Journal of Rajasthan Academy of Physical Sciences
ISSN : 0972-6306; URL : http://raops.org.in
Vol.15, No.4, December, 2016, 315-325

CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH QUASI-SUBORDINATION

Prem Pratap Vyas and Shashi Kant
Department of Mathematics, Government Dungar College, Bikaner – 334001, India

Abstract: In this paper, we introduce and investigate two new subclasses of the function class $\sum$ of bi-univalent function defined in the open unit disk which are associated with the quasi-subordination. We find estimates on the Taylor-Maclaurin coefficient $|a_2|$ and $|a_3|$ for functions in these subclasses. Several known and new consequences of these results are also pointed out.

Key words: Bi-univalent functions, quasi-subordination, coefficient estimates.

2010 Mathematics Subject Classification: 30C45.

1. Introduction

Let A denote the class of analytic functions in the unit disk U= {z ∈C | |z| < 1} that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and let S be the class of all functions from A which are univalent in U.

The Koebeone quarter theorem [5] states that the image of U under every function f from S contains a disk of radius ¼. Thus such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \qquad (z \in U)$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_o(f),\ r_o(f) \geq 1/4).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)\, w^3 - (5a_2^3 - 5a_2a_3 + a_4)\, w^4 + \ldots = g(w) \tag{2}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U.

Let $\sum$ denotes the class of bi-univalent functions defined in the unit disk U.

Robertson [12] introduced the concept of quasi-subordination which is generalization of subordination and majorization. For two analytic functions *f* and *h*, the function *f* is quasi subordination to *h* written as

$$f(z) \prec_q h(z) \qquad (z \in U) \tag{3}$$

If there exist analytic functions ϕ and ω, with $|\phi(z)| \le 1, \omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = \phi(z)h(\omega(z))$. Observe that if $\phi(z) = 1$, then $f(z) = h(\omega(z))$, so that

$f(z)$ $h(z)$ in U. also if $\omega(z) = z$, then $f(z) = \phi(z)h(z)$ and it is said that $f(z)$ is majorized by $h(z)$ and written as $f(z) << h(z)$ in U.

In 1967, Lewin [7] investigated the class ∑ of bi-univalent functions and obtained the bound for the second coefficient$|a_2|$. Brannan and Taha [3] considered certain subclasses of bi-univalent functions similar to the familiar subclasses of univalent functions consisting of starlike, strongly starlike and convex functions. They introduced the bi-starlike function, bi-convex function classes and obtained non sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

Recently Ali *et al.* [1], Deniz [4], Tang *et al*. [14], Peng *et al.*[9], Ramchandran *et al.* [11], Peng *et al.* [8] etc. have introduced and investigated Ma-Minda type subclasses of bi-univalent functions class ∑.

Motivated by the work of Goyal [6] we define subclasses of ∑ by means of quasi-subordination.

Throughout this paper it is assumed that *h* is analytic in U with $h(0) = 1$ and let

$$\phi(z) = A_o + A_1 z + A_2 z^2 + \ldots \quad (|\phi(z)| \le 1,\ z \in U) \tag{4}$$

$$h(z) = 1 + B_1 z + B_2 z^2 + \ldots \quad B_1 \in R^+ \tag{5}$$

Definition 1.1. For$\alpha \ge 1$ and $\gamma \ge 0$ a function $f \in \Sigma$ is said to be in class

$M_{\Sigma}^{q}(\tau, \alpha, \gamma; h)$ if the following quasi-subordination holds :

$$\frac{1}{\tau}\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \gamma z f''(z) - 1\right\} \prec_q (h(z)-1)$$

and

$$\frac{1}{\tau}\left\{(1-\alpha)\frac{g(w)}{w} + \alpha g'(w) + \gamma w g''(w) - 1\right\} \prec_q (h(w)-1) \tag{6}$$

Definition 1.2. A function $f \in \Sigma$ is said to be in class $N_{\Sigma}(\eta, h)$ if the following quasi-subordination holds:

$$\frac{zF'(z)}{F(z)} - 1 \prec_q (h(z) - 1)$$

and

$$\frac{wG'(w)}{G(w)} - 1 \prec_q (h(w) - 1) \tag{7}$$

where $F(z)$ and $G(w)$ are as follows:

$$\frac{1}{F(z)} = \frac{1-\eta}{f(z)} + \frac{\eta}{zf'(z)}$$

and

$$\frac{1}{G(w)} = \frac{1-\eta}{g(w)} + \frac{\eta}{wg'(w)} \qquad (\eta \in C)$$

The above classes of functions defined in terms of the quasi-subordination are associated with the classes of functions with positive real parts.

