| \leq 1, z \in U)$$  (4)

$$h(z) = 1 + B_1z + B_2z^2 + ... \quad B_i \in \mathbb{R}^+$$  (5)

**Definition 1.** For $\alpha \geq 1$ and $\gamma \geq 0$ a function $f \in \Sigma$ is said to be in class

$$M_\Sigma^q (\tau, \alpha, \gamma ; h )$$

if the following quasi-subordination holds :

$$\frac{1}{\tau} \left\{ (1 - \alpha) \frac{f(z)}{z} + \alpha f'(z) + \gamma f^\prime\prime(z) - 1 \right\} \prec_q (h(z) - 1)$$

and

$$\frac{1}{\tau} \left\{ (1 - \alpha) \frac{g(w)}{w} + \alpha g'(w) + \gamma wg''(w) - 1 \right\} \prec_q (h(w) - 1)$$  (6)

**Definition 1.2.** A function $f \in \Sigma$ is said to be in class $N_\Sigma(\eta, \ h)$ if the following quasi-subordination holds:
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\[
\frac{zF'(z)}{F(z)} - 1 \prec \frac{\eta}{q} (h(z) - 1)
\]

and

\[
\frac{wG'(w)}{G(w)} - 1 \prec \frac{\eta}{q} (h(w) - 1)
\]

where \( F(z) \) and \( G(w) \) are as follows:

\[
\frac{1}{F(z)} = \frac{1 - \eta}{f'(z)} + \frac{\eta}{zf'(z)}
\]

and

\[
\frac{1}{G(w)} = \frac{1 - \eta}{g'(w)} + \frac{\eta}{wg'(w)}
\]

(\( \eta \in \mathbb{C} \))

The above classes of functions defined in terms of the quasi-subordination are associated with the classes of functions with positive real parts.

In the present paper, the coefficient bounds of \(|a_2|\) and \(|a_3|\) for function in the class \( M_{q}^q (\tau, \alpha, \gamma ;h) \) and \( N_{\Xi}(\eta, h) \) are obtained.

In order to derive our main results, we have to recall here the following Lemma.

**Lemma 1.** [10] Let \( p \in P \) the family of all functions \( p \) analytic in \( U \) for which \( R\{p(z)\} > 0 \) and have the form \( p(z) = 1 + p_1z + p_2z^2 + \ldots \) for \( z \in U \), then \(|p_n| \leq 2\), for each \( n \).

**2. Main Results**

**Theorem 2.1.** If \( f \in M_{q}^q (\tau, \alpha, \gamma ;h) \), then

\[
|a_2| \leq \frac{|\tau||A_o||B_1|^{\frac{1}{2}}}{\sqrt[\gamma]{\tau(1 + 2\alpha + 6\gamma)A_oB_1^2 + (1 + \alpha + 2\gamma)(B_1 - B_2)}}
\]

(8)

and

\[
|a_3| \leq \frac{|\tau||B_1|}{(1 + 2\alpha + 6\gamma)} \left( A_1 + |A_o| \left( \frac{(1 + 2\alpha + 6\gamma)|\tau||A_o||B_1|}{(1 + \alpha + 2\gamma)^2} \right) \right)
\]

(9)
Proof. Since \( f \in M_{\Sigma}^N (\tau, \alpha, \gamma ; h) \) and \( g = f^{-1} \). Then there exists two analytic functions \( r, s : U \rightarrow U \) with \( r(0) = 0 = s(0) \), \( |r(z)| < 1 \) and \( |s(z)| < 1 \) and a function \( \phi \) in \( U \) defined by (4) satisfying.

\[
\frac{1}{\tau} \left( 1 - \alpha \right) \frac{f(z)}{z} + \alpha f' (z) + \gamma zf'' (z) - 1 = \phi(z) (h(r(z)) - 1) \tag{10}
\]

and

\[
\frac{1}{\tau} \left( 1 - \alpha \right) \frac{g(w)}{w} + \alpha g' (w) + \gamma wg'' (w) - 1 = \phi(w) (h(r(w)) - 1) \tag{11}
\]

