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MULTIPLICATIVELY PERFECT AND RELATED NUMBERS

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Abstract: Sandor [4] has discussed multiplicatively perfect (Mperfect) and multiplicatively +perfect numbers (called as M +perfect). In this paper, we have discussed a new perfect number called as multiplicatively – perfect numbers (also called as $M-$ perfect). Further we study about Abundant, Deficient, Harmonic and Unitary analogue of harmonic numbers.

Keywords and Phrases: Multiplicatively perfect numbers, Abundant numbers, Deficient numbers, Harmonic numbers and unitary divisor.

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1. Introduction

Sandor and Egri [3], Sandor [4] have defined multiplicatively perfect and multiplicatively + perfect numbers as follows,

A positive integer $n \geq 1$ is called multiplicatively perfect if

$$R(n) = n^2, \tag{1}$$

where, $R(n)$ is product of divisor function. Let $d_1, d_2, d_3, \dots d_r$ are the divisors of n then

$R(n) = d_1 . d_2 . d_3 . \dots d_r$.

Also, $$R(n) = n^{\frac{d(n)}{2}} \tag{2}$$

Let $R_+(n)$ denotes the product of even divisors of n. We say that n is $M+$ perfect number if

$$R_+(n) = n^2. \tag{3}$$

Now, in this paper we define $M-$ perfect numbers as follows,

let $R_-(n)$ denotes the product of odd divisors of n .We say that n is $M-$ perfect if

$$R_{-}(n) = n^2. \quad (4)$$

Sandor and Egri[3] have discussed all forms of Mperfect and $M+$perfect numbers by following theorem.

Theorem 1.1 *All Mperfect numbers are of the form* $n = p_1 p_2$ *or* $n = {p_1}^3$, *where* $p_1 \neq p_2$ *are arbitrary primes.*

Theorem 1.2 *All* $M+$*perfect numbers are of the form* $n = 8p_1p_2$ *or* $n = 8{p_1}^3$, *where* $p_1 \neq p_2$ *are distinct odd primes*.

In this paper we discuss the form of $M-$ perfect numbers in section 2.

Sandor and Egri [3] called n to be nobly abundant if both $\sigma(n)$ and $d(n)$ are abundant numbers. Similarly n is called nobly deficient if both $\sigma(n)$ and $d(n)$ are deficient, n is called abundant if $\sigma(n) > 2n$ and deficient if $\sigma(n) < 2n$. Sandor and Egri [3] introduced a number $n \in N$ is called harmonic number if $\sigma(n)/nd(n)$.

Many results on harmonic numbers have been proved in Sandor and Egri[3] introduced by Cohen and Sorli [2] or Cohen and Deng [1].

Let $\sigma_e(n)$ and $d_e(n)$ denotes the sum and product of all e - divisors of n. Also that $\sigma_e(1) = 1, d_e(1) = 1$. Straus and Subbarao [5] called n to be e- perfect if

$$\sigma_e(n) = 2n\,. \quad (5)$$

Further, Sandor and Egri [3] introduced that n is called e -Harmonic of type 1, if

$$\sigma_e(n)/nd_e(n), \quad (6)$$

and e -Harmonic of type 2, if $p_e(n)/nd_e(n)$ where

$$p_e(n) = \prod_{i=1}^{r}\left(\sum_{d_i/a_i} {p_i}^{a_i - d_i}\right), \quad n = {p_1}^{a_1}.{p_2}^{a_2} \ldots .. {p_r}^{a_r} \quad (7)$$

And $$p_e(n) = \sum_{d/a} p^{a-d}, n = p^a\,. \quad (8)$$

Sandor defined that $n \in N$ is Modified e -perfect number if

$$p_e(n)/n\;. \quad (9)$$

Sandor and Egri [3] defined Geometric numbers as follows,

let $G(n)$ is geometric mean of divisors of an integer n given by

$$G(n) = [R(n)]^{\frac{1}{d(n)}}\;. \quad (10)$$

A positive integer n is called geometric number if $G(n)$ is an integer.It is clear using (1.1) that

$$G(n) = \sqrt{n}\,, \quad (11)$$

so all Geometric numbers are perfect squares.

In Mladen, Vassilev Missana and Krassimir Atanassov [6] definition of e-perfect number is given as follows:

A positive integer n is called e -perfect number if $\sigma_e(n) = 2n$, (12)

where $\sigma_e(n)$ is the sum of all e divisors of n.

In Cohen and Sorli [2] many results on harmonic numbers has been given, if we consider unitary divisors of n then unitary harmonic numbers can be defined as, A number $n \geq 1$ is called unitary

harmonic if $\sigma^*(n)/nd^*(n)$. (13)

where $d^*(n)$ is the number of unitary divisors of n which is defined as

$$d^*(n) = 2^r, \quad (14)$$

if $n = {p_1}^{\alpha_1}.{p_2}^{\alpha_2} \ldots\ldots {p_r}^{\alpha_r}$, and $\sigma^*(n)$ is unitary divisor function.