In the present paper, the coefficient bounds of $|a_2|$ and $|a_3|$ for function in the class $M_{\Sigma}^{q}(\tau, \alpha, \gamma; h)$ and $N_{\Sigma}(\eta, h)$ are obtained.

In order to derive our main results, we have to recall here the following Lemma.

Lemma 1. [10] Let $p \in P$ the family of all functions p analytic in U for which

$R\{p(z)\} > 0$ and have the form $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ for $z \in U$, then $|p_n| \leq 2$, for each n.

2. Main Results

Theorem 2.1. If $f \in M_{\Sigma}^{q}(\tau, \alpha, \gamma; h)$, then

$$|a_2| \leq \frac{|\tau||A_o|B_1\sqrt{B_1}}{\sqrt{|\tau(1 + 2\alpha + 6\gamma)A_o B_1^2 + (1 + \alpha + 2\gamma)^2(B_1 - B_2)|}} \tag{8}$$

and

$$|a_3| \leq \frac{|\tau|B_1}{(1 + 2\alpha + 6\gamma)} \left(|A_1| + |A_o| \left(1 + \frac{(1 + 2\alpha + 6\gamma)|\tau||A_o|B_1}{(1 + \alpha + 2\gamma)^2} \right) \right) \tag{9}$$

Proof. Since $f \in M_{\Sigma}^{q}(\tau, \alpha, \gamma; h)$ and $g = f^{-1}$. Then there exists two analytic functions $r, s : U \to U$ with $r(0) = 0 = s(0)$, $|r(z)| < 1$ and $|s(z)| < 1$ and a function ϕ in U defined by (4) satisfying.

$$\frac{1}{\tau}\left\{(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)+\gamma z f''(z)-1\right\}=\phi(z)\,(h(r(z))-1) \tag{10}$$

and

$$\frac{1}{\tau}\left\{(1-\alpha)\frac{g(w)}{w}+\alpha g'(w)+\gamma w f''(w)-1\right\}=\phi(w)\,(h(r(w))-1) \tag{11}$$

Defined the function p and q by

$$p(z)=\frac{1+r(z)}{1-r(z)}=1+c_1 z+c_2 z^2+\ldots \tag{12}$$

$$q(z)=\frac{1+s(z)}{1-s(z)}=1+d_1 z+d_2 z^2+\ldots \tag{13}$$

or equivalently

$$r(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left(c_1 z+\left[c_2-\frac{c_1^2}{2}\right]z^2+\ldots\right) \tag{14}$$

$$s(z)=\frac{q(z)-1}{q(z)+1}=\frac{1}{2}\left(d_1 z+\left[d_2-\frac{d_1^2}{2}\right]z^2+\ldots\right) \tag{15}$$

It is clear that p and q are analytic in U and $p(0) = 1 = q(0)$. Also p and q have positive real part in U, and hence by Lemma 1, we have $|c_i| \le 2$ and $|d_i| \le 2$.

In the view of (10), (11), (14) and (15), clearly

$$\frac{1}{\tau}\left\{(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)+\gamma z\, f''(z)-1\right\}=\phi(z)\left(h\left(\frac{p(z)-1}{p(z)+1}\right)-1\right) \tag{16}$$

and

$$\frac{1}{\tau}\left\{(1-\alpha)\frac{g(w)}{w}+\alpha g'(w)+\gamma w\, g''(w)-1\right\}=\phi(w)\left(h\left(\frac{p(w)-1}{p(w)+1}\right)-1\right) \tag{17}$$

Since $f \in \Sigma$ has the Maclaurin series given by (1), and $g = f^{-1}$ has the expansion given by (2), hence we have

$$\frac{1}{\tau}\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \gamma z f''(z) - 1\right\}$$

$$= \frac{1}{\tau}\{(1+\alpha+2\gamma)\,a_2 z + (1+2\alpha+6\gamma)\,a_3 z^2 + \ldots\} \tag{18}$$

and

$$\frac{1}{\tau}\left\{(1-\alpha)\frac{g(w)}{w} + \alpha g'(w) + \gamma w g''(w) - 1\right\}$$

$$= \frac{1}{\tau}\{-(1+\alpha+2\gamma)\,a_2 w + (1+2\alpha+6\gamma)\,(2a_2^2 - a_3)w^2 + \ldots\}. \tag{19}$$