Defined the function \( p \) and \( q \) by

\[
p(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + c_1 z + c_2 z^2 + ... \tag{12}
\]

\[
q(z) = \frac{1 + s(z)}{1 - s(z)} = 1 + d_1 z + d_2 z^2 + ... \tag{13}
\]

or equivalently

\[
r(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left( c_1 z + \left[ c_2 - \frac{c_1^2}{2} \right] z^2 + ... \right) \tag{14}
\]

\[
s(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left( d_1 z + \left[ d_2 - \frac{d_1^2}{2} \right] z^2 + ... \right) \tag{15}
\]

It is clear that \( p \) and \( q \) are analytic in \( U \) and \( p(0) = 1 = q(0) \). Also \( p \) and \( q \) have positive real part in \( U \), and hence by Lemma 1, we have \( |c_i| \leq 2 \) and \( |d_i| \leq 2 \).

In the view of (10), (11), (14) and (15), clearly

\[
\frac{1}{\tau} \left( 1 - \alpha \right) \frac{f(z)}{z} + \alpha f' (z) + \gamma zf'' (z) - 1 = \phi(z) \left( h \left( \frac{p(z) - 1}{p(z) + 1} \right) - 1 \right) \tag{16}
\]

and

\[
\frac{1}{\tau} \left( 1 - \alpha \right) \frac{g(w)}{w} + \alpha g' (w) + \gamma wg'' (w) - 1 = \phi(w) \left( h \left( \frac{p(w) - 1}{p(w) + 1} \right) - 1 \right) \tag{17}
\]
Since \( f \in \Sigma \) has the Maclaurin series given by (1), and \( g = f^{-1} \) has the expansion given by (2), hence we have

\[
\frac{1}{\tau} \left\{ \frac{f(z)}{1 - \alpha} + \alpha f'(z) + \gamma zf''(z) - 1 \right\} = \frac{1}{\tau} \left\{ (1 + \alpha + 2\gamma) a_2 z + (1 + 2\alpha + 6\gamma) a_3 z^2 + \ldots \right\}
\]

and

\[
\frac{1}{\tau} \left\{ \frac{g(w)}{\alpha - w} + \alpha g'(w) + \gamma wg''(w) - 1 \right\} = \frac{1}{\tau} \left\{ -(1 + \alpha + 2\gamma) a_3 w + (1 + 2\alpha + 6\gamma) (2a_3^2 - a_3)w^2 + \ldots \right\}.
\]

Using (12) and (13) together with (4) and (5), it is evident that

\[
\phi(z) \left( \frac{p(z) - 1}{p(z) + 1} \right) = \frac{1}{2} A_0 B_1 c_1 z + \left\{ \frac{1}{2} A_1 B_1 c_1 + \frac{1}{2} A_0 B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{A_0 B_2 c_1^2}{4} \right\} z^2 + \ldots
\]

and

\[
\phi(w) \left( \frac{q(w) - 1}{q(w) + 1} \right) = \frac{1}{2} A_0 B_1 d_1 w + \left\{ \frac{1}{2} A_1 B_1 d_1 + \frac{1}{2} A_0 B_1 \left( d_2 - \frac{d_1^2}{2} \right) + \frac{A_0 B_2 c_1^2}{4} \right\} w^2 + \ldots
\]

Now using (18) and (20) in (16) and comparing the coefficient of \( z \) and \( z^2 \), we get

\[
\frac{(1 + \alpha + 2\gamma)}{\tau} a_2 = \frac{1}{2} A_0 B_1 c_1
\]

and

\[
\frac{(1 + 2\alpha + 6\gamma)}{\tau} a_3 = \frac{1}{2} A_1 B_1 c_1 + \frac{1}{2} A_0 B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{A_0 B_2 c_1^2}{4}
\]

Similarly it follows from (17), (19) and (21) that

\[
\frac{1 + 2\alpha + 6\gamma}{\tau} (2a_3^2 - a_3) = \frac{1}{2} A_1 B_1 d_1 + \frac{1}{2} A_0 B_1 \left( d_2 - \frac{d_1^2}{2} \right) + \frac{A_0 B_2 d_1^2}{4}
\]
From (22) and (24), it follows that
\[ c_1 = -d_1 \]  
and (23), (25) and (26), yields
\[ a_2^2 = \frac{\tau^2 A_0^2 B_1^3 (c_2 + d_2)}{4[\tau(1 + 2\alpha + 6\gamma)A_0 B_1^2 - (1 + \alpha + 2\gamma)^2 (B_2 - B_1)]} \]  
(27)

Thus the desired estimate on \( |a_3| \) as asserted in (8), follows by Lemma 1.

