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A SUBCLASS OF HARMONIC MULTIVALENT FUNCTIONS WITH VARYING ARGUMENTS DEFINED BY GENERALIZED SALAGEAN DERIVATIVE OPERATOR

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Abstract: Making use of generalized Salagean derivative operator, we introduce a new subclass of multivalent harmonic functions in the open unit disc. Coefficient bounds, neighborhood and extreme points for this generalized class of functions have been investigated.

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1. Introduction

A continuous complex valued function $f = u + iv$ defined in simple connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simple connected domain we can write

$$f = h + \bar{g}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ for all z in D (see Clunie and Shiel-Small[3]).

Let H denotes the class of functions $f = h + \bar{g}$ which are harmonic univalent and sense preserving in the unit disc $U = \{z: |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. then for $f = h + \bar{g} \in H$, we may express the analytic function h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad z \in U, |b_1| < 1. \tag{1}$$

Recently , Ahuja and Jahangiri [1] defined the class $H_p(k)$, $(p,k \in N = \{1,2,3,\ldots\ldots\})$ consisting of all p-valent harmonic functions $f = h + \bar{g}$ which are sense preserving in U and h and g are of the form

$$h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \quad g(z) = \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \tag{2}$$

where $z \in U, |b_p| < 1$.

Note that H *and* $H_p(k)$ reduce to the class S and $S_p(k)$ of analytic univalent and multivalent functions, respectively, if the co-analytic part of its members are zero. For these classes $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{3}$$

and

$$f(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}. \tag{4}$$

In 1984, Clunie and Sheli-Small [3] investigated the class S_H as well as its geometric sub classes and obtained some coefficient bounds (see also [4] and [9]). Since then, there have been several related papers on S_H and its subclasses. In 2012, Eljamal and Darus [5] introduced a generalized derivative operator for $f = h + \bar{g}$ given by (1) for fixed positive natural number m and $\lambda_2 \geq \lambda_1 \geq 0$, they proved that for $\lambda_1 = \lambda_2 = 0$ their subclass reduces to Salagean operator but it is not true. In fact the condition $\lambda_2 \geq \lambda_1 \geq 0$ should be replaced by $\lambda_1, \lambda_2 \geq 0$.Thus we will introduce a generalized derivative operator for $f = h + \bar{g}$ given by (2) for fixed positive natural number m and $\lambda_1, \lambda_2 \geq 0$,

$$D_{\lambda_1,\lambda_2}^{m,k} f(z) = D_{\lambda_1,\lambda_2}^{m,k} h(z) + \overline{D_{\lambda_1,\lambda_2}^{m,k} g(z)}, \qquad z \in U, \tag{5}$$

where

$$D_{\lambda_1,\lambda_2}^{m,k} h(z) = z^p + \sum_{k=2}^{\infty} \left(\frac{1 + (\lambda_1 + \lambda_2)(k + p - 2)}{p\{1 + \lambda_2(k + p - 2)\}} \right)^m a_{k+p-1}\, z^{k+p-1},$$

$$D_{\lambda_1,\lambda_2}^{m,k} g(z) = \sum_{k=1}^{\infty} \left(\frac{1 + (\lambda_1 + \lambda_2)(k + p - 2)}{p\{1 + \lambda_2(k + p - 2)\}} \right)^m b_{k+p-1}\, z^{k+p-1}.$$

We note that the specializing the parameters, especially when $\lambda_1 = 1, \lambda_2 = 0$, $D_{\lambda_1,\lambda_2}^{m,k}$ reduces to modified Salagean operator introduced by Jahangiri et al. [6], and when $\lambda_1 = 1, \lambda_2 = 0$, and $\mathrm{p} = 1, D_{\lambda_1,\lambda_2}^{m,k}$ reduces to Salagean operator which was studied by Salagean in [11].

Now we consider the following definition.