Now, $R^*(n)$ is the product of unitary divisors of n which is defined as,

$$R^*(n) = (n)^{\frac{d^*(n)}{2}}. \quad (15)$$

Now, unitary geometric mean of a positive integer is defined as,

$$G^*(n) = (R^*(n))^{\frac{1}{d^*(n)}}, \quad (16)$$

and a positive integer n is called unitary geometric number if $G^*(n)$ is an integer so using (15), we have

$$G^*(n) = \sqrt{n} . \quad (17)$$

Hence a unitary geometric number is infact a perfect square. In section 2 we prove some results about the related numbers discussed as above in this section.

2. Results

Theorem 2.1 *All $M-$ perfect numbers are of the form $n = p_1p_2$ or $n = {p_1}^3$, where $p_1 \neq p_2$ are distinct odd primes.*

Proof: Let $n = 2^kN$ is a $M-$ perfect number, where N is odd positive integer. Then all odd divisors d of n are $d = N_1$,where N_1/N .

Therefore, $R_-(n) = \prod_{N_1/N} N_1 = R(N)$.

Using (1.4) we get $R(N) = 2^{2k}N^2$, because $R(N)$ is always odd as N is odd therefore $k = 0$, so $n = N$, this implies $R(N) = N^2$, thus N is odd M perfect number.

According to theorem (1.1), $n = p_1p_2$ or $n = {p_1}^3$.

Theorem 2.2 *Let $r \geq 3$ and $r = 2^k - 1$ be a mersenne prime then $n = 2r$ has the property that $d(n)$ is deficient , $\sigma(n)$ is abundant and if $r \geq 7$ then n is deficient. If $r < 7$ then n is a perfect number.*

Proof: Let $n = 2r$, where $r \geq 3$ and $r = 2^k - 1$ be a mersenne prime. Then,

$\sigma\big(d(n)\big) = \sigma(4) = 7 < 2.\, d(n) = 8$, so, $\sigma\big(d(n)\big) < 2d(n)$. Hence $d(n)$ is deficient.

Now, $\sigma\big(\sigma(n)\big) = \sigma\big(\sigma(2r)\big) = \sigma(\sigma(r)\sigma(2)) = \sigma\big(3.\, 2^k\big) = \sigma(3)\sigma\big(2^k\big) = 4(2^{k+1} - 1)$.

Since, $2\,\sigma(2r)$=$2\sigma(2)\sigma(r)$= $3.\, 2^{k+1} < 4\big(2^{k+1} - 1\big), \forall\, k \geq 2$. Therefore, $\sigma\big(\sigma(n)\big) > 2\sigma(n)$.

Hence, $\sigma(n)$ is abundant. Since, $\sigma(n) = \sigma(2r) = 3.\, 2^k < 4\big(2^k - 1\big) = 2n,\ \forall\ k > 2$.

Therefore, $\sigma(n) < 2n$. Hence, n is deficient. Since, $r = 3$ and $n = 6$ for $k = 2$ therefore n is a perfect number. Hence, for $r < 7$, n is a perfect number.

Theorem 2.3 *Let $r \geq 3$ and $q = r^2 + r + 1$ is a prime then $n = 2r^2$ has property that $\sigma(n)$ is deficient, $d(n)$ is perfect and n is abundant.*

Proof: Let $n = 2r^2$ **,** $r \geq 3$ is a prime number. Then,

$\sigma(n) = \sigma(2r^2) = \sigma(2)\sigma(r^2) = 3(r^2 + r + 1) = 3q$, where, $q = r^2 + r + 1$ be a prime.

Now, $\sigma\big(\sigma(n)\big) = \sigma(3.\, q) = \sigma(3)\sigma(q) = 4(q + 1) = 4(r^2 + r + 2)$, and

$2\sigma(n) = 2\sigma(2r^2) = 2.3(r^2 + r + 1) = 6(r^2 + r + 1)$. Since, it is clear that

$4(r^2 + r + 2) < 6(r^2 + r + 1)$. For $r \geq 3$. Therefore, $\sigma\big(\sigma(n)\big) < 2\sigma(n)$. Hence $\sigma(n)$ is deficient.

Since,

$\sigma\big(d(n)\big) = \sigma\big(d(2r^2)\big) = \sigma\big(d(2)d(r^2)\big) = \sigma(2.3) = \sigma(6) = 12 = 2.\, d(2r^2)\sigma\big(d(n)\big) = 2d(n)$.

Hence $d(n)$ is perfect. Since, $\sigma(2r^2) = \sigma(2)\sigma(r^2) = 3(r^2 + r + 1) > 4r^2$, for $r \geq 3$.

Therefore, $\sigma(n) > 2n$. Hence, n is abundant.

Preposition 2.1 *If $n = 8$ then $\sigma(n)$ and $d(n)$ both are deficient i.e. $n = 8$ is a nobly deficient number.*

Proof: Since, $\sigma\big(\sigma(8)\big) = \sigma(15) = 24 < 2\sigma(8) = 2.15 = 30$, therefore, $\sigma(8)$ is deficient.