Now further computation (22) to (26) leads to
\[ a_3 = \frac{\tau}{(1 + 2\alpha + 6\gamma)} \left( \frac{A_1 B_1 c_1}{2} + \frac{A_0 B_1}{4} (c_2 - d_2) + \frac{\tau(1 + 2\alpha + 6\gamma)}{4(1 + \alpha + 2\gamma)^2 A_0^2 B_1^2 c_1^2} \right) \]  
(28)

We readily get the estimate given in (9) by applying Lemma 1.

Remarks 2.2.

(i) Putting \( \alpha = 1 \) in Theorem 2.1, we obtain the corresponding result given by Goyal et al. [6].

(ii) Putting \( \alpha = 1 \) and \( \phi(z) = 1 \) in Theorem 2.1, we obtain the corresponding result given by Deniz [4].

(iii) Putting \( \alpha = \lambda - 2 \gamma \) and \( \phi(z) = 1 \) in Theorem 2.1, we obtain the corresponding result given by Ramachandran et al. [11].

(iv) Putting \( \tau = 1, \gamma = 0 \) and \( \phi(z) = 1 \) in Theorem 2.1, we obtain the corresponding result given earlier by Ali et al. [1].

(v) Putting \( \tau = 1, \gamma = 0 \) and \( \phi(z) = \left( \frac{1 + z}{1 - z} \right)^{\beta} \) in Theorem 2.1, we obtain the corresponding result given earlier by Srivastava and Bansal [13].

For \( \alpha = 1 \) and \( \gamma = 0 \), the above Theorem 2.1 reduces to

**Corollary 2.3.** If
\[ \frac{1}{\tau}(f'(z) - 1) \prec_q (h(z) - 1) \]
and
\[ \frac{1}{\tau}(g'(w) - 1) \prec_q (h(w) - 1), \]
then
\[
|a_3| \leq \frac{|\tau| |A_0| B_1 \sqrt{B_1}}{\sqrt{3 \tau A_0 B_1^2 + 4(B_1 - B_2)|}}
\]
and
\[
|a_3| \leq \frac{|B_1|}{(1 + \eta^2) A_0 B_1^2 + (1 + \eta^2)(B_1 - B_2)|}
\]

**Theorem 2.4.** If \( f \in N_\mathcal{S}(\eta, h) \), then
\[
|A_0| B_1 \sqrt{B_1}
\]
and
\[
|a_3| \leq \frac{B_1}{(1 + 2 \eta)} \left( \frac{|A_1|}{2} + |A_0| \left( \frac{1}{2} + \frac{(1 + 2 \eta)}{(1 + \eta)^2} \right) |A_0| B_1 \right) \]

**Proof.** Since \( f \in N_\mathcal{S}(\eta, h) \) and \( g = f^{-1} \). Then there exists two analytic functions \( r, s : U \to U \) with \( r(0) = 0 = s(0) \), \( |r(z)| < 1 \) and \( |s(z)| < 1 \) and a function \( \phi \) in \( U \) defined by (4) satisfying
\[
\frac{zF'(z)}{F(z)} - 1 = \phi(z)(h(r(z)) - 1)
\]
and
\[
\frac{wG'(w)}{G(w)} - 1 = \phi(w)(h(s(w)) - 1),
\]
where \( r(z), s(z) \) are defined by (14) and (15) respectively.