Definition 1.1 : For $0 \le l < 1, \beta \ge 0$, let $G_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$ denote the subfamily of starlike harmonic function $f \in H_p(k)$ of the form (1.02) such that

$$Re\left\{(1+\beta e^{i\varphi})\frac{z\left(D^{m,k}_{\lambda_1,\lambda_2}f(z)\right)'}{z'\left(D^{m,k}_{\lambda_1,\lambda_2}f(z)\right)} - \beta e^{i\varphi}\right\} \ \ge \ l \tag{6}$$

For a suitable φ and $z \in U$ where $\left(D^{m,k}_{\lambda_1,\lambda_2}f(z)\right)' = \left(\frac{\partial}{\partial\theta}\right)(D^{m,k}_{\lambda_1,\lambda_2}f\left(re^{i\theta}\right),\ z' = \left(\frac{\partial}{\partial\theta}\right)\left(z = re^{i\theta}\right)$.

We also let $V_H(l,\beta,p,m,k,\lambda_1,\lambda_2) = G_H(l,\beta,p,m,k,\lambda_1,\lambda_2) \cap V_H^p$ where V_H^p is the class of harmonic function consisting of functions of the form (1.02) in $H_p(k)$ for which there exist a real number φ such that

$$\eta_k + (k-1)\varphi \equiv \pi(mod\ 2\pi), \delta_k + (k-1)\varphi \equiv 0 \qquad (k \ge 2), \tag{7}$$

where $\eta_k = \arg(a_k)$ and $\delta_k = \arg(b_k)$. The same class was introduced in [8] with different differential operator.

In this paper we obtain a sufficient coefficient condition for the functions f given by (2) to be in the class $G_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$. It is shown that this coefficient condition is necessary also for functions belonging to the class $V_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$. Further extreme points for functions in $V_H(l,\beta.p,m,k,\lambda_1,\lambda_2)$ are also obtained.

2. Main Result : We begin deriving a sufficient condition for the functions belonging to the class $G_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$. This result is contained in the following.

Theorem 2.1. Let $f = h + \bar{g}$ given by (2). Furthermore, let

$$\sum_{k=2}^{\infty}\Big(\frac{k+p-1+(k+p-2)\beta-l}{p-l}\left|a_{k+p-1}\right|$$
$$+\frac{k+p-1+(k+p)\beta+l}{p-l}\left|b_{k+p-1}\right|\Big)\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m$$
$$\le 1 - \frac{p+(p+1)\beta+l}{p+(p+1)\beta-l}\left(\frac{1+(\lambda_1+\lambda_2)(p-1)}{p\{1+\lambda_2(p-1)\}}\right)^m \left|b_p\right|, \tag{8}$$

where $0 \le l < 1$, then $f \in G_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$.

Proof. Using $f = h + \bar{g}$ in (6), we get

$$Re\left\{(1+\beta e^{i\varphi})\frac{z\left(D^{m,k}_{\lambda_1,\lambda_2}h(z)\right)' - \overline{z\left(D^{m,k}_{\lambda_1,\lambda_2}g(z)\right)'}}{\left(D^{m,k}_{\lambda_1,\lambda_2}h(z)\right)+\overline{\left(D^{m,k}_{\lambda_1,\lambda_2}g(z)\right)}} - \beta e^{i\varphi}\right\} = Re\frac{A(z)}{B(z)} \tag{9}$$

where

$$A(z) = \left(1+\beta e^{i\varphi}\right)\left[z\left(D^{m,k}_{\lambda_1,\lambda_2}h(z)\right)' - \overline{z\left(D^{m,k}_{\lambda_1,\lambda_2}g(z)\right)'}\right] - \beta e^{i\varphi}\left[\left(D^{m,k}_{\lambda_1,\lambda_2}h(z)\right) + \overline{\left(D^{m,k}_{\lambda_1,\lambda_2}g(z)\right)}\right] \tag{10}$$

and

$$B(z) = \left[\left(D^{m,k}_{\lambda_1,\lambda_2}h(z)\right) + \overline{\left(D^{m,k}_{\lambda_1,\lambda_2}g(z)\right)}\right]. \tag{11}$$