Using (12) and (13) together with (4) and (5), it is evident that

$$\phi(z)\left(h\left(\frac{p(z)-1}{p(z)+1}\right)-1\right) = \frac{1}{2}A_0B_1c_1z + \left\{\frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0B_2c_1^2}{4}\right\}z^2 + \ldots \tag{20}$$

and

$$\phi(w)\left(h\left(\frac{q(w)-1}{q(w)+1}\right)-1\right) = \frac{1}{2}A_0B_1d_1w + \left\{\frac{1}{2}A_1B_1d_1 + \frac{1}{2}A_0B_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{A_0B_2d_1^2}{4}\right\}w^2 + \ldots \tag{21}$$

Now using (18) and (20) in (16) and comparing the coefficient of z and z^2, we get

$$\frac{(1+\alpha+2\gamma)}{\tau}a_2 = \frac{1}{2}A_0B_0c_1 \tag{22}$$

$$\frac{(1+2\alpha+6\gamma)}{\tau}a_3 = \frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0B_2}{4}c_1^2 \tag{23}$$

Similarly it follows from (17), (19) and (21) that

$$\frac{-(1+\alpha+2\gamma)}{\tau}a_2 = \frac{1}{2}A_0B_1d_1 \tag{24}$$

$$\frac{(1+2\alpha+6\gamma)}{\tau}(2a_2^2 - a_3) = \frac{1}{2}A_1B_1d_1 + \frac{1}{2}A_0B_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{A_0B_2}{4}d_1^2 \tag{25}$$

From (22) and (24), it follows that

$$c_1 = -d_1 \tag{26}$$

and (23), (25) and (26), yields

$$a_2^2 = \frac{\tau^2 A_0^2 B_1^3 (c_2 + d_2)}{4[\tau(1+2\alpha+6\gamma)A_0 B_1^2 - (1+\alpha+2\gamma)^2 (B_2 - B_1)]} \tag{27}$$

Thus the desired estimate on $|a_2|$ as asserted in (8), follows by Lemma 1.

Now further computation (22) to (26) leads to

$$a_3 = \frac{\tau}{(1+2\alpha+6\gamma)}\left(\frac{A_1 B_1 c_1}{2} + \frac{A_0 B_1}{4}(c_2 - d_2) + \frac{\tau(1+2\alpha+6\gamma)}{4(1+\alpha+2\gamma)^2} A_0^2 B_1^2 c_1^2\right) \tag{28}$$

We readily get the estimate given in (9) by applying Lemma 1.

Remarks 2.2.

(i) Putting $\alpha = 1$ in Theorem 2.1, we obtain the corresponding result given by Goyal *et al*. [6].

(ii) Putting $\alpha = 1$ and $\phi(z) = 1$ in Theorem 2.1, we obtain the corresponding result given by Deniz [4].

(iii) Putting $\alpha = \lambda - 2\gamma$ and $\phi(z) = 1$ in Theorem 2.1, we obtain the corresponding result given by Ramachandran *et al*. [11].

(iv) Putting $\tau = 1$, $\gamma = 0$ and $\phi(z) = 1$ in Theorem 2.1, we obtain the corresponding result given earlier by Ali *et al*. [1].

(v) Putting $\tau = 1$, $\gamma = 0$ and $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\beta}$ in Theorem 2.1, we obtain the corresponding result given earlier by Srivastava and Bansal [13].

