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COEFFICIENT INEQUALITIES FOR CLASSES OF q-STARLIKE AND q-CONVEX FUNCTIONS USING q-DERIVATIVE

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Abstract: In view of classes for uniformly starlike and convex functions in the open unit disc U, some coefficient inequalities for theses classes using q-derivative are discussed.

Keywords and phrases: q-starlike and q-convex functions, q-differential operator.

1. Introduction

Quantum calculus or q-calculus began with Frank Hilton Jackson in the early 20th century, but this kind of calculus had already been worked out by Euler and Jacobi. Recently it aroused interest due to high demand of mathematics that models quantum computing q-calculus, which appeared as a connection between mathematics and physics. It has a lot of applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hyper geometric functions, quantum theory, mechanics and the theory of relativity.

The q-analogue of derivative of $f(z)$ [5] is given by

$$\partial_q f(z) = \frac{f(qz)-f(z)}{z(q-1)} \quad ,(z \neq 0 \ , 0 < q < 1) \tag{1}$$

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let $S_q^*(\beta)$ denote the subclass of A consisting of functions $f(z)$ which satisfy

$$Re\left\{\frac{z\partial_q f(z)}{f(z)}\right\} > \beta , \quad z \in U \tag{3}$$

For some $\beta\ (0 \le \beta < 1)$. A function $f(z) \in S_q{}^*(\beta)$ is said to be q-starlike of order $\beta\ in\ U$.

Let $K_q(\beta)$ be the subclass of A. A function $f(z) \in K_q(\beta)$ is said to be q- convex of order β if it satisfies

$$Re\left\{\frac{\partial_q(z\,\partial_q f(z))}{\partial_q(z)}\right\} > \beta\ ,\quad z \in U. \tag{4}$$

For some $\beta\ (0 \le \beta < 1)$.

Note that as $q \to 1$ the classes $S_q^*(\beta)$ and $K_q(\beta)$ reduce to the classes $S^*(\beta)$ and $K(\beta)$ respectively.

We introduce the class $SD_q(\alpha, \beta)$ as the subclass of A consisting of all functions $f(z)$ satisfying

$$Re\left\{\frac{z\partial_q f(z)}{f(z)}\right\} > \alpha\left|\frac{z\partial_q f(z)}{f(z)} - 1\right| + \beta\quad ,\ z \in U \tag{5}$$

for some $\alpha \ge 0$ and $\beta\ (0 \le \beta < 1)$.

We also introduce the class $KD_q(\alpha, \beta)$ as the subclass of A consisting of all functions $f(z)$ which obeying the condition

$$Re\left\{\frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)}\right\} > \alpha\left|\frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)} - 1\right| + \beta\,,\qquad z \in U \tag{6}$$

for some $\alpha \ge 0$ and $\beta\ (0 \le \beta < 1)$.Then we note that $f(z) \in KD_q(\alpha, \beta) \Leftrightarrow z\partial_q f(z) \in SD_q(\alpha, \beta)$

As $q \to 1$ we get the classes $SD(\alpha, \beta)$ and $KD(\alpha, \beta)$ introduced by Owa and Polatoglu [3].

Now we prove coefficient inequalities for functions in these classes.

2. Main Results

Theorem 2.1. If $f(z) \in SD_q(\alpha, \beta)$ with $0 \le \alpha \le \beta\ or\ \alpha > \frac{1+\beta}{2}$ then $f(z) \in S_q{}^*\left(\frac{\beta-\alpha}{1-\alpha}\right)$.

Proof : Since $Re(w) \le |w|$ for any complex number w , $f(z) \in SD_q(\alpha, \beta)$ implies that

$$Re\left\{\frac{z\partial_q f(z)}{f(z)}\right\} > \alpha\left|\frac{z\partial_q f(z)}{f(z)} - 1\right| + \beta \tag{7}$$

or that

$$Re\left\{\frac{z\partial_q f(z)}{f(z)}\right\} > \frac{\beta-\alpha}{1-\alpha}\quad ,\ z \in U\,. \tag{8}$$

If $0 \le \alpha \le \beta$, then we have that

$$0 \le \frac{\beta - \alpha}{1 - \alpha} < 1\,,\quad z \in U\,.$$

and if $\alpha > \frac{1+\beta}{2}$, then we have

$$-1 < \frac{\alpha - \beta}{\alpha - 1} \le 0.$$

Corollary 2.2 : As $q \to 1$ in the above Theorem we get the Theorem 2.1 in [3] which states that

If $f(z) \in SD\,(\alpha, \beta)$ with $0 \le \alpha \le \beta$ or $\alpha > \frac{1+\beta}{2}$ then $f(z) \in \; S^*\left(\frac{\beta-\alpha}{1-\alpha}\right)$

Corollary 2.3 : If $f(z) \in KD_q(\alpha\,, \beta)$ with $0 \le \alpha \le \beta \;\; or \; \alpha > \frac{1+\beta}{2}$, then $f(z) \in K_q\left(\frac{\beta-\alpha}{1-\alpha}\right)$.