In view of the simple assertion that $Re(\omega) \geq l$ if and only if $|1-l+\omega| \geq |1+l-\omega|$, it is sufficient to show that

$$|A(z)+(1-l)B(z)| - |A(z)-(1+l)B(z)| \geq 0. \tag{12}$$

Substituting for $A(z)$ and $B(z)$ the appropriate expressions in (12) we get

$$|A(z)+(1-l)B(z)| - |A(z)-(1+l)B(z)| =$$

$$\left|\left(1+\beta e^{i\varphi}\right)\left[z\left(D^{m,k}_{\lambda_1,\lambda_2}h(z)\right)' - \overline{z\left(D^{m,k}_{\lambda_1,\lambda_2}g(z)\right)'}\right] - \beta e^{i\varphi}\left[\left(D^{m,k}_{\lambda_1,\lambda_2}h(z)\right) + \overline{\left(D^{m,k}_{\lambda_1,\lambda_2}g(z)\right)}\right] + (1-l)\left[\left(D^{m,k}_{\lambda_1,\lambda_2}h(z)\right) + \overline{\left(D^{m,k}_{\lambda_1,\lambda_2}g(z)\right)}\right]\right|$$

$$-\left|\left(1+\beta e^{i\varphi}\right)\left[z\left(D^{m,k}_{\lambda_1,\lambda_2}h(z)\right)' - \overline{z\left(D^{m,k}_{\lambda_1,\lambda_2}g(z)\right)'}\right] - \beta e^{i\varphi}\left[\left(D^{m,k}_{\lambda_1,\lambda_2}h(z)\right) + \overline{\left(D^{m,k}_{\lambda_1,\lambda_2}g(z)\right)}\right] - (1+l)\left[\left(D^{m,k}_{\lambda_1,\lambda_2}h(z)\right) + \overline{\left(D^{m,k}_{\lambda_1,\lambda_2}g(z)\right)}\right]\right|$$

$$= \left| (1+\beta e^{i\varphi})\left[pz^p + \sum_{k=2}^{\infty}(k+p-1)\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m a_{k+p-1}z^{k+p-1} - \sum_{k=1}^{\infty}(k+p-1)\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m b_{k+p-1}z^{k+p-1}\right] \right.$$
$$- \beta e^{i\varphi}\left[z^p + \sum_{k=2}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m a_{k+p-1}z^{k+p-1} + \sum_{k=1}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m b_{k+p-1}z^{k+p-1}\right]$$
$$\left. + (1-l)\left[z^p + \sum_{k=2}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m a_{k+p-1}z^{k+p-1} + \sum_{k=1}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m b_{k+p-1}z^{k+p-1}\right]\right|$$

$$- \left| (1+\beta e^{i\varphi})\left[pz^p + \sum_{k=2}^{\infty}(k+p-1)\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m a_{k+p-1}z^{k+p-1} - \sum_{k=1}^{\infty}(k+p-1)\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m b_{k+p-1}z^{k+p-1}\right] \right.$$
$$- \beta e^{i\varphi}\left[z^p + \sum_{k=2}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m a_{k+p-1}z^{k+p-1} + \sum_{k=1}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m b_{k+p-1}z^{k+p-1}\right]$$
$$\left. - (1+l)\left[z^p + \sum_{k=2}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m a_{k+p-1}z^{k+p-1} + \sum_{k=1}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m b_{k+p-1}z^{k+p-1}\right]\right|$$

$$= \left| \begin{array}{c} [(p+1-l)+\beta(p-1)e^{i\varphi}]z^p + \sum_{k=2}^{\infty}[(k+p-l)+\beta(k+p-2)e^{i\varphi}]\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m a_{k+p-1}\, z^{k+p-1} \\ -\sum_{k=1}^{\infty}[(k+p-2+l)+\beta(k+p)e^{i\varphi}]\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m b_{k+p-1}\, z^{k+p-1} \end{array} \right|$$

$$- \left| \begin{array}{c} [(p-1-l)+\beta(p-1)e^{i\varphi}]z^p + \sum_{k=2}^{\infty}[(k+p-2-l)+\beta(k+p-2)e^{i\varphi}]\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m a_{k+p-1}\, z^{k+p-1} \\ -\sum_{k=1}^{\infty}[(k+p+l)+\beta(k+p)e^{i\varphi}]\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m b_{k+p-1}\, z^{k+p-1} \end{array} \right|$$