Under the same restrictions for \( p(z), q(z), c_i, \) and \( d_i \) as mentioned in the Theorem 2.1, we have
\[
\frac{zF'(z)}{F(z)} - 1 = \phi(z) \left( h \left( \frac{p(z) - 1}{p(z) + 1} \right) \right) - 1
\]
and
\[
\frac{wG(w)}{G(w)} - 1 = \phi(w) \left( \frac{q(w)-1}{q(w)+1} \right) - 1 \tag{34}
\]

Since
\[
\frac{zF'(z)}{F(z)} - 1 = (1 + \eta)a_2z + \left\{ 2(1 + 2\eta) a_3 + (\eta^2 - 4\eta - 1) a_2^2 \right\} z^2 + \ldots \tag{35}
\]

and
\[
\frac{wG(w)}{G(w)} - 1 = - (1 + \eta)a_2w + \left\{ -2(1 + 2\eta) a_3 + (\eta^2 + 4\eta + 3) a_2^2 \right\} w^2 + \ldots \tag{36}
\]

The right-hand sides of (33) and (34) are given by (20) and (21), respectively.

Now using (20) and (35) in (33) and comparing the coefficients of \( z \) and \( z^2 \), we get
\[
\frac{1}{(1 + \eta)} a_2 = 2A_0B_1c_1 \tag{37}
\]
\[
2(1 + 2\eta) a_3 + (\eta^2 - 4\eta - 1) a_2^2 = \frac{1}{2} A_1B_1c_1 + \frac{1}{2} A_0B_1(c_2 - \frac{c_1^2}{2}) + \frac{A_0B_2}{4} c_1^2 \tag{38}
\]

Similarly it follows from (21), (34) and (36) that
\[
\frac{1}{(1 + \eta)} a_2 = \frac{1}{2} A_0B_1d_1 \tag{39}
\]
\[
-2(1 + 2\eta) a_3 + (\eta^2 + 4\eta + 3) a_2^2 = \frac{1}{2} A_1B_1d_1 + \frac{1}{2} A_0B_1(d_2 - \frac{d_1^2}{2}) + \frac{A_0B_2}{4} d_1^2 \tag{40}
\]

From (37) and (39), it follows that
\[
c_1 = -d_1 \tag{41}
\]

and, (38), (40) and (41), yields
\[
a_2^2 = \frac{A_0^2B_1^2(c_2 + d_2)}{4 \left[ (1 + \eta^2)A_0B_1^2 - (1 + \eta)^2(B_2 - B_1) \right]} \tag{42}
\]

Thus the desired estimate on \( |a_2| \) as asserted in (29) follows using the Lemma 1 that
\[
|c_2| \leq 2 \text{ and } |d_2| \leq 2.
\]

Now further computation (37) to (41) leads to
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\[ a \frac{1}{1 + 2\eta} \left( \frac{A_1 B_1 c_1}{4} + \frac{1}{8} A_0 B_1 (c_2 - d_2) + \frac{1}{4} \frac{(1 + 2\eta)}{(1 + \eta)} \right) A_2^2 B_2^2 c_2^2 \]  \tag{43}

We readily get the estimate given in (30) by applying Lemma 1.

**Remark 2.5.** Putting \( \phi(z) = 1 \) in Theorem 2.4, we obtain the corresponding result given by Deniz [4].

For \( \eta = 0 \), the above Theorem 2.4 reduces to

**Corollary 2.6.** If

\[ \frac{zf'(z)}{f(z)} - 1 \prec_q (h(z) - 1) \]

and

\[ \frac{wg'(w)}{g(w)} - 1 \prec_q (h(z) - 1) \]

then

\[ |a_3| \leq \frac{|A_0|B_1 \sqrt{B_1}}{\sqrt{|A_0 B_1^2 + (B_1 - B_2)|}} \]

and

\[ |a_3| \leq B_1 \left( \frac{|A_1|}{2} + |A_0| \left( \frac{1}{2} + |A_0|B_1 \right) \right) \]

For \( \eta = 1 \), the above theorem 2.4 reduces to

**Corollary 2.7.** If

\[ \frac{zf''(z)}{f'(z)} - 1 \prec_q (h(z) - 1) \]

and

\[ \frac{wg''(w)}{g'(w)} - 1 \prec_q (\eta(\omega) - 1), \]

then
\[ |a_2| \leq \frac{|A_0|B_1 \backslash B_1}{\sqrt{|2A_0B_1^2 + 4(B_1 - B_2)|}} \]

and

\[ |a_3| \leq \frac{B_1}{3} \left( \frac{|A_1|}{2} + |A_0| \left( \frac{1}{2} + \frac{3}{4} |A_0| B_1 \right) \right) \]

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**References**