Since, $\sigma\big(d(8)\big) = \sigma(4) = 7 < 2d(8) = 2.4 = 8$, therefore, $d(8)$ is deficient. Hence $n = 8$ is nobly deficient.

Theorem 2.5 *All multiplicatively Perfect numbers of the form $n = p_1 p_2$ are modified e-Perfect numbers.*

Proof: Using (8), since, $p_e(p_1 p_2) = p_e(p_1)p_e(p_2) = 1$, therefore, (9) holds true.

Hence n is a Modified e-Perfect number.

Theorem 2.6 *All multiplicatively Perfect numbers of the form $n = p_1p_2$ are e-Harmonic of both types.*

Proof: If n is product of distinct primes then n is e-Harmonic of both types (proof has been given in Sandor and E. Egri [8]). Hence $n = p_1p_2$ is e-Harmonic of both types.

Theorem 2.7 *If p is an arbitrary prime, then $n = p$ is e-Harmonic of both types and modified e-Perfect.*

Proof: For, $n = p$, $p_e(p) = 1$. Hence (9) holds true. Also by the proof given in Sandor and Egri [3], $n = p$ is e-Harmonic of both types.

Theorem 2.8 *If $n = p^{\alpha}$, where α is an even positive integer.Then n is a geometric number.*

Proof: Let $n = p^{\alpha}$, where α is an even positive integer. Let $\alpha = 2r$, then $n = p^{2r}$ using (1.1), we get

$R(n) = (p^{2r})^{\frac{2r+1}{2}}$, now using (1.10), we get $G(n) = [(p^{2r})^{\frac{(2r+1)}{2}}]^{\frac{1}{(2r+1)}} = (p^{2r})^{\frac{1}{2}} = p^r = \sqrt{p^{2r}}$,

which is always an integer. Hence $n = p^{\alpha}$, where α is an even positive integer and is a geometric number.

Theorem 2.9 *All prime numbers are modified e-Perfect numbers but not a geometric number.*

Proof: Let $n = p$, where p is a prime number. Since $p_e(p) = 1$, therefore, using (9) n is a modified e-Perfect number.

Since $R(p) = p$, therefore, using (10), $G(n)$ is not an integer. Hence $n = p$ is not a geometric number.

Remark- Since, $p_e(36) = 12$ and $p_e(144) = 36$ therefore using (1.9), $2^2.3^2$ and $2^4 3^2$ are modified e-Perfect numbers. Also both are perfect squares therefore, these are geometric numbers too.

Theorem 2.10 *All almost perfect numbers which are even power of 2 are geometric numbers too.*

Proof: Let $n = 2^r$, where r is an even positive integer. By the definition of almost perfect numbers introduced in Mladen V. Vassilev Missana and Krassimir T. Atanassov [6], if n is even power of 2, then it is almost perfect number. Now it is clear by theorem 2.8 that n is a geometric number.

Preposition 2 $n = 36$ *is a positive integer which is e-Perfect, e Harmonic of type 1, e Harmonic of type 2, modified e-Perfect and geometric numbers too.*

Proof: Now $n = 36 = 2^2.3^2$. Since, $\sigma_e(2^2.3^2) = 2.3^2 + 2^2.3 + 2^2.3^2 + 2.3 = 72$,

therefore using (12) n is a e-Perfect number. Now $d_e(2^2.3^2) = 4$ and $\sigma_e(2^2.3^2) = 72$.

Using (1.6) n is e-Harmonic of type 1. Since, $p_e(2^2.3^2) = (2+1)(3+1) = 12$ therefore, using (17) n is e -Harmonic of type 2. Using (9), n is also modified e-Perfect number. Since $n = 36$ is a perfect square number therefore using (1.11) n is also a geometric number.

Theorem 2.11 *All unitary perfect numbers are unitary harmonic too.*

Proof: Let $n = p_1^{\alpha_1}.p_2^{\alpha_2} \dots \dots p_r^{\alpha_r}$, $\alpha_i \geq 1$ be an unitary perfect number. Using (14) we have

$d^*(n) = 2^r$ and $\sigma^*(n) = 2n$. So, $\sigma^*(n)/n.d^*(n)$. Hence n is an unitary harmonic number.

Theorem 2.12 *If $n = p^{2\alpha}$ with $\alpha \geq 1$ and p is prime then n is an unitary geometric number.*

Proof: Let $n = p^{2\alpha}$, using (14), (15), (16), (17), we have $d^*(n) = 2$ and $R^*(n) = (p^{2\alpha})^{2/2}$.

So, $G^*(n) = (p^{2\alpha})^{\frac{1}{2}} = p^\alpha = \sqrt{n}$.

Conclusion

A new perfect number called multiplicatively perfect number (also called M-perfect) is discussed. We have also studied about abundant, deficient, harmonic and unitary analogue of harmonic numbers.

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