$$\geq [(\{p+1-l-\beta(p-1)\}+\{p-1-l+\beta(p-1)\})|z|^p - \sum_{k=2}^{\infty}[\{k+p-l+\beta(k+p-2)\} + \{k+p-2-l + \beta(k+p-2)\}]\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m |a_{k+p-1}|\,|z|^{k+p-1} - \sum_{k=1}^{\infty}[\{k+p-2+l+\beta(k+p)\} + \{k+p+l + \beta(k+p)\}]\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m |b_{k+p-1}|\,|z|^{k+p-1}]$$

$$= 2(p-l)|z|^p - \sum_{k=2}^{\infty} 2[k+p-1+\beta(k+p-2) - l]\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m |a_{k+p-1}|\,|z|^{k+p-1} - \sum_{k=1}^{\infty} 2[k+p-1+\beta(k+p) + l]\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m |b_{k+p-1}|\,|z|^{k+p-1}$$

$$\geq$$

$$2(p-l)|z|^p[1-\left(\frac{p+\beta(p+1)+l}{p-l}\right)\left(\frac{1+(\lambda_1+\lambda_2)(p-1)}{p\{1+\lambda_2(p-1)\}}\right)^m|b_p|$$
$$-\sum_{k=2}^{\infty}\left\{\left(\frac{k+p-1+\beta(k+p-2)-l}{p-l}\right)\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m|a_{k+p-1}|\right.$$
$$\left.+\left(\frac{k+p-1+\beta(k+p)+l}{p-l}\right)\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m|b_{k+p-1}|\right\}]$$
$$\geq 0.$$

By virtue of inequality (8). This implies that $f \in G_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$.

Now we obtain the necessary and sufficient condition for the function $f = h + \bar{g}$ be given by (2) with condition (7).

Theorem 2.2. Let $f = h + \bar{g}$ be given by (1.02).Then $f \in V_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$ if and only if

$$\sum_{k=2}^{\infty}\Big(\frac{k+p-1+(k+p-2)\beta-l}{p-l}|a_{k+p-1}|$$
$$+\frac{k+p-1+(k+p)\beta+l}{p-l}|b_{k+p-1}|\Big)\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m$$
$$\leq 1-\frac{p+(p+1)\beta+l}{p+(p+1)\beta-l}\left(\frac{1+(\lambda_1+\lambda_2)(p-1)}{p\{1+\lambda_2(p-1)\}}\right)^m|b_p|, \qquad (13)$$

where $0 \leq l < 1$.

Proof. Since $V_H(l,\beta,p,m,k,\lambda_1,\lambda_2) \subset G_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$, we only need to prove 'only if' part of the theorem.

To this end, for functions f of the form (1.02) with condition (1.07), we notice that condition

$$Re\left\{(1+\beta e^{i\varphi})\frac{z\left(D^{m,k}_{\lambda_1,\lambda_2}h(z)\right)'-\overline{z\left(D^{m,k}_{\lambda_1,\lambda_2}g(z)\right)'}}{\left(D^{m,k}_{\lambda_1,\lambda_2}h(z)\right)+\overline{\left(D^{m,k}_{\lambda_1,\lambda_2}g(z)\right)}}-(\beta e^{i\varphi}+l)\right\}\geq 0. \qquad (14)$$

The above inequality is equivalent to

$$Re\left[\left\{(1+\beta e^{i\varphi})\left[pz^p+\sum_{k=2}^{\infty}(k+p-1)\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m a_{k+p-1}\,z^{k+p-1}\right.\right.\right.$$
$$\left.-\sum_{k=1}^{\infty}(k+p-1)\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m b_{k+p-1}\,\bar{z}^{k+p-1}\right]-(\beta e^{i\varphi}$$
$$+l)\left[z^p+\sum_{k=2}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m a_{k+p-1}\,z^{k+p-1}\right.$$
$$\left.\left.+\sum_{k=1}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m b_{k+p-1}\,\bar{z}^{k+p-1}\right]\right\}$$
$$\times\left[z^p+\sum_{k=2}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m a_{k+p-1}\,z^{k+p-1}\right.$$
$$\left.\left.+\sum_{k=1}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m b_{k+p-1}\,\bar{z}^{k+p-1}\right]^{-1}\right]$$