For $\alpha = 1$ and $\gamma = 0$, the above Theorem 2.1 reduces to

Corollary 2.3. If

$$\frac{1}{\tau}(f'(z) - 1) \prec_q (h(z) - 1)$$

and

$$\frac{1}{\tau}(g'(w) - 1) \prec_q (h(w) - 1),$$

then

$$|a_2| \leq \frac{|\tau||A_0|B_1\sqrt{B_1}}{\sqrt{|3\tau A_0 B_1^2 + 4(B_1 - B_2)|}}$$

and

$$|a_3| \leq \frac{|\tau|B_1}{3}\left(|A| + |A_0|\left(1 + \frac{3}{4}|\tau||A_0|B_1\right)\right)$$

Theorem 2.4. If $f \in N_\Sigma(\eta, h)$, then

$$|a_2| \leq \frac{|A_0|B_1\sqrt{B_1}}{\sqrt{|(1+\eta^2)A_0 B_1^2 + (1+\eta)^2(B_1 - B_2)|}} \tag{29}$$

and

$$|a_3| \leq \frac{B_1}{(1+2\eta)}\left(\frac{|A_1|}{2} + |A_0|\left(\frac{1}{2} + \frac{(1+2\eta)}{(1+\eta)^2}|A_0|B_1\right)\right) \tag{30}$$

Proof. Since $f \in N_\Sigma(\eta, h)$ and $g = f^{-1}$. Then there exists two analytic functions

$r, s : U \to U$ with $r(0) = 0 = s(0)$, $|r(z)| < 1$ and $|s(z)| < 1$ and a function ϕ in U defined by (4) satisfying

$$\frac{zF'(z)}{F(z)} - 1 = \phi(z)(h(r(z)) - 1) \tag{31}$$

and

$$\frac{wG'(w)}{G(w)} - 1 = \phi(w)(h(s(w)) - 1), \tag{32}$$

where $r(z)$, $s(z)$ are defined by (14) and (15) respectively.

Under the same restrictions for $p(z)$, $q(z)$, c_i and d_i as mentioned in the Theorem 2.1, we have

$$\frac{zF'(z)}{F(z)} - 1 = \phi(z)\left(h\left(\frac{p(z)-1}{p(z)+1}\right)\right) - 1 \tag{33}$$

and

$$\frac{wG'(w)}{G(w)} - 1 = \phi(w)\left(h\left(\frac{q(w)-1}{q(w)+1}\right) - 1\right) \tag{34}$$

Since

$$\frac{zF'(z)}{F(z)} - 1 = (1+\eta)a_2 z + \{2(1+2\eta)\, a_3 + (\eta^2 - 4\eta - 1)\, a_2^2\} z^2 + \ldots \tag{35}$$

and

$$\frac{wG'(w)}{G(w)} - 1 = -(1+\eta)a_2 w + \{-2(1+2\eta)\, a_3 + (\eta^2 + 4\eta + 3)\, a_2^2\} w^2 + \ldots \tag{36}$$

The right-hand sides of (33) and (34) are given by (20) and (21), respectively.

Now using (20) and (35) in (33) and comparing the coefficients of z and z^2, we get

$$(1+\eta)\, a_2 = \frac{1}{2} A_0 B_1 c_1 \tag{37}$$

$$2(1+2\eta)\, a_3 + (\eta^2 - 4\eta - 1)\, a_2^2 = \frac{1}{2} A_1 B_1 c_1 + \frac{1}{2} A_0 B_1 \left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0 B_2}{4} c_1^2 \tag{38}$$

Similarly it follows from (21), (34) and (36) that

$$-(1+\eta)\, a_2 = \frac{1}{2} A_0 B_1 d_1 \tag{39}$$

$$-2(1+2\eta)\, a_3 + (\eta^2 + 4\eta + 3)\, a_2^2 = \frac{1}{2} A_1 B_1 d_1 + \frac{1}{2} A_0 B_1 \left(d_2 - \frac{d_1^2}{2}\right) + \frac{A_0 B_2}{4} d_1^2 \tag{40}$$

From (37) and (39), it follows that

$$c_1 = -d_1 \tag{41}$$

and, (38), (40) and (41), yields

$$a_2^2 = \frac{A_0^2 B_1^3 (c_2 + d_2)}{4\left[(1+\eta^2) A_0 B_1^2 - (1+\eta)^2 (B_2 - B_1)\right]} \tag{42}$$

Thus the desired estimate on $|a_2|$ as asserted in (29) follows using the Lemma 1 that $|c_2| \le 2$ and $|d_2| \le 2$.

Now further computation (37) to (41) leads to

$$a\frac{1}{(1+2\eta)}\left(\frac{A_1B_1c_1}{4}+\frac{1}{8}A_0B_1(c_2-d_2)+\frac{1}{4}\frac{(1+2\eta}{(1+\eta)}\right)A_2^2B_2^2c_2^2 \tag{43}$$

We readily get the estimate given in (30) by applying Lemma 1.