Corollary 2.4 : As $q \to 1$ in the above corollary we get the Corollary 2.2 in [3] which reads as

If $f(z) \in KD\,(\alpha, \beta)$ with $0 \le \alpha \le \beta \;\; or \; \alpha > \frac{1+\beta}{2}$, then $f(z) \in K\left(\frac{\beta-\alpha}{1-\alpha}\right)$.

Theorem 2.5 : If $f(z) \in SD_q(\alpha\,, \beta)$ then

$$|a_2| \le \frac{2(1-\beta\,)}{(\,[2]_q - 1\,)\,|1-\alpha|} \tag{9}$$

and

$$|a_n| \le \frac{2(1-\beta)}{(\,[n]_q - 1\,)|1-\alpha|} \prod_{j=1}^{n-2}\left[1 + \frac{2\,(1-\beta)}{(\,[j+1]_q - 1)|1-\alpha|}\right], (n \ge 3) \tag{10}$$

Proof : Note that, for $f(z) \in SD_q(\,\alpha\,, \beta\,)$,

$$Re\left\{\frac{z\partial_q f(z)}{f(z)}\right\} > \frac{\beta - \alpha}{1 - \alpha}, \qquad (z \in U)$$

If we define the function $p(z)$ by

$$p(z) = \frac{(1-\alpha)\frac{z\partial_q f(z)}{f(z)} - (\,\beta - \alpha)}{1-\beta}\;, \quad (\,z \in U\,) \tag{11}$$

then $p(z)$ is analytic in U with $Re\{p(z)\} > 0$, $(\,z \in U)$.

Letting $(z) = 1 + p_1 z + p_2 z^2 + \,.\,.\,.\,.\,$, we have

$$z\,\partial_q f(z) = f(z)\left(1 + \frac{1-\beta}{1-\alpha}\sum_{n=1}^{\infty} p_n z^n\right) \tag{12}$$

Therefore, equation (12) implies that

$$\left(\,[n]_q - 1\right)a_n = \frac{1-\beta}{1-\alpha}\,(p_{n-1} + a_2 p_{n-2} + a_3 p_{n-3} + .\;\;.\;\;\; . + a_{n-1} p_1). \tag{13}$$

Applying the coefficient estimates such that $|p_n| \le 2\,(\,n \ge 1)$ [1] for Caratheodary functions, we obtain that

$$|a_n| \le \frac{2(1-\beta)}{([n]_q-1)|1-\alpha|}\,[\,1+|a_2|+|a_3|+\ldots\ldots+|a_{n-1}|\,]. \tag{14}$$

Therefore, for $n = 2$,

$$|a_2| \le \frac{2(1-\beta)}{\big([2]_q-1\big)\,|1-\alpha|}$$

which proves (9), and for $n = 3$,

$$|a_3| \le \frac{2(1-\beta)}{\big([3]_q-1\big)|1-\alpha|}\left[1+\frac{2(1-\beta)}{\big([2]_q-1\big)|1-\alpha|}\right].$$

Thus (10) holds true for $n = 3$.

Suppose that (10) is true for $n = 3, 4, 5, \ldots, k$, we see that

$|a_{k+1}|$

$$\le \frac{2(1-\beta)}{\big([k+1]_q-1\big)|1-\alpha|}\left\{\begin{array}{c} 1+\dfrac{2(1-\beta)}{\big([2]_q-1\big)|1-\alpha|}+\dfrac{2(1-\beta)}{\big([3]_q-1\big)|1-\alpha|}\left(1+\dfrac{2(1-\beta)}{\big([2]_q-1\big)|1-\alpha|}\right) \\ +\ \ldots+\dfrac{2(1-\beta)}{\big([k]_q-1\big)|1-\alpha|}\displaystyle\prod_{j=1}^{k-2}\left(1+\dfrac{2(1-\beta)}{\big([j+1]_q-1\big)|1-\alpha|}\right) \end{array}\right\}$$

$$= \frac{2(1-\beta)}{\big([k+1]_q-1\big)|1-\alpha|}\prod_{j=1}^{k-1}\left(1+\frac{2(1-\beta)}{\big([j+1]_q-1\big)|1-\alpha|}\right).$$

Consequently , using mathematical induction, we have proved that (10) holds true for all $n \ge 3$.

Corollary 2.6 : As $q \to 1$ in the above theorem we obtain the Theorem 2.3 in [3] which states that

If $f(z) \in SD(\alpha, \beta)$, then

$$|a_2| \le \frac{2(1-\beta)}{|1-\alpha|}$$

and

$$|a_n| \le \frac{2(1-\beta)}{(n-1)|1-\alpha|}\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{j|1-\alpha|}\right),\ (n \ge 3).$$

Corollary 2.7 : Putting $\alpha = 0$ in the Corollary 2.6, we get

$$|a_n| \le \frac{\prod_{j=2}^{n}(j-2\beta)}{(n-1)!}, \qquad (n \ge 2).$$

which was given by Robertson [2] .