$=$

$$Re\left\{\left[\{p+\beta(p-1)e^{i\varphi}-l\}\right.\right.$$
$$+\sum_{k=2}^{\infty}\{k+p-1+\beta(k+p-2)e^{i\varphi}$$
$$-l\}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m a_{k+p-1}\,z^{k-1}$$
$$-\frac{\bar{z}^p}{z^p}\sum_{k=1}^{\infty}\{k+p-1+\beta(k+p)e^{i\varphi}$$
$$\left.+l\}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m b_{k+p-1}\,\bar{z}^{k-1}\right]$$
$$\times\left[1+\sum_{k=2}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m a_{k+p-1}\,z^{k-1}\right.$$
$$\left.\left.+\frac{\bar{z}^p}{z^p}\sum_{k=1}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m b_{k+p-1}\,\bar{z}^{k-1}\right]^{-1}\right\}$$

$$\geq\quad 0. \tag{15}$$

Thus condition must hold for all values of z, such that $|z|=r<1$. Upon choosing φ according to (7) and noting that $Re\left(-\beta e^{-i\varphi}\right)\geq-\beta\left|e^{i\varphi}\right|=-\beta$ the above inequality reduces to

$$\left[\begin{array}{c} \{p+\beta(p-1)-l\}-\{p+\beta(p+1)+l\}\left(\frac{1+(\lambda_1+\lambda_2)(p-1)}{p\{1+\lambda_2(p-1)\}}\right)^m b_p \\ -(\sum_{k=2}^{\infty}\{k+p-1+\beta(k+p-2)-l\}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m |a_{k+p-1}|\, r^{k-1} \\ +\{k+p-1+\beta(k+p)+l\}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m |b_{k+p-1}| r^{k-1}) \end{array}\right]$$
$$\times\left[1-\sum_{k=2}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m |a_{k+p-1}|\, r^{k-1} + \sum_{k=1}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m |b_{k+p-1}|\, r^{k-1}\right]^{-1}$$
$$\geq\ 0. \tag{16}$$

If (13) does not hold, then the numerator in (16) is negative for r sufficiently close to 1. Therefore, there exist a point $z_0 = r_0$ in (0,1) for which the quotient in (10) is negative. This contradicts our assumptions that $f \in V_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$. We thus conclude that it is both necessary and sufficient that the coefficient bounds inequality (13) holds true when $f \in V_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$. This completes the proof of Theorem 2.2.

Theorem 2.3 : The close convex hull of $f \in V_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$ (denote by clco $V_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$) is

$$\left\{f(z) = z^p + \sum_{k=2}^{\infty}|a_{k+p-1}|z^{k+p-1} + \sum_{k=1}^{\infty}|b_{k+p-1}|\,\bar{z}^{k+p-1} \;:\; \sum_{k=2}^{\infty}(k+p-1)[|a_k|+|b_k|] \leq 1-b_p\right\}. \tag{17}$$

For $\lambda_k = \frac{p-l}{\{k+p-1+\beta(k+p-2)-l\}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m}$,

and $\mu_k = \frac{p-l}{\{k+p-1+\beta(k+p)+l\}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m}$, and b_p fixed, the extreme points for $clcoV_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$ are

$$\left\{z^p + \lambda_k x z^{k+p-1} + \overline{b_p z^p}\right\} \cup \left\{z^p + \overline{b_p z^p + \mu_k y z^{k+p-1}}\right\} \tag{18}$$

where $k \geq 2$ and $|x| = 1 - |b_p|$.