Remark 2.5. Putting $\phi(z) = 1$ in Theorem 2.4, we obtain the corresponding result given by Deniz [4].

For$\eta = 0$, the above Theorem 2.4 reduces to

Corollary 2.6. If

$$\frac{zf^{'}(z)}{f^{'}(z)}-1\prec_q (h(z)-1)$$

and

$$\frac{wg'(w)}{g(w)}-1\prec_q (h(z)-1)$$

then

$$|a_2| \leq \frac{|A_0|B_1\sqrt{B_1}}{\sqrt{|A_0B_1^2+(B_1-B_2)|}}$$

and

$$|a_3|\leq B_1\left(\frac{|A_1|}{2}+|A_0|\left(\frac{1}{2}+|A_0|B_1\right)\right)$$

For $\eta = 1$, the above theorem 2.4 reduces to

Corollary 2.7. If

$$\frac{zf^{''}(z)}{f^{'}(z)}-1\prec_q (h(z)-1)$$

and

$$\frac{wg''(w)}{g'(w)}-1\prec_q (\eta(\omega)-1),$$

then

$$|a_2| \leq \frac{|A_0|B_1\sqrt{B_1}}{\sqrt{|2A_0 B_1^2 + 4(B_1 - B_2)|}}$$

and

$$|a_3| \leq \frac{B_1}{3}\left(\frac{|A_1|}{2} + |A_0|\left(\frac{1}{2} + \frac{3}{4}|A_0|B_1\right)\right)$$

Acknowledgement : The authors are thankful to the Referee for valuable suggestions.

References

[1] Ali, R.M., Lee, S.K., Ravichandran, V. and Subramaniam, S. (2012). Coefficient estimates for bi-univalent Ma-Mindastarlike and convex functions, *Appl. Math. Lett*. 25(3), 344-351.

[2] Bansal, D. and Sokol, J. (2014). Coefficient bound for a new class of analytic and bi-univalent functions, *J. Fract. Calc. Appl*. 5(1), 122-128.

[3] Brannan, D.A. and Taha, T.S. (1986). On some classes of bi-univalent functions, *Studia Univ. Babes-Bolyai Math*. 31(2), 70-77.

[4] Deniz, E. (2013). Certain subclasses of bi-univalent functions satisfying subordinate conditions, *J. Classical Ana*. 2(1), 49-60.

[5] Duren, P.L. (1983). *Univalent functions*, Grundlehren der mathematischen Wissenschaften, 259, Springer – Verlag, New York, Berlin, Heidelbery and Tokyo.

[6] Goyal, S.P., Singh, O. and Mukherjee, R. (2016). Certain results on a subclass of analytic and bi-univalent functions associated with coefficient estimates and quasi-subordination, *Palestine Journal of Mathematics*. 5(1), 79-85.

[7] Lewin, M. (1967). On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc*. 18, 63-68.

[8] Murugusundaramoorthy, G., Janani, T. and Cho, N.E. (2016). bi-univalent functions of complex order based on subordinate conditions involving Hurwitz – LerchZeta function, *East Aasian Math*. J. 32(1), 47-59.

[9] Peng, Z., Murugusundaramoorthy, G. and Janani, T. (2014). Coefficient estimate of bi-univalent functions of complex order associated with Hohlovoperato, *Journal of Complex analysis*. (2014) Art. 693908, 6 pages.

[10] Pommerenke, Ch. (1975).*Univalent Functions*. Vandenhoeck and Ruprecht, Gottingen.

[11] Ramachandran, C., Ambroseprabhu, R. and Magesh, N. (2015). Initial coefficient estimates for certain subclasses of bi-univalent functions of Ma-minda type, *Applied Mathematical Sci*ences. 9(47), 2299-2308.

[12] Robertson, M.S. (1970). Quasi-subordination and coefficient conjecture, *Bull. Amer. Math. Soc*. 76, 1-9.

[13] Srivastava, H.M., Bansal, D. (2015). Coefficient estimates for a subclass of analytic and bi-univalent functions, *J. Egypt. Math. Soc*. 23, 242-246.

[14] Tang, H., Deng, G. and Li, S. (2013). Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions, *J. Ineq. Appl*. 2013 Art. 317, 10 pages.