Since $f(z)\in KD_q(\alpha,\beta)\Leftrightarrow zD_qf(z)\in SD_q(\alpha,\beta)$, we have

Corollary 2.8. $If\ f(z)\in KD_q(\alpha,\beta)$, then

$$|a_2|\le\frac{2(1-\beta)}{[2]_q\big([2]_q-1\big)|1-\alpha|}$$

and

$$|a_n|\le\frac{2(1-\beta)}{[n]_q\big([n]_q-1\big)|1-\alpha|}\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{\big([j+1]_q-1\big)|1-\alpha|}\right),(n\ge 3).$$

Corollary 2.9 : As $q\to 1$ in the above Corollary ,we obtain Corollary 2.5 in [3] which reads as

If $f(z)\in KD(\alpha,\beta)$, then

$$|a_2|\le\frac{1-\beta}{|1-\alpha|}$$

and

$$|a_n|\le\frac{2(1-\beta)}{n(n-1)|1-\alpha|}\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{j|1-\alpha|}\right),(n\ge 3).$$

Corollary 2.10 : Putting $\alpha=0$ in Corollary 2.9 , we see that

$$|a_n|\le\frac{\prod_{j=2}^{n}(j-2\beta)}{n!}\quad(n\ge 2).$$

a result by Robertson [2].

Theorem 2.11 : If $f(z)\in SD_q(\alpha,\beta)$, then

$$max\left\{0,|z|-\frac{2(1-\beta)}{\big([2]_q-1\big)|1-\alpha|}|z|^2 -\sum_{n=3}^{\infty}\frac{2(1-\beta)}{\big([n]_q-1\big)|1-\alpha|}\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{\big([j+1]_q-1\big)|1-\alpha|}\right)|z|^n\right\}\le|f(z)|$$

$$\leq |z| + \frac{2(1-\beta)}{([2]_q - 1)|1-\alpha|}|z|^2 + \sum_{n=3}^{\infty} \frac{2(1-\beta)}{([n]_q - 1)|1-\alpha|} \prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{([j+1]_q - 1)|1-\alpha|}\right)|z|^n$$

and

$$max\left\{0\,, 1 - \frac{2[2]_q(1-\beta)}{([2]_q - 1)|1-\alpha|}\;|z| - \sum_{n=3}^{\infty} \frac{2[n]_q(1-\beta)}{([n]_q - 1)|1-\alpha|} \prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{([j+1]_q - 1)|1-\alpha|}\right)|z|^{n-1}\right\}$$

$$\leq \left|\partial_q(z)\right|$$

$$\leq\; 1 + \frac{2[2]_q(1-\beta)}{([2]_q - 1)|1-\alpha|}|z| + \sum_{n=3}^{\infty} \frac{2[n]_q(1-\beta)}{([n]_q - 1)|1-\alpha|} \prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{([j+1]_q - 1)|1-\alpha|}\right)|z|^{n-1}$$

Remark 2.12: As $q \to 1$ in the above Theorem we get the Theorem (2.7) in [3]

Corollary 2.13 : If $f(z) \in KD_q(\alpha\,,\beta)$, then

$$max\left\{0\,, |z| - \frac{2(1-\beta)}{[2]_q([2]_q - 1)|1-\alpha|}\;|z|^2 - \sum_{n=3}^{\infty} \frac{2(1-\beta)}{[n]_q([n]_q - 1)|1-\alpha|} \prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{([j+1]_q - 1)|1-\alpha|}\right)|z|^n\right\}$$

$$\leq |f(z)|$$

$$\leq |z| + \frac{2(1-\beta)}{[2]_q([2]_q - 1)|1-\alpha|}\;|z|^2 + \sum_{n=3}^{\infty} \frac{2(1-\beta)}{[n]_q([n]_q - 1)|1-\alpha|} \prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{([j+1]_q - 1)|1-\alpha|}\right)|z|^n$$

and

$$max\left\{0\,,1-\frac{2(1-\beta)}{\left([2]_q-1\right)|1-\alpha|}|z| - \sum_{n=3}^{\infty}\frac{2(1-\beta)}{\left([n]_q-1\right)|1-\alpha|}\prod_{j=1}^{n-1}\left(1+\frac{2(1-\beta)}{\left([j+1]_q-1\right)|1-\alpha|}\right)|z|^{n-1}\right\}$$

$$\leq \left|\partial_q(z)\right|$$

$$\leq 1+\frac{2(1-\beta)}{\left([2]_q-1\right)|1-\alpha|}|z| + \sum_{n=3}^{\infty}\frac{2(1-\beta)}{\left([n]_q-1\right)|1-\alpha|}\prod_{j=1}^{n-1}\left(1+\frac{2(1-\beta)}{\left([j+1]_q-1\right)|1-\alpha|}\right)|z|^{n-1}$$

Remark 2.14: As $q \to 1$ in the corollary 2.13 we obtain the Corallay 2.8 in [3].

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