Proof. Any function f in clco $V_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$ may be expressed as

$$f(z) = z^p + \sum_{k=2}^{\infty} |a_{k+p-1}| e^{i\eta_k} z^{k+p-1} + \overline{b_p z^p} + \overline{\sum_{k=2}^{\infty} |b_{k+p-1}| e^{i\delta_k} z^{k+p-1}}, \tag{19}$$

where the coefficient satisfy the inequality (8) . Set

$h_1(z) = z^p$, $g_1(z) = b_p z^p$, $h_k(z) = z^p + \lambda_k e^{i\eta_k} z^{k+p-1}$, $g_k(z) = b_p z^p + \mu_k e^{i\delta_k} z^{k+p-1}$ for $k = 2,3,...$ writing

$x_k = \frac{|a_{k+p-1}|}{\lambda_k}$, $y_k = \frac{|b_{k+p-1}|}{\mu_k}$, $k = 2,3,...$ and

$x_1 = 1 - \sum_{k=2}^{\infty} x_{k+p-1}$, $y_1 = 1 - \sum_{k=2}^{\infty} y_{k+p-1}$ we get

$$f(z) = \sum_{k=1}^{\infty} \{x_{k+p-1} h_k(z) + y_{k+p-1} g_k(z)\}. \tag{20}$$

In particular, setting $f_1(z) = z^p + \overline{b_p z^p}$,

$$f_k(z) = z^p + \lambda_k x z^{k+p-1} + \overline{b_p z^p} + \overline{\mu_k y z^{k+p-1}}, \quad (k \geq 2, |x| + |y| = 1 - |b_p|). \tag{21}$$

We see that extreme points of $V_H(l, \beta, p, m, k, \lambda_1, \lambda_2) \subset \{f_k(z)\}$.

To see that $f_1(z)$ is not in extreme points , note that $f_1(z)$ may be written as

$$f_1(z) = \frac{1}{2}\{f_1(z) + \lambda_2(1 - |b_p|)z^{1+p}\} + \frac{1}{2}\{f_1(z) - \lambda_2(1 - |b_p|)z^{1+p}\}, \tag{22}$$

a convex linear combination of function in *clco* $V_H(l, \beta, p, m, k, \lambda_1, \lambda_2)$.

To see that f_k is not an extreme point if both $|x| \neq 0$ and $|y| \neq 0$ we will show that it can also be expressed as a convex linear combination of function in clco $V_H(l, \beta, p, m, k, \lambda_1, \lambda_2)$. Without loss of generality, assume $|x| \geq |y|$ choose $\epsilon > 0$ small enough so that $\epsilon > \frac{|x|}{|y|}$.

Set $A = 1 + \epsilon$ and $B = 1 - \left|\epsilon \frac{x}{y}\right|$. We then show that both

$$t_1(z) = z^p + \lambda_k A x z^{k+p-1} + \overline{b_p z^p + \mu_k y B z^{k+p-1}} \tag{23}$$

$$t_2(z) = z^p + \lambda_k (2 - A) x z^{k+p-1} + \overline{b_p z^p + \mu_k y (2 - B) z^{k+p-1}}$$

are in clco $V_H(l, \beta, p, m, k, \lambda_1, \lambda_2)$ and that

$$f_k(z) = \frac{1}{2}\{t_1(z) + t_2(z)\}. \tag{24}$$

The extremal coefficient bounds show that functions of the form (2.14) are the extreme points for clco $V_H(l, \beta, p, m, k, \lambda_1, \lambda_2)$ and so the proof is complete.

Following Avci and Zlotkiewicz [2], we refer to the δ-neghiborhood of the functions $f(z)$ defined by (1.01) to be set of functions F for which

$$N_\delta(f)=\{F(z)=z+\sum_{k=2}^{\infty}A_k z^k+\sum_{k=1}^{\infty}\overline{B_k z^k},$$
$$\sum_{k=2}^{\infty}k[|a_k-A_k|+|b_k-B_k|+|b_1-B_1|]\le\delta\}$$

Following Eljamal and M. Darus [5] and ([2], [8]), we refer to the δ-neghiborhood of the functions $f(z)$ defined by (1.01) to be set of functions F for which

$$N_\delta(f)=\{F(z):\sum_{k=2}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)}\right)^m[(2k-1-l)|a_k-A_k|+(2k+1+l)|b_k-B_k|+(1-l)|b_1-B_1|]\le(1-l)\delta\}. \tag{25}$$

In our case, let us define the generalized δ-neghibourhood of f to be the set

$$N_\delta(f)=\{F(z)=z^p+\sum_{k=2}^{\infty}A_{k+p-1}z^{k+p-1}+\sum_{k=1}^{\infty}\overline{B_{k+p-1}z^{k+p-1}},$$
$$\sum_{k=2}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m[(k+p-1+\beta\{k+p-2\}-l)\left|a_{k+p-1}-A_{k+p-1}\right|+(k+p-1+\beta\{k+p\}+l)\left|b_{k+p-1}-B_{k+p-1}\right|+(p-l)\left|b_p-B_p\right|]\le(p-l)\delta\}. \tag{26}$$

Theorem 2.4 : Let f be given by (2). If f satisfies conditions

$$\sum_{k=2}^{\infty}(k+p-1)(k+p-1+\beta\{k+p-2\}-l)\left|a_{k+p-1}\right|\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m+\sum_{k=1}^{\infty}(k+p-1)(k+p-1+\beta\{k+p\}+l)|b_k|\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m\le(p-l) \tag{27}$$

where $0\le l<1$ and $\delta=\frac{p-l}{p+(p+1)\beta-l}\left[1-\frac{p+(p+1)\beta+l}{p-l}\left(\frac{1+(\lambda_1+\lambda_2)(p-1)}{p\{1+\lambda_2(p-1)\}}\right)^m\left|b_p\right|\right]$ (28)

Then $N(f)\subset G_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$.

Proof. Let f satisfies (2.21) and $F(z)$ is given by

$$F(z) = z^p + \overline{b_p z^p} + \sum_{k=2}^{\infty}[A_{k+p-1}z^{k+p-1} + \overline{B_{k+p-1}z^{k+p-1}}] \tag{29}$$

which belong to $N(f)$. We obtain

$$[p+(p+1)\beta+l]|B_p| + \sum_{k=2}^{\infty}[\,(k+p-1+\beta\{k+p-2\}-l)|A_{k+p-1}| + (k+p-1+\beta\{k+p\} + l)|B_{k+p-1}|]\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m \leq$$

$$[p+(p+1)\beta+l]|B_p - b_p| + [p+(p+1)\beta+l]|b_p| + \sum_{k=2}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m [(k+p-1+\beta\{k+p-2\} - l)|A_{k+p-1} - a_{k+p-1}| + (k+p-1+\beta\{k+p\} + l)|B_{k+p-1} - b_{k+p-1}|] + \sum_{k=2}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m [(k+p-1+\beta\{k+p-2\} - l)|a_{k+p-1}| + (k+p-1+\beta\{k+p\}+l)|b_{k+p-1}|]$$

$\leq$

$$(p-l)\delta + [p+(p+1)\beta+l]|b_p| + \frac{1}{p+(p+1)\beta-l}\sum_{k=2}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(k+p-2)}{p\{1+\lambda_2(k+p-2)\}}\right)^m [(k+p-1 + \beta\{k+p-2\}-l)|a_{k+p-1}| + (k+p-1+\beta\{k+p\}+l)|b_{k+p-1}|]$$

$\leq$

$$(p-l)\delta + [p+(p+1)\beta+l]|b_p| + \frac{1}{p+(p+1)\beta-l}[(p-l) - (p+(p+1)\beta + l)\left(\frac{1+(\lambda_1+\lambda_2)(p-1)}{p\{1+\lambda_2(p-1)\}}\right)^m |b_p|] \leq \quad p-l$$

Hence for $\delta = \frac{p-l}{p+(p+1)\beta-l}[1 - \frac{p+(p+1)\beta+l}{p-l}\left(\frac{1+(\lambda_1+\lambda_2)(p-1)}{p\{1+\lambda_2(p-1)\}}\right)^m |b_p|]$

we infer that $F(z) \in G_H(l,\beta,p,m,k,\lambda_1,\lambda_2)$ which concludes the proof of Theorem 2.4.

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